

**SOME NEW SUBCLASSES OF MEROMORPHICALLY
MULTIVALENT FUNCTIONS DEFINED BY MEANS OF THE
LIU-SRIVASTAVA OPERATOR**

(COMMUNICATED BY H.M.SRIVASTAVA)

LI-NA MA, SHU-HAI LI

ABSTRACT. In this paper, we introduce and investigate the various properties and characteristics of the class $M_{a,c}(p; \beta; A, B)$ and its subclass $M_{a,c}^+(p; \beta; A, B)$ of meromorphically p -valent functions of order $\beta(p\beta > 1)$, which are defined by means of the Liu-Srivastava operator. In particular, several inclusion relations, coefficients estimates, distortion theorems, neighborhoods partial, Hadamard products and fractional calculus are proven here for each of these function classes.

1. INTRODUCTION AND DEFINITIONS

Let \mathcal{A} denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by \mathcal{S} the subclass of \mathcal{A} , consisting of functions which are also univalent in \mathbb{U} .

A function $f \in \mathcal{S}$ is said to be reverse starlike of order α if it satisfies

$$\Re\left(\frac{f(z)}{zf'(z)}\right) > \alpha,$$

which is equivalent to

$$\frac{1}{(1-\alpha)}\left(\frac{f(z)}{zf'(z)} - \alpha\right) \prec \frac{1+z}{1-z},$$

for $0 \leq \alpha < 1$ and for all $z \in \mathbb{U}$.

In 1992, Yamakawa[15] first considered it and J.S.Kang et al.[5] further studied it in 1996.

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S.Najafzadeh and S.R.Kulkarni[10] and M.K.Aouf et al.[1] characterized and discussed some subclasses of this kind of function with two different operators.

Recently, more and more researchers are interested in the reverse case of the starlike functions.

In this paper, we consider some new subclass of meromorphically p-valent functions with the reverse case.

Let Σ_p denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathbb{N} = \{1, 2, \dots\}), \quad (1.2)$$

which are analytic and p-valent in the punctured open unit disc

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

For functions $f \in \Sigma_p$ given by (1.2) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \quad (p \in \mathbb{N}), \quad (1.3)$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) := z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z). \quad (1.4)$$

The Liu-Srivastava linear operator $L_p(a, c)$ is defined as follows(see[8])

$$L_p(a, c)f(z) := \phi_p(a, c; z) * f(z) \quad (f \in \Sigma_p), \quad (1.5)$$

and $\phi_p(a, c; z)$ is defined by

$$\phi_p(a, c; z) := z^{-p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p}, \quad (1.6)$$

$$(z \in \mathbb{U}^*; a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = 0, -1, -2, \dots),$$

where $(\lambda)_n$ is the Pochhammer symbol(or the shifted factorial) defined (in terms of the Gamma function) by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1), & n \in \mathbb{N}. \end{cases} \quad (1.7)$$

It is easily verified from the definitions (1.5) and (1.6) that

$$z(L_p(a, c)f(z))' = aL_p(a + 1, c)f(z) - (a + p)L_p(a, c)f(z). \quad (1.8)$$

The Liu-Srivastava operator $L_p(a, c)$ was considered by Liu and Srivastava [8]. A linear operator, analogous to $L_p(a, c)$, was introduced earlier by Saitoh [12]. In[8], making use of the linear operator $L_p(a, c)$, Liu and Srivastava discussed the subclass of Σ_p such that

$$\frac{z(L_p(a, c)f(z))'}{pL_p(a, c)f(z)} \prec -\frac{1 + Az}{1 + Bz}. \quad (1.9)$$

By using the Liu-Srivastava operator $L_p(a, c)$, we introduce a new subclass $M_{a,c}(p; \beta; A, B)$ such that the following subclass of meromorphically p-valent functions for $f \in \Sigma_p$, $p\beta > 1$, $-1 \leq B < A \leq 1$ and

$$\frac{p}{1 - p\beta} \left(\frac{L_p(a, c)f(z)}{z(L_p(a, c)f(z))'} + \beta \right) \prec -\frac{1 + Az}{1 + Bz}. \quad (1.10)$$

According to subordination theory, (1.10) is equivalent to

$$\frac{z(L_p(a, c)f(z))'}{pL_p(a, c)f(z)} \prec -\frac{1 + Bz}{1 + [A - p\beta(A - B)]z},$$

or equivalently, the following inequality holds true

$$\left| -\frac{pL_p(a, c)f(z) + z(L_p(a, c)f(z))'}{B(pL_p(a, c)f(z) + z(L_p(a, c)f(z))' + (1 - p\beta)(A - B)z(L_p(a, c)f(z))'} \right| < 1. \quad (1.11)$$

Remark. For $a = c$, $A = 1 - 2\rho$, $B = -1$, $0 \leq \rho < 1$, $L_p(a, c)f(z) = f(z)$, if $f \in M_{a,a}(p; \beta; 1 - 2\rho, -1)$, then

$$\frac{p}{1 - p\beta} \left(\frac{f(z)}{zf'(z)} + \beta \right) \prec -\frac{1 + (1 - 2\rho)z}{1 - z},$$

which is equivalent to

$$\frac{p}{p\beta - 1} \Re \left(\frac{f(z)}{zf'(z)} + \beta \right) > \rho.$$

Especially, for $p = 1$, if $f \in M_{a,a}(1; \beta; 1 - 2\rho, -1)$, then

$$\frac{1}{\beta - 1} \Re \left(\frac{f(z)}{zf'(z)} + \beta \right) > \rho.$$

Furthermore, we say that a function $f \in M_{a,c}(p; \beta; A, B)$ is in the analogous class $M_{a,c}^+(p; \beta; A, B)$ whenever $f(z)$ is of the form

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k \quad (p \in \mathbb{N}). \quad (1.12)$$

2. INCLUSION PROPERTIES OF THE CLASS $M_{a,c}(p; \beta; A, B)$

We begin by recalling the following result (popularly known as Jack's Lemma), which we shall apply in proving our first inclusion theorem (Theorem 2.1 below).

Lemma 2.1. (Jack [4]) *Let the (nonconstant) function $\omega(z)$ be analytic in \mathbb{U} with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then*

$$z_0 \omega'(z_0) = \gamma \omega(z_0)$$

where γ is a real number and $\gamma \geq 1$.

Theorem 2.1. *If*

$$a \geq \frac{p(A - B)(p\beta - 1)}{1 + [A - p\beta(A - B)]}, \quad (-1 \leq B < A \leq 1; \frac{1}{p} < \beta < \frac{1 + A}{p(A - B)}; p \in \mathbb{N}),$$

then

$$M_{a+1,c}(p; \beta; A, B) \subset M_{a,c}(p; \beta; A, B).$$

Proof. Let $f \in M_{a+1,c}(p; \beta; A, B)$ and suppose that

$$\frac{p}{1 - p\beta} \left(\frac{L_p(a, c)f(z)}{z(L_p(a, c)f(z))'} + \beta \right) = -\frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad (2.1)$$

where the function $\omega(z)$ is either analytic or meromorphic in \mathbb{U} with $\omega(0) = 0$. Then, by using (1.8) and (2.1), we have

$$\frac{aL_p(a+1, c)f(z)}{L_p(a, c)f(z)} = \frac{a + \{(a+p)[A - p\beta(A-B)] - Bp\}\omega(z)}{1 + [A - p\beta(A-B)]\omega(z)}. \quad (2.2)$$

Upon differentiating both sides of (2.2) with respect to z logarithmically, if we make use of (1.8) once again, we obtain

$$\begin{aligned} \frac{z(L_p(a+1, c)f(z))'}{L_p(a+1, c)f(z)} &= \frac{p(A-B)(1-p\beta)z\omega'(z)}{(a + \{(a+p)[A - p\beta(A-B)] - Bp\}\omega(z))(1 + [A - p\beta(A-B)]\omega(z))} \\ &\quad - \frac{p(1 + B\omega(z))}{1 + [A - p\beta(A-B)]\omega(z)}. \end{aligned} \quad (2.3)$$

If we suppose that

$$\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1 \quad (z_0 \in \mathbb{U}), \quad (2.4)$$

and apply Jack's Lemma, we find that

$$z_0\omega'(z_0) = \gamma\omega(z_0) \quad (\gamma \geq 1). \quad (2.5)$$

Now, upon setting

$$\omega(z_0) = e^{i\theta} \quad (0 \leq \theta < 2\pi).$$

If we put $z = z_0$ in (2.3), we get

$$\begin{aligned} &\left| -\frac{pL_p(a+1, c)f(z_0) + z_0(L_p(a+1, c)f(z_0))'}{B[pL_p(a+1, c)f(z_0) + z_0(L_p(a+1, c)f(z_0))'] + (1-p\beta)(A-B)z_0(L_p(a+1, c)f(z_0))'} \right|^2 - 1 \\ &= \left| \frac{(\gamma + a) + \{(a+p)[A - p\beta(A-B)] - Bp\}e^{i\theta}}{a + \{[A - p\beta(A-B)](a+p-\gamma) - Bp\}e^{i\theta}} \right|^2 - 1 \\ &= \frac{\gamma^2 + 2a\gamma + \gamma\{(2a+2p-\gamma)M - 2Bp\}M + 2\gamma((2a+p)M - Bp)\cos\theta}{|a + \{M(a+p-\gamma) - Bp\}e^{i\theta}|^2}. \end{aligned}$$

$(M := A - p\beta(A-B))$

Set

$$g(t) = \gamma^2 + 2a\gamma + \gamma\{(2a+2p-\gamma)M - 2Bp\}M + 2\gamma((2a+p)M - Bp)t. \quad (2.6)$$

Then, by conditions, we have

$$g(1) = \gamma(1+M)\{2a(1+M) + \gamma(1-M) - 2p(B-M)\} > 0$$

and

$$g(-1) = \gamma(1-M)\{2a(1-M) + \gamma(1+M) + 2p(B-M)\} > 0,$$

which, together, imply that

$$g(\cos\theta) \geq 0 \quad (0 \leq \theta < 2\pi). \quad (2.7)$$

In view of (2.6) and (2.7), it would obviously contradict our hypothesis that

$$f \in M_{a+1, c}(p; \beta; A, B).$$

Thus we must have

$$|\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

and we conclude from (2.1) that

$$f \in M_{a, c}(p; \beta; A, B),$$

which evidently completes the proof of Theorem 2.1. \square

Next we prove an inclusion property associated with a certain integral transform.

Theorem 2.2. *Let λ be a complex number such that*

$$\Re(\lambda) \geq \frac{p(A-B)(p\beta-1)}{1+[A-p\beta(A-B)]} \quad (-1 \leq B < A \leq 1; \frac{1}{p} < \beta < \frac{1+A}{p(A-B)}; p \in \mathbb{N}).$$

If $f(z) \in M_{a,c}(p; \beta; A, B)$, then the function $F(z)$ defined by

$$F(z) := \frac{\lambda}{z^{\lambda+p}} \int_0^z t^{\lambda+p-1} f(t) dt \quad (2.8)$$

also belongs to the class $M_{a,c}(p; \beta; A, B)$

Proof. Suppose that $f \in M_{a,c}(p; \beta; A, B)$ and put

$$\frac{p}{1-p\beta} \left(\frac{L_p(a, c)F(z)}{z(L_p(a, c)F(z))'} + \beta \right) = -\frac{1+A\omega(z)}{1+B\omega(z)}. \quad (2.9)$$

From (2.8), we have

$$\frac{z(L_p(a, c)F(z))'}{L_p(a, c)F(z)} = \frac{\lambda L_p(a, c)f(z)}{L_p(a, c)F(z)} - (\lambda + p) \quad (2.10)$$

where the function $\omega(z)$ is either analytic or meromorphic in \mathbb{U} with $\omega(0) = 0$.

Then by using (2.9) and (2.10), we find after some computations that

$$\begin{aligned} & \frac{z(L_p(a, c)f(z))'}{L_p(a, c)f(z)} \\ &= \frac{p(A-B)(1-p\beta)z\omega'(z)}{(\lambda + \{(\lambda+p)[A-p\beta(A-B)] - Bp\}\omega(z))(1+[A-p\beta(A-B)]\omega(z))} \\ & \quad - \frac{p(1+B\omega(z))}{1+[A-p\beta(A-B)]\omega(z)} \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} & \frac{z(L_p(a, c)f(z))'}{L_p(a, c)f(z)} + p \\ &= \frac{p(A-B)(1-p\beta)z\omega'(z)}{(\lambda + \{(\lambda+p)[A-p\beta(A-B)] - Bp\}\omega(z))(1+[A-p\beta(A-B)]\omega(z))} \\ & \quad + \frac{p(A-B)(1-p\beta)\omega(z)}{1+[A-p\beta(A-B)]\omega(z)}. \end{aligned} \quad (2.12)$$

Now we follow the lines of the proof of Theorem 2.1 and assume that (2.4) and (2.5) hold true. Thus, by writing $\omega(z_0) = e^{i\theta}$ ($0 \leq \theta < 2\pi$) and setting $z = z_0$ in (2.11) and (2.12), we obtain

$$\begin{aligned} & \left| -\frac{pL_p(a, c)f(z_0) + z_0(L_p(a, c)f(z_0))'}{B[pL_p(a, c)f(z_0) + z_0(L_p(a, c)f(z_0))'] + (1-p\beta)(A-B)z_0(L_p(a, c)f(z_0))'} \right|^2 - 1 \\ &= \left| -\frac{p + \frac{z_0(L_p(a, c)f(z_0))'}{L_p(a, c)f(z_0)}}{B(p + \frac{z_0(L_p(a, c)f(z_0))'}{L_p(a, c)f(z_0)}) + (1-p\beta)(A-B)\frac{z_0(L_p(a, c)f(z_0))'}{L_p(a, c)f(z_0)}} \right|^2 - 1 \\ &= \left| \frac{(\gamma + \lambda) + \{(\lambda+p)[A-p\beta(A-B)] - Bp\}e^{i\theta}}{\lambda + \{[A-p\beta(A-B)](\lambda+p-\gamma) - Bp\}e^{i\theta}} \right|^2 - 1 \\ &= \frac{\Omega(\theta)}{|\lambda + \{[A-p\beta(A-B)](\lambda+p-\gamma) - Bp\}e^{i\theta}|^2}, \end{aligned} \quad (2.13)$$

where, for convenience,

$$\begin{aligned}\Omega(\theta) &:= |(\gamma + \lambda) + \{(\lambda + p)M - Bp\}e^{i\theta}|^2 - |\lambda + \{M(\lambda + p - \gamma) - Bp\}e^{i\theta}|^2 \\ &= 2\gamma\Re(\lambda) + \gamma^2 + \gamma(2\Re(\lambda) + 2p - \gamma)M^2 \\ &\quad - 2pB\gamma M + 2\gamma \cos \theta (M(2\Re(\lambda) + p) - Bp) \\ (M &:= A - p\beta(A - B); -1 \leq B < A \leq 1; \gamma \geq 1; 0 \leq \theta < 2\pi).\end{aligned}\quad (2.14)$$

Set

$$g(t) = \gamma^2 + 2\gamma\Re(\lambda) + \gamma\{(2\Re(\lambda) + 2p - \gamma)M - 2Bp\}M + 2\gamma((2\Re(\lambda) + p)M - Bp)t \quad (2.15)$$

By conditions

$$\Re(\lambda) \geq \frac{p(A - B)(p\beta - 1)}{1 + [A - p\beta(A - B)]},$$

so that

$$g(1) = \gamma(1 + M)\{2\Re(\lambda)(1 + M) + \gamma(1 - M) - 2p(B - M)\} \geq 0$$

and

$$g(-1) = \gamma(1 - M)\{2\Re(\lambda)(1 - M) + \gamma(1 + M) + 2p(B - M)\} \geq 0,$$

which imply that

$$g(\cos \theta) \geq 0 \quad (0 \leq \theta < 2\pi). \quad (2.16)$$

So, we have

$$\Omega(\theta) \geq 0 \quad (0 \leq \theta < 2\pi). \quad (2.17)$$

In view of (2.17), (2.13) would obviously contradict our hypothesis that

$$f \in M_{a,c}(p; \beta; A, B).$$

Hence, we must have

$$|\omega(z)| < 1 \quad (z \in \mathbb{U})$$

and we conclude from (2.9) that

$$F \in M_{a,c}(p; \beta; A, B),$$

where the function $F(z)$ is given by (2.8).

The proof of Theorem 2.2 is thus completed. \square

Theorem 2.3. *Set $-1 \leq B < A \leq 1; p\beta > 1; a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = 0, -1, -2, \dots$, the function $f \in M_{a,c}(p; \beta; A, B)$ if and only if the function $F(z)$ given by*

$$F(z) := \frac{a}{z^{a+p}} \int_0^z t^{a+p-1} f(t) dt \quad (2.18)$$

belongs to the class $M_{a+1,c}(p; \beta; A, B)$.

Proof. By using of (2.18), we have

$$af(z) = (a + p)F(z) + zF'(z),$$

which, according to (1.8), implies

$$aL_p(a, c)f(z) = (a + p)L_p(a, c)F(z) + z(L_p(a, c)F(z))' = aL_p(a + 1, c)F(z).$$

Therefore, we have

$$L_p(a, c)f(z) = L_p(a + 1, c)F(z)$$

and the desired result follows at once. \square

3. BASIC PROPERTIES OF THE CLASS $M_{a,c}^+(p; \beta; A, B)$

In this section, we assume further that

$$a > 0, c > 0.$$

We first determine a necessary and sufficient condition for a function $f \in \Sigma_p$ of the form (1.12) to be in the class $M_{a,c}^+(p; \beta; A, B)$ of meromorphically p -valent functions with positive coefficients.

Theorem 3.1. *Let $-1 \leq B < A \leq 1, p\beta > 1, p \in \mathbb{N}$ and $f \in \Sigma_p$ be given by (1.12). Then $f \in M_{a,c}^+(p; \beta; A, B)$ if and only if*

$$\sum_{k=p}^{\infty} [(k+p)(1-B) + k(A-B)(p\beta-1)] \frac{(a)_{k+p}}{(c)_{k+p}} |a_k| \leq p(A-B)(p\beta-1). \quad (3.1)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{p(A-B)(p\beta-1)}{[(k+p)(1-B) + k(A-B)(p\beta-1)]} \frac{(c)_{k+p}}{(a)_{k+p}} z^k \quad (3.2)$$

$(k = p, p+1, p+2, \dots; p \in \mathbb{N})$

Proof. Suppose that $f \in M_{a,c}^+(p; \beta; A, B)$ is given by (1.12). Then, making use of (1.11) and (1.12), we obtain

$$\begin{aligned} & \left| \frac{pL_p(a, c)f(z) + z(L_p(a, c)f(z))'}{B[pL_p(a, c)f(z) + z(L_p(a, c)f(z))'] + (A-B)(1-p\beta)z(L_p(a, c)f(z))'} \right| \\ &= \left| \frac{\sum_{k=p}^{\infty} (k+p) \frac{(a)_{k+p}}{(c)_{k+p}} |a_k| z^{k+p}}{p(A-B)(p\beta-1) + \sum_{k=p}^{\infty} (B(k+p) - k(A-B)(p\beta-1)) \frac{(a)_{k+p}}{(c)_{k+p}} |a_k| z^{k+p}} \right| \\ &< 1 \quad (z \in \mathbb{U}) \end{aligned} \quad (3.3)$$

Since $|\Re(z)| \leq |z|$ for any z , choosing z to be real and letting $z \rightarrow 1^-$ through real values, (3.3) yields

$$\sum_{k=p}^{\infty} (k+p) \frac{(a)_{k+p}}{(c)_{k+p}} |a_k| \leq p(A-B)(p\beta-1) + \sum_{k=p}^{\infty} (B(k+p) - k(A-B)(p\beta-1)) \frac{(a)_{k+p}}{(c)_{k+p}} |a_k| \quad (3.4)$$

So, we can obtain the desired inequality (3.1).

In order to prove the converse, we assume that the inequality (3.1) holds true. Then, if we let $z \in \partial\mathbb{U}$ we find from (1.12) and (3.1) that

$$\begin{aligned} & \left| \frac{pL_p(a, c)f(z) + z(L_p(a, c)f(z))'}{B[pL_p(a, c)f(z) + z(L_p(a, c)f(z))'] + (A-B)(1-p\beta)z(L_p(a, c)f(z))'} \right| \\ & < \frac{\sum_{k=p}^{\infty} (k+p) \frac{(a)_{k+p}}{(c)_{k+p}} |a_k|}{p(A-B)(p\beta-1) + \sum_{k=p}^{\infty} (B(k+p) - k(A-B)(p\beta-1)) \frac{(a)_{k+p}}{(c)_{k+p}} |a_k|} \\ & \leq 1 \quad (z \in \partial\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}) \end{aligned} \quad (3.5)$$

Hence, by the Maximum Modulus Theorem, we have $f(z) \in M_{a,c}^+(p; \beta; A, B)$.

By observing that the function $f(z)$ given by (3.2) is indeed an extremal function for the assertion (3.1), we complete the proof of Theorem 3.1. \square

By applying Theorem 3.1, we obtain the following sharp coefficient estimates.

Corollary 3.1. *Let $-1 \leq B < A \leq 1$, $p\beta > 1$, $p \in \mathbb{N}$ and $f \in \Sigma_p$ be given by (1.12). If $f(z) \in M_{a,c}^+(p; \beta; A, B)$, then*

$$|a_k| \leq \frac{p(A-B)(p\beta-1)}{(1-B)(k+p) + k(A-B)(p\beta-1)} \cdot \frac{(c)_{k+p}}{(a)_{k+p}} \quad (3.6)$$

$$(k = p, p+1, p+2, \dots; p \in \mathbb{N}).$$

Each of these inequalities is sharp, with the extremal function $f(z)$ given by (3.2).

Next we prove the following growth and distortion properties of the class $M_{a,c}^+(p; \beta; A, B)$.

Theorem 3.2. *Let $f \in M_{a,c}^+(p; \beta; A, B)$, $-1 \leq B < A \leq 1$, $p\beta > 1$, $p \in \mathbb{N}$. If the sequence $\{C_k\}$ is nondecreasing, then*

$$r^{-p} - \frac{p(A-B)(p\beta-1)}{C_p} r^p \leq |f(z)| \leq r^{-p} + \frac{p(A-B)(p\beta-1)}{C_p} r^p, \quad (0 < |z| = r < 1) \quad (3.7)$$

where

$$C_k = [(k+p)(1-B) + k(A-B)(p\beta-1)] \frac{(a)_{k+p}}{(c)_{k+p}}. \quad (k = p, p+1, p+2, \dots; p \in \mathbb{N}) \quad (3.8)$$

If the sequence $\{C_k/k\}$ is nondecreasing, then

$$pr^{-p-1} - \frac{p^2(A-B)(p\beta-1)}{C_p} r^{p-1} \leq |f'(z)| \leq pr^{-p-1} + \frac{p^2(A-B)(p\beta-1)}{C_p} r^{p-1} \quad (0 < |z| = r < 1) \quad (3.9)$$

Each of these inequalities is sharp, with the extremal function $f(z)$ given by (3.2).

Proof. Let the function $f(z)$ of the form (1.12) be in the class $M_{a,c}^+(p; \beta; A, B)$. If the sequence $\{C_k\}$ is nondecreasing, then (by Theorem 3.1) we have

$$\sum_{k=p}^{\infty} |a_k| \leq \frac{p(A-B)(p\beta-1)}{C_p}. \quad (3.10)$$

On the other hand, if the sequence $\{C_k/k\}$ is nondecreasing, Theorem 3.1 also implies

$$\sum_{k=p}^{\infty} k|a_k| \leq \frac{p^2(A-B)(p\beta-1)}{C_p} \quad (3.11)$$

Thus, assertions (3.7) and (3.9) follow immediately.

Finally, it is easy to see that the bounds in (3.7) and (3.9) are attained for the function $f(z)$ given by (3.2) with $k = p$. \square

Next we determine the radii of meromorphically p -valent starlikeness and meromorphically p -valent convexity of the class $M_{a,c}^+(p; \beta; A, B)$. We first state our results as

Theorem 3.3. Let $-1 \leq B < A \leq 1$, $p\beta > 1$, $p \in \mathbb{N}$ and $f(z)$ be given by (1.12) be in the class $M_{a,c}^+(p; \beta; A, B)$. Then we have

(i) f is meromorphically p -valent starlike of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_1$, that is,

$$\Re \left(-\frac{zf'(z)}{pf(z)} \right) > \delta \quad (|z| < r_1; 0 \leq \delta < 1; p \in \mathbb{N}) \quad (3.12)$$

where

$$r_1 := \inf_{k \geq p} \left\{ \frac{(1-\delta)[(k+p)(1-B) + k(A-B)(p\beta-1)]}{(A-B)(p\beta-1)(k+\delta p)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \right\}^{\frac{1}{k+p}}; \quad (3.13)$$

(ii) f is meromorphically p -valent convex of order δ ($0 \leq \delta < 1$) in the disk $|z| < r_2$, that is,

$$\Re \left(-\frac{(zf'(z))'}{pf'(z)} \right) > \delta \quad (|z| < r_2; 0 \leq \delta < 1; p \in \mathbb{N}) \quad (3.14)$$

where

$$r_2 = \inf_{k \geq p} \left\{ \frac{p(1-\delta)[(k+p)(1-B) + k(A-B)(p\beta-1)]}{k(A-B)(p\beta-1)(k+\delta p)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \right\}^{\frac{1}{k+p}}. \quad (3.15)$$

Each of these inequalities is sharp, with the extremal function $f(z)$ given by (3.2).

Proof. (i) From the definition (1.12), we easily get

$$\left| \frac{1 + \frac{zf'(z)}{pf(z)}}{\frac{zf'(z)}{pf(z)} + 2\delta - 1} \right| \leq \frac{\sum_{k=p}^{\infty} (k+p)|a_k||z|^{k+p}}{2p(1-\delta) - \sum_{k=p}^{\infty} [k - (1-2\delta)p]|a_k||z|^{k+p}}. \quad (3.16)$$

Thus we have the desired inequality

$$\left| \frac{1 + \frac{zf'(z)}{pf(z)}}{\frac{zf'(z)}{pf(z)} + 2\delta - 1} \right| \leq 1 \quad (0 \leq \delta < 1; p \in \mathbb{N}), \quad (3.17)$$

if

$$\sum_{k=p}^{\infty} \frac{(k+\delta p)}{p(1-\delta)} |a_k||z|^{k+p} \leq 1, \quad (3.18)$$

that is, if

$$\frac{(k+\delta p)}{p(1-\delta)} |z|^{k+p} \leq \frac{(k+p)(1-B) + k(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}}, \quad (3.19)$$

$$(k = p, p+1, p+2, \dots; p \in \mathbb{N})$$

where we have made use of the assertion (3.1) of Theorem 3.1, since

$$f \in M_{a,c}^+(p; \beta; A, B).$$

The last inequality (3.19) leads us immediately to the disk $|z| < r_1$, where r_1 is given by (3.13).

(ii) In order to prove the second assertion of Theorem 3.3, we find from the definition (1.12) that

$$\left| \frac{1 + \frac{(zf'(z))'}{pf'(z)}}{\frac{(zf'(z))'}{pf'(z)} + 2\delta - 1} \right| \leq \frac{\sum_{k=p}^{\infty} k(k+p)|a_k||z|^{k+p}}{2p^2(1-\delta) - \sum_{k=p}^{\infty} k[k+p(2\delta-1)]|a_k||z|^{k+p}}. \quad (3.20)$$

Thus we have the desired inequality

$$\left| \frac{1 + \frac{(zf'(z))'}{pf'(z)}}{\frac{(zf'(z))'}{pf'(z)} + 2\delta - 1} \right| \leq 1 \quad (0 \leq \delta < 1; p \in \mathbb{N}), \quad (3.21)$$

if

$$\sum_{k=p}^{\infty} \frac{k(k+\delta p)}{p^2(1-\delta)} |a_k||z|^{k+p} \leq 1, \quad (3.22)$$

that is, if

$$\frac{k(k+\delta p)}{p^2(1-\delta)} |z|^{k+p} \leq \frac{(k+p)(1-B) + k(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \quad (3.23)$$

$$(k = p, p+1, p+2, \dots; p \in \mathbb{N}).$$

The last inequality (3.23) readily yields the disk $|z| < r_2$ with r_2 defined by (3.15), and the proof of Theorem 3.3 is completed by merely verifying that each assertion is sharp for the function $f(z)$ given by (3.2). \square

Theorem 3.4. *Let $-1 \leq B < A \leq 1, p\beta > 1, a > 0, c > 0, \nu > 0$. If $f \in M_{a,c}^+(p; \beta; A, B)$, then $F(z)$ defined by*

$$F(z) = \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu+p-1} f(t) dt$$

belongs to $M_{a,c}^+(p; \beta; A, B)$.

Proof. If $f(z)$ of the form (1.12) be in the class $M_{a,c}^+(p; \beta; A, B)$, by Theorem 3.1, we have

$$\sum_{k=p}^{\infty} \frac{(k+p)(1-B) + k(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} |a_k| \leq 1.$$

For

$$F(z) = \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu+p-1} f(t) dt = z^{-p} + \sum_{k=p}^{\infty} \frac{\nu}{\nu+k+p} |a_k| z^k,$$

we obtain

$$\begin{aligned} & \sum_{k=p}^{\infty} \frac{(k+p)(1-B) + k(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \cdot \frac{\nu}{\nu+k+p} |a_k| \\ & \leq \sum_{k=p}^{\infty} \frac{(k+p)(1-B) + k(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} |a_k| \\ & \leq 1, \end{aligned}$$

which implies that $F(z) \in M_{a,c}^+(p; \beta; A, B)$.

This evidently completes the proof of Theorem 3.4. \square

4. NEIGHBORHOODS AND PARTIAL SUMS

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman[3] and Ruscheweyh[11], and (more recently) by J.L.Liu and H.M.Srivastava[8], and M.S.Liu and N.S.Song[7], we begin by introducing here the δ -neighborhood of a function $f(z) \in \Sigma_p$ of the form (1.2) by means of the definition :

$$N_\delta(f) := \{g \in \Sigma_p : g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \text{ and}$$

$$\sum_{k=1}^{\infty} \frac{k(1 + |A - p\beta(A - B)|) + p(A - B)(p\beta - 1)}{p(A - B)(p\beta - 1)} \cdot \frac{(a)_k}{(c)_k} |a_k - b_k| \leq \delta$$

$$(a > 0; c > 0; 1 \leq B < A \leq 1; \delta \geq 0; p\beta > 1)\}$$
 (4.1)

Theorem 4.1. *Let $f \in M_{a,c}(p; \beta; A, B)$ ($-1 \leq B < A \leq 1; p\beta > 1$) be given by (1.2). If f satisfies the condition*

$$\frac{f(z) + \epsilon z^{-p}}{1 + \epsilon} \in M_{a,c}(p; \beta; A, B) \quad (\epsilon \in \mathbb{C}; |\epsilon| < \delta; \delta > 0)$$
 (4.2)

then

$$N_\delta(f) \subset M_{a,c}(p; \beta; A, B).$$
 (4.3)

Proof. It is obvious from (1.11) that $g \in M_{a,c}(p; \beta; A, B)$ if and only if

$$-\frac{pL_p(a, c)g(z) + z(L_p(a, c)g(z))'}{B[pL_p(a, c)g(z) + z(L_p(a, c)g(z))'] + (A - B)(1 - p\beta)z(L_p(a, c)g(z))'} \neq \sigma$$

$$(z \in \mathbb{U}^*, \sigma \in \mathbb{C}; |\sigma| = 1),$$
 (4.4)

which is equivalent to

$$\frac{(g * h)(z)}{z^{-p}} \neq 0 \quad (z \in \mathbb{U}^*),$$
 (4.5)

where, for convenience,

$$h(z) := z^{-p} + \sum_{k=1}^{\infty} c_k z^{k-p}$$

$$= z^{-p} + \sum_{k=1}^{\infty} \frac{k[1 + \sigma(A - p\beta(A - B))] + p\sigma(A - B)(p\beta - 1)}{p\sigma(A - B)(p\beta - 1)} \cdot \frac{(a)_k}{(c)_k} z^{k-p}$$
 (4.6)

We find from (4.6) that

$$|c_k| = \left| \frac{k[1 + \sigma(A - p\beta(A - B))] + p\sigma(A - B)(p\beta - 1)}{p\sigma(A - B)(p\beta - 1)} \cdot \frac{(a)_k}{(c)_k} \right|$$

$$\leq \frac{k(1 + |A - p\beta(A - B)|) + p(A - B)(p\beta - 1)}{p(A - B)(p\beta - 1)} \cdot \frac{(a)_k}{(c)_k}$$
 (4.7)
$$(k = p, p + 1, p + 2, \dots; p \in \mathbb{N})$$

Under the hypothesis of Theorem 4.1, (4.5) yields

$$\left| \frac{(f * h)(z)}{z^{-p}} \right| \geq \delta \quad (z \in \mathbb{U}^*; \delta > 0).$$
 (4.8)

By letting

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \in N_{\delta}(f), \quad (4.9)$$

we have

$$\begin{aligned} & \left| \frac{(f(z) - g(z)) * h(z)}{z^{-p}} \right| \\ &= \left| \sum_{k=1}^{\infty} (a_k - b_k) c_k z^k \right| \\ &\leq |z| \sum_{k=1}^{\infty} \frac{k(1 + |A - p\beta(A - B)|) + p(A - B)(p\beta - 1)}{p(A - B)(p\beta - 1)} \cdot \frac{(a)_k}{(c)_k} |a_k - b_k| \\ &< \delta \quad (z \in \mathbb{U}; \delta > 0). \end{aligned} \quad (4.10)$$

Thus we have (4.5), and hence also (4.4) for any $\sigma \in \mathbb{C}$ such that $|\sigma| = 1$, which implies that $g \in M_{a,c}(p; \beta; A, B)$. This evidently proves the assertion (4.3) of Theorem 4.1. \square

Theorem 4.2. *Let $-1 \leq B < A \leq 1, p\beta > 1$ and $f \in \Sigma_p$ be given by (1.2) and define the partial sums $s_1(z)$ and $s_n(z)$ by*

$$s_1(z) := z^{-p} \quad \text{and} \quad s_n(z) := z^{-p} + \sum_{k=1}^{n-1} a_k z^{k-p} \quad (n \in \mathbb{N} \setminus \{1\}). \quad (4.11)$$

Suppose that

$$\sum_{k=1}^{\infty} d_k |a_k| \leq 1 \quad (d_k = \frac{k(1 + |A - p\beta(A - B)|) + p(A - B)(p\beta - 1)}{p(A - B)(p\beta - 1)} \cdot \frac{(a)_k}{(c)_k}) \quad (4.12)$$

(i) If $a > 0$ and $c > 0$, then $f(z) \in M_{a,c}(p; \beta; A, B)$;

(ii) If $a > c > 0$, then

$$\Re \left(\frac{f(z)}{s_n(z)} \right) > 1 - \frac{1}{d_n} \quad (z \in \mathbb{U}; n \in \mathbb{N}) \quad (4.13)$$

and

$$\Re \left(\frac{s_n(z)}{f(z)} \right) > \frac{d_n}{1 + d_n} \quad (z \in \mathbb{U}; n \in \mathbb{N}) \quad (4.14)$$

Each of the bounds in (4.13) and (4.14) is the best possible for each $n \in \mathbb{N}$.

Proof. (i) It is not difficult to see that

$$z^{-p} \in M_{a,c}(p; \beta; A, B) \quad (p \in \mathbb{N})$$

Thus, from Theorem 4.1 and the hypothesis (4.12), we have

$$f(z) \in N_1(z^{-p}) \subset M_{a,c}(p; \beta; A, B) \quad (a > 0; c > 0; p \in \mathbb{N}) \quad (4.15)$$

as asserted by Theorem 4.2.

(ii) For the coefficients d_k given by (4.12), it is easy to verify that

$$d_{k+1} > d_k > 1 \quad (a > c > 0; k = p, p+1, p+2, \dots; p \in \mathbb{N}) \quad (4.16)$$

So, we have

$$\sum_{k=1}^{n-1} |a_k| + d_n \sum_{k=n}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} d_k |a_k| \leq 1 \quad (4.17)$$

by using the hypothesis (4.12) again.

By setting

$$g_1(z) := d_n \left[\frac{f(z)}{s_n(z)} - \left(1 - \frac{1}{d_n}\right) \right] = 1 + \frac{d_n \sum_{k=n}^{\infty} a_k z^k}{1 + \sum_{k=1}^{n-1} a_k z^k} \quad (4.18)$$

and applying (4.17), we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{d_n \sum_{k=n}^{\infty} |a_k|}{2 - 2 \sum_{k=1}^{n-1} |a_k| - d_n \sum_{k=n}^{\infty} |a_k|} \leq \frac{d_n \sum_{k=n}^{\infty} |a_k|}{1 - \sum_{k=1}^{n-1} |a_k|} \leq 1 \quad (z \in \mathbb{U}) \quad (4.19)$$

which readily yields the assertion (4.13) of Theorem 4.2.

If we take

$$f(z) = z^{-p} - \frac{1}{d_n} z^{n-p}, \quad (4.20)$$

then

$$\frac{f(z)}{s_n(z)} = 1 - \frac{z^n}{d_n} \rightarrow 1 - \frac{1}{d_n} \quad \text{as } z \rightarrow 1^-,$$

which shows that the bound in (4.13) is the best possible for each $n \in \mathbb{N}$.

Similarly, if we put

$$g_2(z) := (1 + d_n) \left(\frac{s_n(z)}{f(z)} - \frac{d_n}{1 + d_n} \right) = 1 - \frac{(1 + d_n) \sum_{k=n}^{\infty} a_k z^k}{1 + \sum_{k=1}^{\infty} a_k z^k} \quad (4.21)$$

and make use of (4.17), we can deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + d_n) \sum_{k=n}^{\infty} |a_k|}{2 - 2 \sum_{k=1}^{n-1} |a_k| + (1 - d_n) \sum_{k=n}^{\infty} |a_k|} \leq 1 \quad (z \in \mathbb{U}), \quad (4.22)$$

which leads us immediately to the assertion (4.14) of Theorem 4.2.

The bound in (4.14) is sharp for each $n \in \mathbb{N}$, with the extremal function $f \in \Sigma_p$ given by (4.20). The proof of Theorem 4.2 is thus completed. \square

We now define the δ -neighborhood of a function $f \in \Sigma_p$ of the form (1.12) as follows:

$$N_{\delta}^+(f) := \{g \in \Sigma_p : g(z) = z^{-p} + \sum_{k=p}^{\infty} |b_k| z^k$$

and

$$\sum_{k=p}^{\infty} \frac{(k+p)(1+|B|) + k(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \| |a_k| - |b_k| \| \leq \delta$$

$$(a > 0; c > 0; -1 \leq B < A \leq 1; \delta \geq 0; p\beta > 1) \quad (4.23)$$

Theorem 4.3. *If $f \in M_{a+1,c}^+(p; \beta; A, B)$ and $-1 \leq B \leq 0$, then*

$$N_\delta^+(f) \subset M_{a,c}^+(p; \beta; A, B) \quad (\delta := \frac{2p}{a+2p}). \quad (4.24)$$

The result is sharp.

Proof. Making use of the same method as in the proof of Theorem 4.1, we can show that

$$\begin{aligned} h(z) &:= z^{-p} + \sum_{k=p}^{\infty} c_k z^k \\ &= z^{-p} + \sum_{k=p}^{\infty} \frac{[(k+p)(1-\sigma B) + \sigma k(A-B)(p\beta-1)]}{p\sigma(B-A)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} z^k \end{aligned} \quad (4.25)$$

we find from (4.25) that

$$\begin{aligned} |c_k| &= \left| \frac{[(k+p)(1-\sigma B) + \sigma k(A-B)(p\beta-1)]}{p\sigma(B-A)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \right| \\ &\leq \frac{[(k+p)(1+|B|) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \\ &\quad (k = p, p+1, p+2, \dots; p \in \mathbb{N}). \end{aligned}$$

Thus, under the hypothesis $-1 \leq B \leq 0$, if $f \in M_{a+1,c}^+(p; \beta; A, B)$ is given by (1.12), we obtain

$$\begin{aligned} \left| \frac{(f * h)(z)}{z^{-p}} \right| &= \left| 1 + \sum_{k=p}^{\infty} c_k |a_k| z^{k+p} \right| \\ &\geq 1 - \sum_{k=p}^{\infty} |c_k| |a_k| \\ &\geq 1 - \sum_{k=p}^{\infty} \frac{[(k+p)(1+|B|) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a+1)_{k+p}}{(c)_{k+p}} \cdot \frac{a}{a+k+p} |a_k| \\ &\geq 1 - \frac{a}{a+2p} \sum_{k=p}^{\infty} \frac{[(k+p)(1+|B|) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a+1)_{k+p}}{(c)_{k+p}} |a_k| \\ &= 1 - \frac{a}{a+2p} \sum_{k=p}^{\infty} \frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a+1)_{k+p}}{(c)_{k+p}} |a_k|. \end{aligned}$$

By Theorem 3.1, we obtain

$$\left| \frac{(f * h)(z)}{z^{-p}} \right| \geq 1 - \frac{a}{a+2p} = \frac{2p}{a+2p} =: \delta$$

The remainder of the proof of Theorem 4.3 is similar to that of Theorem 4.1, and we skip the details involved.

In order to show that the assertion (4.24) of Theorem 4.3 is sharp, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(p\beta-1)}{2(1-B) + (A-B)(p\beta-1)} \cdot \frac{(c)_{2p}}{(a+1)_{2p}} z^p \in M_{a+1,c}^+(p; \beta; A, B) \quad (4.26)$$

and

$$g(z) = z^{-p} + \frac{(A-B)(p\beta-1)}{2(1-B) + (A-B)(p\beta-1)} \cdot \left(\frac{(c)_{2p}}{(a+1)_{2p}} + \frac{\delta'(c)_{2p}}{(a)_{2p}} \right) z^p, \quad (4.27)$$

$$(\delta' > \delta := \frac{2p}{a+2p}).$$

We have

$$\begin{aligned} & \sum_{k=p}^{\infty} \frac{(k+p)(1+|B|) + k(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} ||a_k| - |b_k|| \\ &= \frac{2(1-B) + (A-B)(p\beta-1)}{(A-B)(p\beta-1)} \cdot \frac{(a)_{2p}}{(c)_{2p}} \cdot \frac{(A-B)(p\beta-1)}{2(1-B) + (A-B)(p\beta-1)} \cdot \frac{\delta'(c)_{2p}}{(a)_{2p}} \\ &= \delta'. \end{aligned}$$

Therefore $g(z) \in N_{\delta'}^+(f)$.

Thus

$$\begin{aligned} & \frac{2(1-B) + (A-B)(p\beta-1)}{(A-B)(p\beta-1)} \cdot \frac{(a)_{2p}}{(c)_{2p}} \cdot \frac{(A-B)(p\beta-1)}{2(1-B) + (A-B)(p\beta-1)} \cdot \left(\frac{(c)_{2p}}{(a+1)_{2p}} + \frac{\delta'(c)_{2p}}{(a)_{2p}} \right) \\ &= \frac{a}{a+2p} + \delta' \\ &> 1. \end{aligned}$$

By Theorem 3.1, $g(z)$ is not in the class $M_{a,c}^+(p; \beta; A, B)$. So the proof of Theorem 4.3 is completed. \square

Theorem 4.4. *Let $-1 \leq B \leq 0$ and λ be a real number with*

$$\lambda > \frac{p(A-B)(p\beta-1)}{1 + [A - p\beta(A-B)]}.$$

If the function $f(z)$ given by (1.12) is in the class $M_{a,c}^+(p; \beta; A, B)$, then $F(z)$ defined by (2.8) belongs to $N_1^+(f)$. The result is sharp in the sense that the constant 1 cannot be decreased.

Proof. Suppose $f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k \in M_{a,c}^+(p; \beta; A, B)$, Then it follows from (2.8) and Theorem 3.4 that

$$F(z) = z^{-p} + \sum_{k=p}^{\infty} |b_k| z^k = z^{-p} + \sum_{k=p}^{\infty} \frac{\lambda}{\lambda + p + k} |a_k| z^k \in M_{a,c}^+(p; \beta; A, B) \quad (4.28)$$

From the hypothesis of Theorem 4.4, we have

$$\begin{aligned}
& \sum_{k=p}^{\infty} \frac{[(k+p)(1+|B|) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} ||a_k| - |b_k|| \\
&= \sum_{k=p}^{\infty} \frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \cdot \frac{p+k}{\nu+p+k} \cdot |a_k| \\
&\leq \sum_{k=p}^{\infty} \frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \cdot |a_k| \\
&\leq 1 \quad (f \in M_{a,c}^+(p; \beta; A, B)),
\end{aligned}$$

which shows that $F(z) \in N_1^+(f)$.

In order to verify the sharpness of the assertion Theorem 4.4, we consider the function $f(z)$ given by (3.2). From (3.2) and (4.28), we have

$$\begin{aligned}
F(z) &= \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu+p-1} f(t) dt \\
&= \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu+p-1} \left(t^{-p} + \frac{p(A-B)(p\beta-1)}{[(k+p)(1-B) + k(A-B)(p\beta-1)]} \cdot \frac{(c)_{k+p}}{(a)_{k+p}} t^k \right) dt \\
&= z^{-p} + \frac{p(A-B)(p\beta-1)}{[(k+p)(1-B) + k(A-B)(p\beta-1)]} \cdot \frac{(c)_{k+p}}{(a)_{k+p}} \cdot \frac{\nu}{\nu+p+k} z^k \\
&\quad (k = p, p+1, p+2, \dots; p \in \mathbb{N}).
\end{aligned}$$

Thus, by making use of (4.23), we get

$$\begin{aligned}
& \sum_{k=p}^{\infty} \frac{(k+p)(1+|B|) + k(A-B)(p\beta-1)}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} ||a_k| - |b_k|| \\
&= \frac{p+k}{\nu+p+k} \rightarrow 1 (k \rightarrow \infty),
\end{aligned} \tag{4.29}$$

which clearly shows that the constant 1 is the best possible. This evidently completes the proof of Theorem 4.4.

The proof of Theorem 4.5 below is similar to that of Theorem 4.3 and so is omitted. \square

Theorem 4.5. *Let $-1 \leq B \leq 0, p\beta > 1$ and λ be a real number such*

$$\lambda > \frac{p(A-B)(p\beta-1)}{1 + [A - p\beta(A-B)]}.$$

If $f \in M_{a+1,c}^+(p; \beta; A, B)$, then

$$N_{\delta'}^+(F) \subset M_{a,c}^+(p; \beta; A, B) \quad \left(\delta' = \frac{2p(a+\lambda+2p)}{(\lambda+2p)(a+2p)} \right) \tag{4.30}$$

and $F(z)$ defined by (2.8). The result is sharp in the sense that δ' cannot be increased.

5. CONVOLUTION PROPERTIES

In this section, we first set $a > 0, c > 0$ and define function $f_j(z)(j = 1, 2)$ by

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,j}| z^k \quad (j = 1, 2) \quad (5.1)$$

and define the Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$ as

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,1}| \cdot |a_{k,2}| z^k. \quad (5.2)$$

Theorem 5.1. *Let the functions $f_j(z)(j = 1; 2)$ defined by (5.1) be in the class $M_{a,c}^+(p; \beta; A, B)$, if the sequence $\{(k+p) \frac{(c)_{k+p}}{(a)_{k+p}}\}(k \geq p; p \in \mathbb{N})$ is nondecreasing and*

$$2p(A-B)^2(p\beta-1)^2 - p[2(1-B) + (A-B)(p\beta-1)]^2 \frac{(a)_{2p}}{(c)_{2p}} > 0,$$

then $(f_1 * f_2)(z) \in M_{a,c}^+(p; \eta; A, B)$, where

$$\eta = \frac{1}{p} - \frac{2(A-B)(p\beta-1)^2(1-B)}{p(A-B)^2(p\beta-1)^2 - p[2(1-B) + (A-B)(p\beta-1)]^2 \frac{(a)_{2p}}{(c)_{2p}}}. \quad (5.3)$$

The result is sharp for the functions $f_j(z)(j = 1; 2)$ given by

$$f_j(z) = z^{-p} + \frac{(A-B)(p\beta-1)}{[(1-B) + (A-B)(p\beta-1)]} \cdot \frac{(c)_{2p}}{(a)_{2p}} z^k. \quad (5.4)$$

Proof. Employing the techniques used earlier by Schild and Silverman[13], we need to find the largest η such that

$$\sum_{k=p}^{\infty} \frac{[(k+p)(1-B) + k(A-B)(p\eta-1)]}{p(A-B)(p\eta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} |a_{k,1}| \cdot |a_{k,2}| \leq 1. \quad (5.5)$$

For $f_j(z) \in M_{a,c}^+(p; \beta; A, B)(j = 1, 2)$, we have

$$\sum_{k=p}^{\infty} \frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} |a_{k,j}| \leq 1 \quad (j = 1, 2). \quad (5.6)$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty} \frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \sqrt{|a_{k,1}| \cdot |a_{k,2}|} \leq 1. \quad (5.7)$$

This implies that we only need to show that

$$\begin{aligned} & \frac{[(k+p)(1-B) + k(A-B)(p\eta-1)]}{(p\eta-1)} |a_{k,1} a_{k,2}| \\ & \leq \frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]}{(p\beta-1)} \sqrt{|a_{k,1} a_{k,2}|}, \end{aligned} \quad (5.8)$$

or, equivalently that

$$\sqrt{|a_{k,1} a_{k,2}|} \leq \frac{[(k+p)(1-B) + k(A-B)(p\beta-1)](p\eta-1)}{[(k+p)(1-B) + k(A-B)(p\eta-1)](p\beta-1)} \quad (k \geq p). \quad (5.9)$$

Hence, by inequality (5.7), it is sufficient to prove that

$$\begin{aligned} & \frac{p(A-B)(p\beta-1)}{[(k+p)(1-B)+k(A-B)(p\beta-1)]\frac{(a)_{k+p}}{(c)_{k+p}}} \\ & \leq \frac{[(k+p)(1-B)+k(A-B)(p\beta-1)](p\eta-1)}{[(k+p)(1-B)+k(A-B)(p\eta-1)](p\beta-1)}. \end{aligned} \quad (5.10)$$

It follows from (5.10) that

$$\eta \leq \frac{1}{p} - \frac{(A-B)(p\beta-1)^2(k+p)(1-B)}{kp(A-B)^2(p\beta-1)^2 - [(k+p)(1-B)+k(A-B)(p\beta-1)]^2\frac{(a)_{k+p}}{(c)_{k+p}}}. \quad (5.11)$$

Now, defining the function $\tau(k)$ by

$$\tau(k) = \frac{kp(A-B)^2(p\beta-1)^2}{k+p} - \frac{[(k+p)(1-B)+k(A-B)(p\beta-1)]^2\frac{(a)_{k+p}}{(c)_{k+p}}}{k+p} \quad (k \geq p).$$

We see that $\tau(k)$ is an increasing function of k . Therefore, we conclude that

$$\eta \leq \tau(p) = \frac{1}{p} - \frac{2(A-B)(p\beta-1)^2(1-B)}{p(A-B)^2(p\beta-1)^2 - p[2(1-B) + (A-B)(p\beta-1)]^2\frac{(a)_{2p}}{(c)_{2p}}} \quad (k \geq p),$$

which evidently completes the proof of Theorem 5.1. \square

Using the same methods as in our proof of Theorem 5.1, we obtain the result as follows:

Theorem 5.2. *Let the function $f_1(z)$ defined by (5.1) be in the class $M_{a,c}^+(p; \beta; A, B)$, and the function $f_2(z)$ defined by (5.1) be in the class $M_{a,c}^+(p; \mu; A, B)$, then $(f_1 * f_2)(z) \in M_{a,c}^+(p; \xi; A, B)$, where*

$$\xi = \frac{1}{p} - \frac{2(A-B)(1-B)(p\beta-1)(p\mu-1)}{p(A-B)^2(p\beta-1)(p\mu-1) - pMN\frac{(a)_{2p}}{(c)_{2p}}} \quad (k \geq p), \quad (5.12)$$

where $M = [2(1-B) + (A-B)(p\beta-1)]$ and $N = [2(1-B) + (A-B)(p\mu-1)]$. The result is sharp for the functions $f_j(z)$ ($j = 1, 2$) given by

$$f_1(z) = z^{-p} + \frac{(A-B)(p\beta-1)}{[(1-B) + (A-B)(p\beta-1)]} \cdot \frac{(c)_{2p}}{(a)_{2p}} z^k \quad (p \in \mathbb{N}) \quad (5.13)$$

and

$$f_2(z) = z^{-p} + \frac{(A-B)(p\mu-1)}{[(1-B) + (A-B)(p\mu-1)]} \cdot \frac{(c)_{2p}}{(a)_{2p}} z^k \quad (p \in \mathbb{N}). \quad (5.14)$$

Theorem 5.3. *Let the functions $f_j(z)$ ($j = 1, 2$) defined by (5.1) be in the class $M_{a,c}^+(p; \beta; A, B)$, if the sequence $\{(k+p)\frac{(c)_{k+p}}{(a)_{k+p}}\}$ ($k \geq p; p \in \mathbb{N}$) is nondecreasing and*

$$2p(A-B)^2(p\beta-1)^2 - p[2(1-B) + (A-B)(p\beta-1)]^2\frac{(a)_{2p}}{(c)_{2p}} > 0,$$

then the function $h(z)$ defined by

$$h(z) = z^{-p} + \sum_{k=p}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \quad (5.15)$$

belongs to the class $M_{a,c}^+(p; \zeta; A, B)$, where

$$\zeta = \frac{1}{p} - \frac{4(1-B)(A-B)(p\beta-1)^2}{2p(A-B)^2(p\beta-1)^2 - p[2(1-B) + (A-B)(p\beta-1)]^2 \frac{(a)_{2p}}{(c)_{2p}}}.$$

This result is sharp for the functions $f_j(z)$ ($j = 1, 2$) given already by (5.4).

Proof. For $f_j(z) \in M_{a,c}^+(p; \beta; A, B)$ ($j = 1, 2$), we have

$$\sum_{k=p}^{\infty} \frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} |a_{k,j}| \leq 1.$$

Therefore,

$$\begin{aligned} & \sum_{k=p}^{\infty} \left[\frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \right]^2 |a_{k,j}|^2 \\ & \leq \left[\sum_{k=p}^{\infty} \frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} |a_{k,j}| \right]^2 \\ & \leq 1 \quad (j = 1, 2). \end{aligned} \quad (5.16)$$

So,

$$\sum_{k=p}^{\infty} \frac{1}{2} \left[\frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \right]^2 [|a_{k,j}|^2 + |a_{k,j}|^2] \leq 1.$$

In order to obtain our result, we have to find the largest ζ such that

$$\sum_{k=p}^{\infty} \frac{[(k+p)(1-B) + k(A-B)(p\zeta-1)]}{p(A-B)(p\zeta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} [|a_{k,j}|^2 + |a_{k,j}|^2] \leq 1.$$

It is sure if

$$\begin{aligned} & \frac{[(k+p)(1-B) + k(A-B)(p\zeta-1)]}{(p\zeta-1)} \\ & \leq \frac{1}{2} \frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]^2}{p(A-B)(p\beta-1)^2} \cdot \frac{(a)_{k+p}}{(c)_{k+p}}, \end{aligned}$$

so that

$$\zeta \leq \frac{1}{p} - \frac{(k+p)(1-B)(A-B)(p\beta-1)^2}{kp(A-B)^2(p\beta-1)^2 - \frac{1}{2}[(k+p)(1-B) + k(A-B)(p\beta-1)]^2 \frac{(a)_{k+p}}{(c)_{k+p}}}.$$

Now, we define the function $\psi(k)$ by

$$\begin{aligned} \psi(k) &= \frac{kp(A-B)^2(p\beta-1)^2}{k+p} \\ & - \frac{1}{2} \cdot \frac{[(k+p)(1-B) + k(A-B)(p\beta-1)]^2 \frac{(a)_{k+p}}{(c)_{k+p}}}{k+p} \quad (k \geq p). \end{aligned}$$

We observe that $\psi(k)$ is an increasing function of k ($k \geq p$). Thus, we conclude that

$$\zeta \leq \psi(p) = \frac{1}{p} - \frac{4(1-B)(A-B)(p\beta-1)^2}{2p(A-B)^2(p\beta-1)^2 - p[2(1-B) + (A-B)(p\beta-1)]^2 \frac{(a)_{2p}}{(c)_{2p}}},$$

which completes the proof of Theorem 5.3. \square

6. APPLICATION OF FRACTIONAL CALCULUS OPERATOR

In this part, we investigate the application of fractional calculus operator which was defined by M.K.Aouf et al.[2] in the class of $M_{a,c}^+(p; \beta; A, B)$. In our investigation, we will use the operators $J_{\delta,p}$ defined by [6, 9, 14].

Definition 6.1(see [6]) The operators $J_{\delta,p}$ defined by

$$(J_{\delta,p}f)(z) := \frac{\delta - p}{z^\delta} \int_0^z t^{\delta-1} f(t) dt \quad (\delta > p; p \in \mathbb{N}) \quad (6.1)$$

Definition 6.2 (see [2])The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\xi)}{(z - \xi)^{1-\mu}} d\xi \quad (\mu > 0) \quad (6.2)$$

where the function $f(z)$ is analytic in a simply connected domain of the complex z -plane containing the origin and the multiplicity of $(z - \xi)^{\mu-1}$ is removed by requiring $\log(z - \xi)$ to be real when $z - \xi \rightarrow 0$.

Definition 6.3 (see [2])The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^\mu f(z) = \frac{1}{\Gamma(1 - \mu)} \int_0^z \frac{f(\xi)}{(z - \xi)^\mu} d\xi \quad (0 \leq \mu < 1) \quad (6.3)$$

where the function $f(z)$ is constrained, the multiplicity of $(z - \xi)^{-\mu}$ is removed as in Definition 6.2.

Lemma 6.1. For a function $f(z) \in \Sigma_p$ of the form (1.12),by the above definitions,we have

$$J_{\delta,p}f(z) = z^{-p} + \sum_{k=p}^{\infty} \frac{\delta - p}{k + \delta} |a_k| z^k; \quad (6.4)$$

$$D_z^{-\mu} \{(J_{\delta,p}f)(z)\} = \frac{z^\mu}{\mu \Gamma(\mu)} (z^{-p} + \sum_{k=p}^{\infty} \frac{\delta - p}{k + \delta} |a_k| z^k) \quad (\mu > 0); \quad (6.5)$$

$$D_z^\mu \{(J_{\delta,p}f)(z)\} = \frac{z^{1-\mu}}{(1 - \mu) \Gamma(1 - \mu)} (z^{-p} + \sum_{k=p}^{\infty} \frac{\delta - p}{k + \delta} |a_k| z^k) \quad (0 \leq \mu < 1); \quad (6.6)$$

$$J_{\delta,p} \{(D_z^{-\mu} f)(z)\} = \frac{(\delta - p) z^\mu}{\mu \Gamma(\mu) (\delta - p + \mu)} (z^{-p} + \sum_{k=p}^{\infty} \frac{(\delta - p + \mu)}{(\delta + k + \mu)} |a_k| z^k) \quad (\mu > 0); \quad (6.7)$$

$$J_{\delta,p} \{(D_z^\mu f)(z)\} = \frac{(\delta - p) z^{1-\mu}}{(1 - \mu) \Gamma(1 - \mu) (\delta - \mu - p + 1)} (z^{-p} + \sum_{k=p}^{\infty} \frac{(\delta - \mu - p + 1)}{(\delta - \mu + k + 1)} |a_k| z^k) \quad (0 \leq \mu < 1). \quad (6.8)$$

Theorem 6.1. Let the function $f(z)$ defined by (1.12) be in the class $M_{a,c}^+(p; \beta; A, B)$ ($a \geq c > 0$),then

$$|D_z^{-\mu} \{(J_{\delta,p}f)(z)\}| \leq \frac{|z|^\mu}{\mu \Gamma(\mu)} \left(|z|^{-p} + \frac{(\delta - p)(A - B)(p\beta - 1)(c)_{2p}}{(\delta + p)[2(1 - B) + (A - B)(p\beta - 1)](a)_{2p}} |z|^p \right), \quad (6.9)$$

$$|D_z^{-\mu}\{(J_{\delta,p}f)(z)\}| \geq \frac{|z|^\mu}{\mu\Gamma(\mu)} \left(|z|^{-p} - \frac{(\delta-p)(A-B)(p\beta-1)(c)_{2p}}{(\delta+p)[2(1-B)+(A-B)(p\beta-1)](a)_{2p}} |z|^p \right), \quad (6.10)$$

where $\mu > 0, \delta > p, p \in \mathbb{N}, z \in \mathbb{U}^*$, each of the assertions is sharp for the function $f(z)$ given by

$$D_z^{-\mu}\{(J_{\delta,p}f)(z)\} = \frac{z^\mu}{\mu\Gamma(\mu)} \left(z^{-p} + \frac{(\delta-p)(A-B)(p\beta-1)(c)_{2p}}{(\delta+p)[2(1-B)+(A-B)(p\beta-1)](a)_{2p}} z^p \right). \quad (6.11)$$

Proof. By Lemma 6.1,

$$|D_z^{-\mu}\{(J_{\delta,p}f)(z)\}| = \frac{|z|^\mu}{\mu\Gamma(\mu)} \left(|z|^{-p} + \sum_{k=p}^{\infty} \frac{\delta-p}{k+\delta} |a_k| |z|^k \right)$$

Since $T(k) = \frac{\delta-p}{k+\delta}$ is a decreasing function of k for $k \geq p$ when $\delta > p$, we have

$$0 < T(k) \leq T(p) = \frac{\delta-p}{p+\delta}$$

If $f(z) \in M_{a,c}^+(p; \beta; A, B) (a \geq c > 0)$, by the Theorem 3.1, we get

$$\begin{aligned} & \frac{[2(1-B)+(A-B)(p\beta-1)]}{(A-B)(p\beta-1)} \cdot \frac{(a)_{2p}}{(c)_{2p}} \sum_{k=p}^{\infty} |a_k| \\ & \leq \sum_{k=p}^{\infty} \frac{[(k+p)(1-B)+k(A-B)(p\beta-1)]}{p(A-B)(p\beta-1)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} |a_k| \\ & \leq 1 \end{aligned} \quad (6.12)$$

It is that

$$\sum_{k=p}^{\infty} |a_k| \leq \frac{(A-B)(p\beta-1)(c)_{2p}}{[2(1-B)+(A-B)(p\beta-1)](a)_{2p}},$$

then

$$|D_z^{-\mu}\{(J_{\delta,p}f)(z)\}| \leq \frac{|z|^\mu}{\mu\Gamma(\mu)} \left(|z|^{-p} + \frac{(\delta-p)(A-B)(p\beta-1)(c)_{2p}}{(\delta+p)[2(1-B)+(A-B)(p\beta-1)](a)_{2p}} |z|^p \right),$$

$$|D_z^{-\mu}\{(J_{\delta,p}f)(z)\}| \geq \frac{|z|^\mu}{\mu\Gamma(\mu)} \left(|z|^{-p} - \frac{(\delta-p)(A-B)(p\beta-1)(c)_{2p}}{(\delta+p)[2(1-B)+(A-B)(p\beta-1)](a)_{2p}} |z|^p \right).$$

□

Theorem 6.2. Let the function $f(z)$ defined by (1.12) be in the class $M_{a,c}^+(p; \beta; A, B) (a \geq c > 0)$, then

$$|D_z^\mu\{(J_{\delta,p}f)(z)\}| \leq \frac{|z|^{1-\mu}}{(1-\mu)\Gamma(1-\mu)} \left(|z|^{-p} + \frac{(\delta-p)(A-B)(p\beta-1)(c)_{2p}}{(\delta+p)[2(1-B)+(A-B)(p\beta-1)](a)_{2p}} |z|^p \right), \quad (6.13)$$

$$|D_z^\mu\{(J_{\delta,p}f)(z)\}| \leq \frac{|z|^{1-\mu}}{(1-\mu)\Gamma(1-\mu)} \left(|z|^{-p} - \frac{(\delta-p)(A-B)(p\beta-1)(c)_{2p}}{(\delta+p)[2(1-B)+(A-B)(p\beta-1)](a)_{2p}} |z|^p \right), \quad (6.14)$$

$0 \leq \mu < 1, \delta > p, p \in \mathbb{N}, z \in \mathbb{U}^*$, each of the assertions is sharp for the function $f(z)$ given by

$$D_z^\mu \{(J_{\delta,p}f)(z)\} = \frac{z^{1-\mu}}{(1-\mu)\Gamma(1-\mu)} \left(z^{-p} + \frac{(\delta-p)(A-B)(p\beta-1)(c)_{2p}}{(\delta+p)[2(1-B) + (A-B)(p\beta-1)](a)_{2p}} z^p \right) \quad (6.15)$$

With the the same methods,we can obtain the similar results of $J_{\delta,p}\{(D_z^{-\mu}f)(z)\}$ and $J_{\delta,p}\{(D_z^\mu f)(z)\}$,we omit.

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LI-NA MA

DEPARTMENT OF MATHEMATICS AND STATISTICS, CHIFENG UNIVERSITY, CHIFENG, INNER MONGOLIA 024001, PEOPLES REPUBLIC OF CHINA

E-mail address: malina9563@gmail.com

SHU-HAI LI

DEPARTMENT OF MATHEMATICS AND STATISTICS, CHIFENG UNIVERSITY, CHIFENG, INNER MONGOLIA 024001, PEOPLES REPUBLIC OF CHINA

E-mail address: lishms66@sina.com