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COMMON FIXED POINT RESULTS FOR ψ - ϕ CONTRACTIONS IN RECTANGULAR METRIC SPACES

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Abstract. In this paper we obtain common fixed point theorems for two self maps satisfying certain $\psi - \phi$ contractive conditions in a rectangular metric space. Our results includes recent results of Lakzian and Samet [17], Inci *etal*[15] and many others.

1. Introduction

Since the introduction of Banach contraction principle in 1922, the study of existence and uniqueness of fixed points and common fixed points has become a subject of great interest. Many authors proved the Banach contraction Principle in various generalised metric spaces. In the sequel Branciari introduced the concept of rectangular metric space (RMS) by replacing the sum of the right hand side of the triangular inequality in metric space by a three-term expression and proved an analog of the Banach Contraction Principle. Since then many fixed point theorems for various contractions on rectangular metric spaces appeared. (see [2, 3, 4, 6, 7, 8, 9]). In 1969 Boyd and Wong[5] introduced the concept of ϕ contraction mappings and in 1997 Albet *etal*[1] generalised this concept by introducing weak ϕ contractions. Recently Lakzian and Samet[12] proved fixed point theorem for (ψ, ϕ) weakly contractive mappings on rectangular metric spaces, which was further generalised by Inci *etal*[10].

In this paper we have proved common fixed point theorems for two self maps satisfying $\psi - \phi$ contractive conditions in a rectangular metric space. Simple examples given shows that our results include more mappings than that of Lakzian and Samet[12] and Inci *etal*[10]. We have also applied our results to prove the existence of common fixed points of two mappings satisfying certain contractions of integral type.

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2. Preliminaries

Definition 2.1. [4] *Let X be a non empty set. Suppose that the mapping d : $X \times X \to [0, \infty)$ satisfies :*
(rm_1) $d(x,y) = 0$ *if and only if* $x = y$
(rm_2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$
(rm_3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ *for all* $x, u, v, y \in X$

Then d is called a *rectangular metric* on X and (X,d) is called a *rectangular metric space* (in short RMS).

Definition 2.2. *Let (X,d) be a rectangular metric space and $\{x_n\}$ be a sequence in X.*
(a)The sequence $\{x_n\}$ is said to be $RMS-convergent$ to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. (b)The sequence $\{x_n\}$ is said to be RMS-Cauchy sequence if and only if for every $\epsilon \geq 0$, there exists a poitive integer $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $n \geq m \geq N(\epsilon)$. (c) A RMS (X,d) is called complete, if every $RMS - Cauchy$ sequence in X is $RMS - convergent$.

Definition 2.3. *Let $T : X \to X$ and $S : X \to X$ be mappings Mappings in (X,d). An element $x \in X$ is said to be a coincidence point of S and T iff $Sx = Tx = u$ and a common fixed point iff $Sx = Tx = x$. u is called the point of coincidence of S and T The set of coincidence points of S and T is denoted by $C(S,T)$. S and T are said to be weakly compatible if and only if they commute at all coincidence points.*

3. Main results

Let Ψ denote set of all continuous functions Ψ: $[0,\infty) \to [0, \infty)$ for which $\Psi(t) = 0$ if and only if $t = 0$. Non decreasing functions which belong to the class Ψ are also known as altering distance functions.(see [11]). Now we present our main results as follows :

Theorem 3.1. *Let (X,d) be a Hausdorff and complete RMS and let $S,T : X \to X$ be mappings satisfying the following conditions:*

$$T(X) \subseteq S(X) \tag{3.1}$$

$$\psi(d(Tx,Ty)) \leq \psi(M(x,y) - \phi(M(x,y) + Lm(x,y) \tag{3.2}$$

for all $x,y \in X$, ψ, $\phi \in \Psi$, $L \geq 0$, the function ψ is non decreasing and

$$M(x,y) = max\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty)\} \tag{3.3}$$

$$m(x,y) = min\{d(Sx,Tx) + d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\} \tag{3.4}$$

Then $C(S,T) \neq \emptyset$ and the point of coincidence is unique. Further, if S,T are weakly compatible, then S and T has a unique common fixed point.

Proof : Let x_0 be any arbitary point in X. Let us define the sequence $\{y_n\}$ $\subset X$ as $Tx_{n-1} = Sx_n = y_n$, for n= 1,2,3,... If $y_n = y_{n+1}$ for all n, then $\{y_n\}$ is a constant sequence and thus convergent.. Suppose $y_n \neq y_{n+1}$ for all n. Then
$\psi(d(y_n, y_{n+1})) = \psi(d(Tx_{n-1}, Tx_n))$

$$\leq \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n)) + L(m(x_{n-1}, x_n)) \tag{3.5}$$

$$\begin{aligned} m(x_{n-1}, x_n) &= min\{d(Sx_{n-1}, Tx_{n-1}) + d(Sx_n, Tx_n), d(Sx_{n-1}, Tx_n), d(Sx_n, Tx_{n-1})\} \\ &= min\{d(y_{n-1}, y_n) + d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1}), d(y_n, y_n)\} = 0. \end{aligned}$$

Also

$$\begin{aligned} M(x_{n-1},x_n) &= max\{d(Sx_{n-1},Sx_n),d(Sx_{n-1},Tx_{n-1}),d(Sx_n,Tx_n)\} \\ &= max\{d(y_{n-1},y_n),d(y_{n-1},y_n),d(y_n,y_{n+1})\} \\ &= max\{d(y_{n-1},y_n),d(y_n,y_{n+1})\} \end{aligned}$$

Suppose, $M(x_{n-1},x_n)=d(y_n,y_{n+1})$, then 3.5 becomes

$$\begin{aligned} \psi(d(y_n,y_{n+1})) &\leq \psi(d(y_n,y_{n+1}))-\phi(d(y_n,y_{n+1}))+0 \\ \Rightarrow \phi(d(y_n,y_{n+1})) &= 0 \\ \Rightarrow d(y_n,y_{n+1}) &= 0 \end{aligned}$$

which is a contradiction to our assumption $(y_n \neq y_{n+1})$. Therefore, $M(x_{n-1},x_n)=d(y_{n-1},y_n)$. From 3.5 we have

$$\psi(d(y_n,y_{n+1})) \leq \psi(d(y_n,y_{n-1}))-\phi(d(y_n,y_{n-1})) \tag{3.6}$$

$$\begin{aligned} \psi(d(y_n,y_{n+1})) &\leq \psi(d(y_n,y_{n-1})) \\ \Rightarrow d(y_n,y_{n+1}) &\leq d(y_{n-1},y_n)\text{for all } n \geq 1. \end{aligned}$$

i.e. $d(y_n,y_{n+1})$ is decreasing sequence of positive terms and so converges to some $c \geq 0$, i.e. $lim_{n\to\infty}d(y_n,y_{n+1})=c$. As , $n\to\infty$, 3.6 becomes,
$\psi(c) \leq \psi(c)-\phi(c) \Rightarrow \phi(c)=0 \Rightarrow c=0$ Thus,

$$lim_{n\to\infty}d(y_n,y_{n+1})=0 \tag{3.7}$$

Now we shall prove that $\{y_n\}$ is RMS-Cauchy sequence. If not let us assume otherwise. Then there exist $\epsilon > 0$, subsequences $\{y_{n(i)}\}$ and $\{y_{m(i)}\}$ of $\{y_n\}$, $n(i) > m(i) > i$ such that,

$$d(y_{m(i)},y_{n(i)}) \geq \epsilon \tag{3.8}$$

where $n(i)$ is the smallest postive integer, satisfying (3.8), i.e.

$$d(y_{m(i)},y_{n(i)-1}) < \epsilon \tag{3.9}$$

Using Rectangular Inequality of RMS in 3.8 we have,

$$\begin{aligned} \epsilon &\leq d(y_{m(i)},y_{n(i)}) \\ &\leq d(y_{m(i)},y_{n(i)-2})+d(y_{n(i)-2},y_{n(i)-1})+d(y_{n(i)-1},y_{n(i)}) \\ &\leq \epsilon+d(y_{n(i)-2},y_{n(i)-1})+d(y_{n(i)-1},y_{n(i)}). \end{aligned}$$

As $i\to\infty$ and using 3.7 we get $\epsilon \leq d(y_{m(i)},y_{n(i)}) \leq \epsilon$, i.e. $lim_{i\to\infty}d(y_{m(i)},y_{n(i)})=\epsilon$. Also, $d(y_{n(i)},y_{m(i)}) \leq d(y_{n(i)},y_{n(i)-1})+d(y_{n(i)-1},y_{m(i)-1})+d(y_{m(i)-1},y_{m(i)})$ and $d(y_{n(i)-1},y_{m(i)-1}) \leq d(y_{n(i)-1},y_{n(i)})+d(y_{n(i)},y_{m(i)})+d(y_{m(i)},y_{m(i)-1})$
As $i\to\infty$ we get

$$\epsilon \leq d(y_{n(i)-1},y_{m(i)-1}) \leq \epsilon$$

$$\Rightarrow lim_{i\to\infty}d(y_{n(i)-1},y_{m(i)-1})=\epsilon$$

Now using 3.2 we have

$$\begin{aligned} \psi(d(y_{n(i)},y_{m(i)})) &= \psi(d(Tx_{n(i)-1},Tx_{m(i)-1})) \\ &\leq \psi(M(x_{n(i)-1},x_{m(i)-1}))-\phi(M(x_{n(i)-1},x_{m(i)-1}))+L(m(x_{n(i)-1},x_{m(i)-1})) \end{aligned}$$

$$\begin{aligned} m(x_{n(i)-1}, x_{m(i)-1}) &= min\{d(Sx_{n(i)-1}, Tx_{n(i)-1}) + d(Sx_{m(i)-1}, Tx_{m(i)-1}), d(Sx_{n(i)-1}, Tx_{m(i)-1}), \\ &\quad d(Sx_{m(i)-1}, Tx_{n(i)-1})\} \\ &= min\{d(y_{n(i)-1}, y_{n(i)}) + d(y_{m(i)-1}, y_{m(i)}), d(y_{n(i)-1}, y_{m(i)}), d(y_{m(i)-1}, y_{n(i)})\} \\ m(x_{n(i)-1}, x_{m(i)-1}) &\to 0 \text{ as } i \to \infty \end{aligned}$$

$$\begin{aligned} M(x_{n(i)-1}, x_{m(i)-1}) &= max\{d(Sx_{n(i)-1}, Sx_{m(i)-1}), d(Sx_{n(i)-1}Tx_{n(i)-1}), d(Sx_{n(i)-1}, Tx_{m(i)-1})\} \\ &= max\{d(y_{n(i)-1}, y_{m(i)-1}), d(y_{n(i)-1}, y_{n(i)}), d(y_{n(i)-1}, y_{m(i)}) \\ &= max\{\epsilon, 0, \epsilon\} = \epsilon. \end{aligned}$$

Thus we have,

$$\psi(\epsilon) \le \psi(\epsilon) - \phi(\epsilon)$$
$$\Rightarrow \phi(\epsilon) = 0 \Rightarrow \epsilon = 0$$

This is contradiction to our assumption that $\epsilon > 0$. Therefore, $\{y_n\}$ is RMS-Cauchy sequence. Since (X, d) is complete, $\{y_n\}$ converges to a limit say $u \in X$. Thus we have, $lim_{n\to\infty} Sx_n = lim_{i\to\infty} Tx_n = z$. Since SX is closed subset of X, $lim_{n\to\infty} Sx_n = z \in S(X)$. Therefore there exist $u \in X$ such that $Su = z$. Now we claim $Su = Tu = z$. By 3.2

$$\psi(d(Tu, Tx_n)) \le \psi(M(u, x_n)) - \phi(M(u, x_n)) + Lm(u, x_n) \tag{3.10}$$

$$M(u, x_n) = max\{d(Su, Sx_n), d(Su, Tu), d(Sx_n, Tx_n)\}$$

As $n \to \infty$ we have,

$$M(u, x_n) = max\{d(z, z), d(z, Tu), d(z, z)\} = d(z, Tu)$$

Also

$$m(u, x_n) = min\{d(Su, Tu) + d(Sx_n, Tx_n), d(Su, Tx_n), d(Sx_n, Tu)\}$$

As $n \to \infty$ we get,

$$m(u, x_n) = min\{d(z, Tu) + d(z, z), d(z, z), d(z, Tu)\} = 0$$

Therefore, as $n \to \infty$ 3.10 reduces to

$$\psi(d(Tu, Tx_n)) = \psi((d(Tu, z)) \le \psi(d(z, Tu)) - \phi(d(z, Tu)) + 0$$
$$\Rightarrow \phi(d(z, Tu)) = 0 \Rightarrow d(z, Tu) = 0 \Rightarrow Tu = z.$$

Thus we have $Su = Tu = z$, i.e. $C(S, T) \neq \emptyset$ and z is a point of coincidence. We claim that z is the unique point of coincidence. Suppose $w \in X$ such that $w = Sv = Tv$ for some $v \in X$. Then

$$\begin{aligned} \psi(d(z, w)) &= \psi(d(Tu, Tv)) \\ &\le \psi(M(u, v)) - \phi(M(u, v)) + Lm(u, v) \end{aligned}$$

$$M(u, v) = Max\{d(Su, Sv), d(Su, Tu), d(Sv, Tv)\} = d(z, w).$$
$$m(u, v) = Min\{d(Su, Tu) + d(Sv, Tv), d(Su, Tv), d(Sv, Tu)\} = 0.$$

Thus we have

$$\begin{aligned} \psi(d(z, w)) &\le \psi(d(z, w)) - \phi(d(z, w)) \\ \Rightarrow \phi(d(z, w)) &= 0 \text{i.e. } z = w. \end{aligned}$$

If the mappings S and T are weak compatible, we have, $STu = TSu = z \Rightarrow Sz = Tz$. Since z is the unique point of coincidence we have $Sz = Tz = z$. Thus z is the fixed point of S and T. Now we claim the fixed point is unique. If not we assume that there exist another fixed point v such that $Sv = Tv = v$. Then,

$$\psi(d(z,v)) = \psi(d(Tz,Tv)) \leq \psi(M(z,v) - \phi(M(z,v) + Lm(z,v) \tag{3.11}$$

$$\begin{aligned} M(z,v) &= max\{d(Sz,Sv), d(Sz,Tz), d(Sv,Tv)\} \\ &= max\{d(z,v), d(z,v), d(z,z)\} = d(z,v). \end{aligned}$$

$$\begin{aligned} m(z,v) &= min\{d(Sz,Tz) + d(Sv,Tv), d(Sz,Tv), d(Sv,Tz)\} \\ &= min\{d(z,z) + d(v,v), d(z,v), d(v,z)\} = 0. \end{aligned}$$

Therefore, 3.11 reduces to

$$\psi(d(Tz,Tv)) = \psi((d(z,Tv)) \leq \psi(d(z,v)) - \phi(d(z,v)) + 0$$
$$\Rightarrow \phi(d(z,v)) = 0 \Rightarrow d(z,v) = 0 \Rightarrow z = v.$$

Taking $S = Id$ (the Identity Mapping) in Theorem (3.1) we have the following:

Corollary 3.2. *Let (X,d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$\psi(d(Tx,Ty)) \leq \psi(M(x,y) - \phi(M(x,y) + Lm(x,y) \tag{3.12}$$

for all x,y $\in X$, ψ, $\phi \in \Psi$, $L \geq 0$, the function ψ is nondecreasing and

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\} \tag{3.13}$$

$$m(x,y) = min\{d(x,Tx) + d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{3.14}$$

Then T has a unique fixed point in X.

Example 3.3. *Let $X = A \bigcup B$, where $A = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}$ and B = [1,2]. Define $d : X \times X \to R$ as under:*
$d(0, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{6}) = 0.3$, $d(0, \frac{1}{3}) = d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{4}, \frac{1}{5}) = 02$,
$d(0, \frac{1}{4}) = d(\frac{1}{2}, \frac{1}{3}), d(\frac{1}{4}, \frac{1}{6}) = 0.4$, $d(0, \frac{1}{5}) = d(\frac{1}{2}, \frac{1}{6} = d(\frac{1}{3}, \frac{1}{6}) = 0.5$
$d(0, \frac{1}{6}) = d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{3}, \frac{1}{5}) = 0.6$
$d(0,0) = d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = d(\frac{1}{6}, \frac{1}{6}) = 0$
and $d(x,y) = \mid x - y \mid$, if x, $y \in B$ OR $x \in A$, $y \in B$.
Clearly d is a rectangular metric. However $d(\frac{1}{3}, \frac{1}{5}) = 0.6 > d(\frac{1}{3}, \frac{1}{4}) + d(\frac{1}{4}, \frac{1}{5}) =$ 0.5 and so d is not a usual metric. Define S and T by

$$Sx = \begin{cases} \frac{1}{5}, x \in [1,2] \\ \frac{1}{2} - x, x \in \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\} \\ \frac{1}{3}, x = \frac{1}{5} \end{cases} \quad Tx = \begin{cases} \frac{1}{5}, x \in [1,2] \\ \frac{1}{4}, x \in \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\} \\ \frac{1}{3}, x = \frac{1}{5}. \end{cases}$$

Let $\psi(t) = t$ and $\phi(t) = \frac{t}{3}$. Then S and T satisfies all the conditions of Theorem (3.1) and $\frac{1}{4}$ is the unique common fixed point.

Remark 3.4. *The following example shows that Corollary 3.2 includes more mappings than Theorem 3.1 of* [10] *and Theorem 4 of* [12].

Example 3.5. *Let $X = A \bigcup B$, where $A = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}$ and B = [1,2]. Define $d : X \times X \to R$ as under:*
$d(0, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{6}) = 0.6$, $d(0, \frac{1}{3}) = d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{4}, \frac{1}{5})$=02,
$d(0, \frac{1}{4}) = d(\frac{1}{2}, \frac{1}{3}), d(\frac{1}{4}, \frac{1}{6})$=0.4, $d(0, \frac{1}{5}) = d(\frac{1}{2}, \frac{1}{6} = d(\frac{1}{3}, \frac{1}{6})$=0.5
$d(0, \frac{1}{6}) = d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{3}, \frac{1}{5})$=0.3
$d(0,0) = d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = d(\frac{1}{6}, \frac{1}{6}) = 0$

and $d(x,y) = |x-y|$, *if* $x, y \in B$ *OR* $x \in A$, $y \in B$.
Clearly d *is a rectangular metric but not a metric as* , $d(\frac{1}{3},\frac{1}{4}) = 0.6 > d(\frac{1}{3},\frac{1}{5}) + d(\frac{1}{5},\frac{1}{4}) = 0.5$. *Define* T *by*

$$Tx = \begin{cases} \frac{1}{5}, x \in [1,2] \\ \frac{1}{2} - x, x \in \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\} \\ \frac{1}{3}, x = \frac{1}{5} \end{cases}$$

Then T *satisfies all the conditions of Corollary 3.2 and* $\frac{1}{4}$ *is the unique fixed point. However* T *does not satisfy Theorem 3.1 of [10] and Theorem 4 of [12] at* $x = \frac{1}{5}$ *and* $x = \frac{1}{4}$ *as,* $d(Tx,Ty) = d(\frac{1}{3},\frac{1}{4} = 0.6$

$$\begin{aligned} M(x,y) &= Max\{d(x,y), d(x,Tx), d(y,Ty)\} \\ &= Max\{d(\frac{1}{5},\frac{1}{4}), d(\frac{1}{5},\frac{1}{3}), d(\frac{1}{4},\frac{1}{4})\} \\ &= Max\{0,2,0.3,0\} = 0.3. \end{aligned}$$

$$\begin{aligned} m(x,y) &= min\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)\} \\ &= min\{d(\frac{1}{5},\frac{1}{4}), d(\frac{1}{5},\frac{1}{3}), d(\frac{1}{4},\frac{1}{3}),\} \\ &= min\{0.2, 0.3, 0, 0.6\} = 0 \end{aligned}$$

Next we prove fixed point results of $\psi - \phi$ contractions involving rational expressions.

Theorem 3.6. *Let* (X,d) *be a Hausdorff and complete RMS and let* $T: X \to X$ *be a self-map satisfying the following:*

$$T(X) \subseteq S(X) \tag{3.15}$$

$$\psi(d(Tx,Ty)) \le \psi(M(x,y)) - \phi(M(x,y)) + Lm(x,y) \tag{3.16}$$

for all x,y $\in X$ *and* ψ, $\phi \in \Psi$ *where* $L \ge 0$, *the function* ψ *is non decreasing and*

$$M(x,y) = max\{d(Sx,Sy), d(Sy,Ty)\frac{1+d(Sx,Tx)}{1+d(Sx,Sy)}\} \tag{3.17}$$

$$m(x,y) = min\{d(Sx,Tx), d(Sx,Ty), d(Sy,Tx)\} \tag{3.18}$$

Then S, T *has a unique fixed point in* X.

Proof Let x_0 be any arbitary point in X. Let us define the sequence $\{y_n\} \subset X$ as $Tx_{n-1} = Sx_n = y_n$, for n= 1,2,3,... Assume $y_n \ne y_{n+1} = Tx_n$ for all $n \ge 1$. Then we have

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n)) \\ &\le \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n)) + L(m(x_{n-1}, x_n)) \end{aligned}$$

$$\begin{aligned} M(x_{n-1}, x_n) &= max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Tx_n)\frac{1+d(Sx_{n-1}, Tx_{n-1})}{1+d(Sx_{n-1}, Tx_{n-1})}\} \\ &= max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\frac{1+d(y_{n-1}, y_n)}{1+d(y_{n-1}, y_n)}\} \\ &= max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \end{aligned}$$

Rest of the proof can be obtained by following the steps in Theorem 3.1.

Taking S to be Identity Mapping and $L = 0$ in the Theorem 3.6 we have

Corollary 3.7. *Let (X, d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)) \tag{3.19}$$

for all x,y $\in X$, ψ, $\phi \in \Psi$, the function ψ is non decreasing and

$$M(x, y) = max\{d(x, y), d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)}\} \tag{3.20}$$

Then T has a unique fixed point in X.

Taking $\psi(t) = t$ and $\phi(t) = (1 - k)t$ in Corollary 3.7 we have

Corollary 3.8. *Let (X, d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$d(Tx, Ty)) \leq kmax\{d(x, y), d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)}\} \tag{3.21}$$

for all $x, y \in X$ and $k \in [0, 1)$. Then T has a unique fixed point in X.

4. Applications

In this section we apply our results to prove fixed points of certain contrations of integral type.
Let $\wedge$ be the set of functions $f : [0, \infty) \to [0, \infty)$ such that:
(a) f is Lebesgue Integrable on each compact subset of $[0, \infty)$
(b) $\int_0^\epsilon f(t)dt > 0$ for every $\epsilon > 0$.

Theorem 4.1. *Let (X, d) be a Hausdorff and complete RMS, $S, T : X \to X$ be mappings satisfying the following conditions:*

$$T(X) \subseteq S(X) \tag{4.1}$$

$$\int_0^{d(Tx,Ty)} f(t)dt \leq \int_0^{M(x,y)} f(t)dt - \int_0^{M(x,y)} g(t)dt + Lm(x, y) \tag{4.2}$$

for all $x, y \in X$, $f, g \in \wedge$, $L \geq 0$, $M(x, y)$ and $m(x, y)$ as given in Theorem (3.1). Then S and T has a coincidence point, i.e. $C(S, T) \neq \emptyset$. Further, if S,T are weakly compatible, then S and T has a unique fixed point.

Proof Let $\psi(t) = \int_0^t f(u)du$ and $\phi(t) = \int_0^t g(u)du$. Then ψ and ϕ are functions of class Ψ and moreover the function ψ is non-decreasing. By Theorem 3.1 S, T has a unique fixed point.

Taking $g(t) = (1 - k)f(t)$ in Theorem 4.1 we have

Corollary 4.2. *Let (X, d) be a Hausdorff and complete RMS and let $T : X \to X$ and $S : X \to X$ be mappings satisfying the following conditions:*

$$T(X) \subseteq S(X) \tag{4.3}$$

$$\int_0^{d(Tx,Ty)} f(t)dt \leq k \int_0^{M(x,y)} f(t)dt + Lm(x, y) \tag{4.4}$$

for all $x, y \in X$, $f \in \wedge$, $0 \leq k < 1$, $L \geq 0$, $M(x,y)$ and $m(x,y)$ as given in Theorem (3.1). Then S and T has a coincidence point, i.e. $C(S,T) \neq \emptyset$. Further, if S,T are weakly compatible, then S and T has a unique fixed point.

Taking S as Identity Mapping in Theorem 4.1 we have

Corollary 4.3. *Let (X,d) be a Hausdorff and complete RMS and let $T : X \to X$ be mapping satisfying the following conditions:*

$$\int_0^{d(Tx,Ty)} f(t)dt \leq \int_0^{M(x,y)} f(t)dt - \int_0^{M(x,y)} g(t)dt + Lm(x,y) \tag{4.5}$$

for all $x, y \in X$, $f \in \wedge$, $L \geq 0$, $M(x,y)$ and $m(x,y)$ as given in Corollary (3.2). Then T has a unique fixed point in X.

Taking $g(t) = (1-k)f(t)$ in Corollary 4.3 we have

Corollary 4.4. *Let (X,d) be a Hausdorff and complete RMS and let $T : X \to X$ be mapping satisfying the following conditions:*

$$\int_0^{d(Tx,Ty)} f(t)dt \leq k \int_0^{M(x,y)} f(t)dt + Lm(x,y) \tag{4.6}$$

for all $x, y \in X$, $f \in \wedge$, $0 \leq k < 1$, $L \geq 0$, $M(x,y)$ and $m(x,y)$ as defined in Corollary (3.2). Then T has a coincidence point in X.

Theorem 4.5. *Let (X,d) be a Hausdorff and complete RMS and let $T : X \to X$ and $S : X \to X$ be mappings satisfying the following conditions:*

$$T(X) \subseteq S(X) \tag{4.7}$$

$$\int_0^{d(Tx,Ty)} f(t)dt \leq \int_0^{M(x,y)} f(t)dt - \int_0^{M(x,y)} g(t)dt + Lm(x,y) \tag{4.8}$$

for all $x, y \in X$, $f, g \in \wedge$, $L \geq 0$

$$M(x,y) = max\{d(Sx,Sy), d(Sy,Ty)\frac{1+d(Sx,Tx)}{1+d(Sx,Sy)}\} \tag{4.9}$$

$$m(x,y) = min\{d(Sx,Tx), d(Sx,Ty), d(Sy,Tx)\} \tag{4.10}$$

Then S, T has a unique common fixed point in X.

Proof Let $\psi(t) = \int_0^t f(u)du$ and $\phi(t) = \int_0^t g(u)du$. Then ψ and ϕ are functions of Ψ and moreover the function ψ is non-decreasing. By Theorem 3.6 S, T has a unique fixed point.

By taking S as identity mapping in Theorem 4.5 we have.

Corollary 4.6. *Let (X,d) be a Hausdorff and complete RMS and let $T : X \to X$ a self mapping satisfying the following conditions:*

$$\int_0^{d(Tx,Ty)} f(t)dt \leq \int_0^{M(x,y)} f(t)dt - \int_0^{M(x,y)} g(t)dt + Lm(x,y) \tag{4.11}$$

for all $x, y \in X$, $f, g \in \wedge$, $L \geq 0$

$$M(x,y) = max\{d(x,y), d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}\} \tag{4.12}$$

$$m(x,y) = min\{d(x,Tx), d(x,Ty), d(y,Tx)\} \tag{4.13}$$

Then T has a unique fixed point in X.

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