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ON GENERALIZED CONFORMAL MAPS

(COMMUNICATED BY KRISHNAN DUGGAL)

A. M. CHERIF, H. ELHENDI AND M. TERBECHE

Abstract. In this paper, we present some new properties of generalized conformal maps as f-biharmonic between equidimensional Riemannian manifolds.

1. Introduction

Consider a smooth map $\varphi : M \longrightarrow N$ between Riemannian manifolds $M = (M^m, g)$ and $N = (N^n, h)$ and $f : M \times N \longrightarrow (0, +\infty)$ be a smooth positive function, then the f-energy functional of φ is defined by

$$E_f(\varphi) = \frac{1}{2}\int_M f(x, \varphi(x))\, |d_x\varphi|^2\, v_g$$

(or over any compact subset $K \subset M$).

A map is called f-harmonic if it is a critical point of the f-energy functional. By [1] the map φ is f-harmonic if and only if

$$\tau_f(\varphi) \quad = \quad f_\varphi \tau(\varphi) + d\varphi(grad^M f_\varphi) - e(\varphi)(grad^N f) \circ \varphi = 0, \qquad (1)$$

where $f_\varphi : M^m \longrightarrow (0, +\infty)$ be a smooth positive function defined by

$$f_\varphi(x) = f(x, \varphi(x)), \quad \forall x \in M,$$

$\tau(\varphi) = trace_g \nabla d\varphi$ is the tension field of φ, and $e(\varphi) = \frac{1}{2}|d\varphi|^2$ is the energy density of φ.

$\tau_f(\varphi)$ is called the f-tension field of φ, and the f-bi-energy functional of φ is defined by

$$E_{2,f}(\varphi) = \frac{1}{2}\int_M |\tau_f(\varphi)|^2\, v_g\, .$$

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A map φ is called f-biharmonic if it is a critical point of the f-energy functional. By [1] the map φ is f-bi-harmonic if and only if

$$\begin{aligned} \tau_{2,f}(\varphi) &= -f_\varphi trace_g R^N(\tau_f(\varphi), d\varphi)d\varphi - trace_g \nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi) \\ &\quad + e(\varphi)(\nabla^N_{\tau_f(\varphi)} grad^N f) \circ \varphi - d\varphi(grad^M \tau_f(\varphi)(f)) \\ &\quad -\tau_f(\varphi)(f)\tau(\varphi) + < \nabla^\varphi \tau_f(\varphi), d\varphi > (grad^N f) \circ \varphi = 0, \end{aligned} \tag{2}$$

where $\langle\, , \rangle$ is the inner product on $T^*M \otimes \varphi^{-1}TN$ and R^N is the curvature tensor of N.

$\tau_{2,f}(\varphi)$ is called the f-bi-tension field of φ.

The f-biharmonic concept is a natural generalization of biharmonic maps ([3][4][7][8]).

2. Conformal f-bi-harmonic maps.

Theorem A. *Let $\varphi : (M^n, g) \longrightarrow (N^n, h)$ be a conformal map with dilation λ. The f-bi-tension field of φ is given by*

$$\begin{aligned} \tau_{2,f}(\varphi) &= F_1 d\varphi(grad^M \ln\lambda) + F_2 d\varphi(grad^M(f_N \circ \varphi)) \\ &\quad +2(n-2)f_\varphi^2 d\varphi(Ricci^M(grad^M \ln\lambda)) - 2f_\varphi d\varphi(Ricci^M(grad^M f_\varphi)) \\ &\quad +nf_\varphi d\varphi(Ricci^M(grad^M(f_N\circ\varphi))) + 2(n-2)f_\varphi^2 < \nabla d\varphi, \nabla d\ln\lambda > \\ &\quad -2f_\varphi < \nabla d\varphi, \nabla df_\varphi > + nf_\varphi < \nabla d\varphi, \nabla d(f_N \circ\varphi) > \\ &\quad +(n-2)f_\varphi^2 d\varphi(grad^M \Delta(\ln\lambda)) - f_\varphi d\varphi(grad^M \Delta(f_\varphi)) \\ &\quad +\frac{n}{2} f_\varphi d\varphi(grad^M \Delta(f_N\circ\varphi)) + 4(n-2)f_\varphi \nabla^\varphi_{grad^M f_\varphi} d\varphi(grad^M \ln\lambda) \\ &\quad -(n-2)^2 f_\varphi^2 \nabla^\varphi_{grad^M \ln\lambda} d\varphi(grad^M\ln\lambda) - \nabla^\varphi_{grad^M f_\varphi} d\varphi(grad^M f_\varphi) \\ &\quad +n\nabla^\varphi_{grad^M f_\varphi} d\varphi(grad^M(f_N\circ\varphi)) - \frac{n(n-2)}{2} f_\varphi \nabla^\varphi_{grad^M(f_N\circ\varphi)} d\varphi(grad^M \ln\lambda) \\ &\quad -\frac{n(n-2)}{2}\nabla^\varphi_{grad^M\ln\lambda} d\varphi(grad^M(f_N\circ\varphi)) - \frac{n^2}{4}\nabla^\varphi_{grad^M(f_N\circ\varphi)} d\varphi(grad^M(f_N\circ\varphi)) \\ &\quad +(n-2)d\varphi(\nabla^M_{grad^M\ln\lambda} grad^M(f_N\circ\varphi)) + (n-2)d\varphi(\nabla^M_{grad^M(f_N\circ\varphi)} grad^M\ln\lambda) \\ &\quad -d\varphi(\nabla^M_{grad^M f_\varphi} grad^M(f_N\circ\varphi)) - d\varphi(\nabla^M_{grad^M(f_N\circ\varphi)} grad^M f_\varphi) \\ &\quad +\frac{n}{2} d\varphi(grad^M(|grad^M(f_N\circ\varphi)|^2), \end{aligned}$$

where

$$\begin{aligned} F_1 &= (n-2)|grad^M f_\varphi|^2 + (n-2)f_\varphi\Delta(f_\varphi) - (n-2)^2 g(grad^M\ln\lambda, grad^M(f_N\circ\varphi)) \\ &\quad +(n-2)g(grad^M f_\varphi, grad^M(f_N\circ\varphi)) - \frac{n(n-2)}{2}|grad^M(f_N\circ\varphi)|^2, \end{aligned}$$

and

$$F_2 = -(n-2)\Delta(\ln\lambda) + \Delta(f_\varphi) - \frac{n}{2}\Delta(f_N\circ\varphi).$$

We need the following lemmas to prove Theorem A.

Lemma A. *Let $\varphi : (M^n, g) \longrightarrow (N^n, h)$ be a conformal map with dilation λ. The f-tension field of φ is given by*

$$\begin{aligned}\tau_f(\varphi) &= (2-n) f_\varphi d\varphi(grad^M \ln\lambda) + d\varphi(grad^M f_\varphi)\\ &\quad -\frac{n}{2} d\varphi\big(grad^M (f_N \circ \varphi)\big),\end{aligned}$$

where, at $x \in M$, $f_N : N \longrightarrow (0,\infty)$ defined by $f_N(y) = f(x,y)$ for all $y \in N$.

Proof. Since $\tau(\varphi) = (2-n)d\varphi(grad^M \ln\lambda)$, $e(\varphi) = \frac{n}{2}\lambda^2$ and let $\{e_i\}_{i=1}^n$ be an orthonormal frame in M, then $\{f_i\}_{i=1}^n$, where $f_i \circ \varphi = \frac{1}{\lambda} d\varphi(e_i)$ for all $i = 1, ..., n$, be an orthonormal frame in N. Then

$$\begin{aligned}(grad^N f)\circ\varphi(x) &= (grad^N f_N)_{\varphi(x)}\\ &= f_i|_{\varphi(x)}(f_N) f_i|_{\varphi(x)}\\ &= \frac{1}{\lambda^2(x)} d\varphi(e_i)_x(f_N) d\varphi(e_i)_x\\ &= \frac{1}{\lambda^2(x)} e_i|_x(f_N\circ\varphi) d\varphi(e_i)_x\\ &= \frac{1}{\lambda^2(x)} d\varphi\big(grad^M (f_N\circ\varphi)\big)_x,\end{aligned}$$

for all $x \in M$. □

Lemma B. *Let $\gamma : M \longrightarrow \mathbb{R}$ be a smooth function, then*

$$\begin{aligned}trace_g(\nabla^\varphi)^2 d\varphi(grad^M\gamma) &= (2-n)\nabla^\varphi_{grad^M\gamma} d\varphi(grad^M \ln\lambda) + 2d\varphi(Ricci^M grad^M\gamma)\\ &\quad -trace_g R^N(d\varphi(grad^M\gamma), d\varphi)d\varphi + 2 < \nabla d\varphi, \nabla d\gamma >\\ &\quad +d\varphi(grad^M \Delta(\gamma)),\end{aligned}$$

where $< \nabla d\varphi, \nabla d\gamma > = \nabla d\varphi(e_i, e_j)\nabla d\gamma(e_i, e_j) = \nabla d\varphi(e_i, e_j) g(\nabla^M_{e_i} grad^M\gamma, e_j)$.

Proof. Fix a point $x_0 \in M$ and let $\{e_i\}_{i=1}^n$ be an orthonormal frame, such that $\nabla^M_{e_i} e_j = 0$ at x_0 for all $i, j = 1, ..., n$. Then calculating at x_0, we have

$$\begin{aligned}trace_g(\nabla^\varphi)^2 d\varphi(grad^M\gamma) &= \nabla^\varphi_{e_i}\nabla^\varphi_{e_i} d\varphi(grad^M\gamma)\\ &= \nabla^\varphi_{e_i}\nabla d\varphi(e_i, grad^M\gamma) + \nabla^\varphi_{e_i} d\varphi(\nabla^M_{e_i} grad^M\gamma) \quad (3)\\ &= \nabla^\varphi_{e_i}\nabla d\varphi(e_i, grad^M\gamma) + \nabla d\varphi(e_i, \nabla^M_{e_i} grad^M\gamma)\\ &\quad +d\varphi(\nabla^M_{e_i}\nabla^M_{e_i} grad^M\gamma)\\ &= \nabla^\varphi_{e_i}\nabla d\varphi(e_i, grad^M\gamma) + < \nabla d\varphi, \nabla d\gamma >\\ &\quad +d\varphi(trace_g(\nabla^M)^2 grad^M\gamma)\end{aligned}$$

Let $\nabla^2 d\varphi$ be the third fundamental form defined by

$$\nabla^2 d\varphi(X, Y, Z) = \nabla^\varphi_X \nabla d\varphi(Y, Z) - \nabla d\varphi(\nabla^\varphi_X Y, Z) - \nabla d\varphi(Y, \nabla^\varphi_X Z), \qquad (4)$$

for all $X, Y, Z \in \Gamma(TM)$. And we have

$$\begin{aligned}\nabla^2 d\varphi(X, Y, Z) &= \nabla^2 d\varphi(Z, Y, X) + d\varphi(R^M(Z, X)Y) \qquad (5)\\ &\quad -R^N(d\varphi(Z), d\varphi(X))d\varphi(Y).\end{aligned}$$

By (4) and (5), the first term of (3) is

$$\begin{aligned}
\nabla^{\varphi}_{e_i}\nabla d\varphi(e_i, grad^M\gamma) &= \nabla^2 d\varphi(e_i, e_i, grad^M\gamma) + \nabla d\varphi(e_i, \nabla^M_{e_i} grad^M\gamma) \\
&= \nabla^2 d\varphi(grad^M\gamma, e_i, e_i) + d\varphi(R^M(grad^M\gamma, e_i)e_i) \\
&\quad -R^N(d\varphi(grad^M\gamma), d\varphi(e_i))d\varphi(e_i) + \nabla d\varphi(e_i, \nabla^M_{e_i} grad^M\gamma) \\
&= \nabla^{\varphi}_{grad^M\gamma}\tau(\varphi) + d\varphi(Ricci^M grad^M\gamma) \\
&\quad -trace_g R^N(d\varphi(grad^M\gamma), d\varphi)d\varphi + <\nabla d\varphi, \nabla d\gamma> .
\end{aligned}$$

From the equation

$$trace_g(\nabla^M)^2 grad^M\gamma = Ricci^M(grad^M\gamma) + grad^M(\Delta\gamma),$$

the Lemma B follows. □

Lemma C. *Let $\gamma : M \longrightarrow \mathbb{R}$ be a smooth function, then*

$$<\nabla d\varphi(grad^M\gamma), d\varphi> = n\lambda^2 g(grad^M\gamma, grad^M \ln\lambda) + \lambda^2\Delta(\gamma).$$

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal frame, then

$$\begin{aligned}
<\nabla d\varphi(grad^M\gamma), d\varphi> &= h(\nabla_{e_i} d\varphi(grad^M\gamma), d\varphi(e_i)) \\
&= e_i(h(d\varphi(grad^M\gamma), d\varphi(e_i))) \\
&\quad -h(d\varphi(grad^M\gamma), \nabla_{e_i} d\varphi(e_i)) \\
&= e_i(\lambda^2 g(grad^M\gamma, e_i)) \\
&\quad -(2-n)h(d\varphi(grad^M\gamma), d\varphi(grad^M \ln\lambda)) \\
&= g(grad^M\gamma, grad^M\lambda^2) \\
&\quad +(n-2)\lambda^2 g(grad^M\gamma, grad^M \ln\lambda) \\
&\quad +\lambda^2 e_i(g(grad^M\gamma, e_i)) \\
&= 2\lambda^2 g(grad^M\gamma, grad^M \ln\lambda) \\
&\quad +(n-2)\lambda^2 g(grad^M\gamma, grad^M \ln\lambda) \\
&\quad +\lambda^2\Delta(\gamma) \\
&= n\lambda^2 g(grad^M\gamma, grad^M \ln\lambda) + \lambda^2\Delta(\gamma).
\end{aligned}$$

□

Proof of Theorem A.

Let $\{e_i\}_{i=1}^n$ be an orthonormal frame, such that $\nabla^M_{e_i} e_j = 0$ at $x_0 \in M$ for all $i, j = 1, ..., n$. Then calculating at x_0, we have

$$\begin{aligned}
-(2-n)trace_g \nabla^\varphi f_\varphi \nabla^\varphi f_\varphi d\varphi(grad^M \ln\lambda) &= (n-2)\nabla^\varphi_{e_i} f_\varphi \nabla^\varphi_{e_i} f_\varphi d\varphi(grad^M \ln\lambda) \\
&= (n-2)\nabla^\varphi_{grad^M f_\varphi} f_\varphi d\varphi(grad^M \ln\lambda) \\
&\quad +(n-2) f_\varphi \nabla^\varphi_{e_i} \nabla^\varphi_{e_i} f_\varphi d\varphi(grad^M \ln\lambda) \\
&= (n-2)|grad^M f_\varphi|^2 d\varphi(grad^M \ln\lambda) \\
&\quad +(n-2) f_\varphi \nabla^\varphi_{grad^M f_\varphi} d\varphi(grad^M \ln\lambda) \\
&\quad +(n-2) f_\varphi \Delta(f_\varphi) d\varphi(grad^M \ln\lambda) \\
&\quad +2(n-2) f_\varphi \nabla^\varphi_{grad^M f_\varphi} d\varphi(grad^M \ln\lambda) \\
&\quad +(n-2) f_\varphi^2 trace_g (\nabla^\varphi)^2 d\varphi(grad^M \ln\lambda),
\end{aligned}$$

and by the lemma B, we obtain

$$\begin{aligned}
-(2-n)trace_g \nabla^\varphi f_\varphi \nabla^\varphi f_\varphi d\varphi(grad^M \ln\lambda) &= (n-2)|grad^M f_\varphi|^2 d\varphi(grad^M \ln\lambda) \\
&\quad +3(n-2) f_\varphi \nabla^\varphi_{grad^M f_\varphi} d\varphi(grad^M \ln\lambda) \\
&\quad +(n-2) f_\varphi \Delta(f_\varphi) d\varphi(grad^M \ln\lambda) \\
&\quad +(n-2) f_\varphi^2 \Big\{ (2-n)\nabla^\varphi_{grad^M \ln\lambda} d\varphi(grad^M \ln\lambda) \\
&\quad +2 d\varphi(Ricci^M grad^M \ln\lambda) \\
&\quad -trace_g R^N(d\varphi(grad^M \ln\lambda), d\varphi) d\varphi \\
&\quad +2 < \nabla d\varphi, \nabla d \ln\lambda > + d\varphi(grad^M \Delta(\ln\lambda)) \Big\}.
\end{aligned} \tag{6}$$

Of the same method, we obtain

$$\begin{aligned}
-trace_g \nabla^\varphi f_\varphi \nabla^\varphi d\varphi(grad^M f_\varphi) &= -\nabla^\varphi_{grad^M f_\varphi} d\varphi(grad^M f_\varphi) \\
&\quad -f_\varphi \Big\{ (2-n)\nabla^\varphi_{grad^M f_\varphi} d\varphi(grad^M \ln\lambda) \\
&\quad +2 d\varphi(Ricci^M grad^M f_\varphi) \\
&\quad -trace_g R^N(d\varphi(grad^M f_\varphi), d\varphi) d\varphi \\
&\quad +2 < \nabla d\varphi, \nabla d f_\varphi > + d\varphi(grad^M \Delta(f_\varphi)) \Big\},
\end{aligned} \tag{7}$$

$$\begin{aligned}
\frac{n}{2} trace_g \nabla^\varphi f_\varphi \nabla^\varphi d\varphi(grad^M (f_N \circ \varphi)) &= \frac{n}{2} \nabla^\varphi_{grad^M f_\varphi} d\varphi(grad^M (f_N \circ \varphi)) \\
&\quad +\frac{n}{2} f_\varphi \Big\{ (2-n)\nabla^\varphi_{grad^M (f_N \circ \varphi)} d\varphi(grad^M \ln\lambda) \\
&\quad +2 d\varphi(Ricci^M grad^M (f_N \circ \varphi)) \\
&\quad -trace_g R^N(d\varphi(grad^M (f_N \circ \varphi)), d\varphi) d\varphi \\
&\quad +2 < \nabla d\varphi, \nabla d(f_N \circ \varphi) > + d\varphi(grad^M \Delta((f_N \circ \varphi))) \Big\},
\end{aligned} \tag{8}$$

by Lemma A we have

$$
\begin{aligned}
e(\varphi)\nabla^N_{\tau_f(\varphi)} grad^N f &= \frac{n}{2}\lambda^2\Big\{(2-n)\nabla^\varphi_{grad^M \ln\lambda}\frac{1}{\lambda^2}d\varphi(grad^M(f_N\circ\varphi)) \\
&\quad +\nabla^\varphi_{grad^M f_\varphi}\frac{1}{\lambda^2}d\varphi(grad^M(f_N\circ\varphi)) \\
&\quad -\frac{n}{2}\nabla^\varphi_{grad^M(f_N\circ\varphi)}\frac{1}{\lambda^2}d\varphi(grad^M(f_N\circ\varphi))\Big\} \\
&= \frac{n}{2}\lambda^2\Big\{\frac{-2(2-n)}{\lambda^2}|grad^M\ln\lambda|^2 d\varphi(grad^M(f_N\circ\varphi)) \qquad (9)\\
&\quad +\frac{2-n}{\lambda^2}\nabla^\varphi_{grad^M\ln\lambda}d\varphi(grad^M(f_N\circ\varphi)) \\
&\quad -\frac{2}{\lambda^2}(grad^M f_\varphi)(\ln\lambda)d\varphi(grad^M(f_N\circ\varphi)) \\
&\quad +\frac{1}{\lambda^2}\nabla^\varphi_{grad^M f_\varphi}d\varphi(grad^M(f_N\circ\varphi)) \\
&\quad +\frac{n}{\lambda^2}(grad^M(f_N\circ\varphi))(\ln\lambda)d\varphi(grad^M(f_N\circ\varphi)) \\
&\quad -\frac{n}{2\lambda^2}\nabla^\varphi_{grad^M(f_N\circ\varphi)}d\varphi(grad^M(f_N\circ\varphi))\Big\}.
\end{aligned}
$$

The function $\tau_f(\varphi)(f)$ is given by

$$
\begin{aligned}
\tau_f(\varphi)(f) &= \tau_f(\varphi)(f_N) \\
&= (2-n)(grad^M\ln\lambda)(f_N\circ\varphi)+(grad^M f_\varphi)(f_N\circ\varphi) \\
&\quad -\frac{n}{2}|grad^M(f_N\circ\varphi)|^2,
\end{aligned}
$$

then

$$
\begin{aligned}
-d\varphi(grad^M\tau_f(\varphi)(f)) &= (n-2)d\varphi(grad^M((grad^M\ln\lambda)(f_N\circ\varphi))) \\
&\quad -d\varphi(grad^M((grad^M f_\varphi)(f_N\circ\varphi))) \\
&\quad +\frac{n}{2}d\varphi(grad^M(|grad^M(f_N\circ\varphi)|^2)) \\
&= (n-2)d\varphi(\nabla^M_{grad^M\ln\lambda}grad^M(f_N\circ\varphi)) \qquad (10)\\
&\quad +(n-2)d\varphi(\nabla^M_{grad^M(f_N\circ\varphi)}grad^M\ln\lambda) \\
&\quad -d\varphi(\nabla^M_{grad^M f_\varphi}grad^M(f_N\circ\varphi)) \\
&\quad -d\varphi(\nabla^M_{grad^M(f_N\circ\varphi)}grad^M f_\varphi) \\
&\quad +\frac{n}{2}d\varphi(grad^M(|grad^M(f_N\circ\varphi)|^2)),
\end{aligned}
$$

$$
\begin{aligned}
-\tau_f(\varphi)(f)\tau(\varphi) &= (n-2)\Big\{(2-n)(grad^M\ln\lambda)(f_N\circ\varphi)+(grad^M f_\varphi)(f_N\circ\varphi) \\
&\quad -\frac{n}{2}|grad^M(f_N\circ\varphi)|^2\Big\}d\varphi(grad^M\ln\lambda). \qquad (11)
\end{aligned}
$$

Then by lemma C, we have

$$\begin{aligned} < \nabla\tau_f(\varphi), d\varphi > \ (grad^N f)\circ\varphi &= \Big\{(2-n)n\lambda^2|grad^M \ln\lambda|^2 + (2-n)\lambda^2\Delta(\ln\lambda) \\ &+n\lambda^2 g(grad^M f_\varphi, grad^M \ln\lambda) + \lambda^2\Delta(f_\varphi) \\ &-\frac{n^2}{2}\lambda^2 g(grad^M(f_N\circ\varphi), grad^M \ln\lambda) \qquad (12) \\ &-\frac{n}{2}\lambda^2\Delta(f_N\circ\varphi)\Big\}\frac{1}{\lambda^2}d\varphi(grad^M(f_N\circ\varphi)). \end{aligned}$$

The Theorem A, follows from (2), (6), (7), (8), (9), (10), (11) and (12).

□

Particular Cases

(1) If $f = 1$, we obtain

$$\begin{aligned} \tau_{2,f}(\varphi) &= \tau_2(\varphi) \\ &= 2(n-2)d\varphi(Ricci^M(grad^M \ln\lambda)) + 2(n-2) < \nabla d\varphi, \nabla d\ln\lambda > \\ &\quad +(n-2)d\varphi(grad^M\Delta(\ln\lambda)) - (n-2)\nabla^\varphi_{grad^M \ln\lambda}d\varphi(grad^M \ln\lambda). \end{aligned}$$

Is the natural bi-tension field of φ.

(2) If $f_1 : M \longrightarrow (0,\infty)$, be a smooth positif function. If $f(x,y) = f_1(x)$ for all $(x,y) \in M\times N$, then the f-bi-tension field of φ is given by

$$\begin{aligned} \tau_{2,f}(\varphi) &= \tau_{2,f_1}(\varphi) \\ &= (n-2)|grad^M f_1|^2 d\varphi(grad^M \ln\lambda) + (n-2)f_1\Delta(f_1)d\varphi(grad^M \ln\lambda) \\ &\quad +2(n-2)f_1^2 d\varphi(Ricci^M(grad^M \ln\lambda)) - 2f_1 d\varphi(Ricci^M(grad^M f_1)) \\ &\quad +2(n-2)f_1^2 < \nabla d\varphi, \nabla d\ln\lambda > -2f_1 < \nabla d\varphi, \nabla df_1 > \\ &\quad +(n-2)f_1^2 d\varphi(grad^M\Delta(\ln\lambda)) - f_1 d\varphi(grad^M\Delta(f_1)) \\ &\quad +4(n-2)f_1\nabla^\varphi_{grad^M f_1}d\varphi(grad^M \ln\lambda) - (n-2)^2 f_1^2\nabla^\varphi_{grad^M \ln\lambda}d\varphi(grad^M \ln\lambda) \\ &\quad -\nabla^\varphi_{grad^M f_1}d\varphi(grad^M f_1), \end{aligned}$$

we recover the result obtained in [6].

Theorem B. *Let $\varphi : (M^n, g) \longrightarrow (N^n, h)$ be a conformal map with dilation λ. Then φ is f-bi-harmonic if and only if*

$$\begin{aligned}
0 \;=\;& F_3\, grad^M \ln\lambda + F_4\, grad^M (f_N \circ \varphi) + F_5\, grad^M f_\varphi \\
&+2(n-2)f_\varphi^2 Ricci^M(grad^M \ln\lambda) - 2f_\varphi Ricci^M(grad^M f_\varphi) \\
&+nf_\varphi Ricci^M(grad^M (f_N\circ\varphi)) - 4f_\varphi \nabla^M_{grad^M \ln\lambda} grad^M f_\varphi \\
&+(n-2)f_\varphi^2 grad^M \Delta(\ln\lambda) - f_\varphi grad^M \Delta(f_\varphi) + \frac{n}{2} f_\varphi grad^M \Delta(f_N\circ\varphi) \\
&+4(n-2)f_\varphi \nabla^M_{grad^M f_\varphi} grad^M \ln\lambda - \frac{n(n-2)}{2} f_\varphi \nabla^M_{grad^M (f_N\circ\varphi)} grad^M \ln\lambda \\
&+(n-2)\nabla^M_{grad^M (f_N\circ\varphi)} grad^M \ln\lambda - \nabla^M_{grad^M (f_N\circ\varphi)} grad^M f_\varphi \\
&-\frac{(n-2)(n-6)}{2} f_\varphi^2 grad^M (|grad^M \ln\lambda|^2) \\
&+\Big(2nf_\varphi - \frac{(n-2)^2}{2}\Big)\nabla^M_{grad^M \ln\lambda} grad^M (f_N\circ\varphi) \\
&-\frac{1}{2} grad^M(|grad^M f_\varphi|^2) + (n-1)\nabla^M_{grad^M f_\varphi} grad^M (f_N\circ\varphi) \\
&-\frac{n(n-2)}{4} grad^M(|grad^M (f_N\circ\varphi)|^2),
\end{aligned}$$

where

$$\begin{aligned}
F_3 \;=\;& F_1 - 2(n-2)f_\varphi^2 \Delta(\ln\lambda) + 2f_\varphi\Delta(f_\varphi) - nf_\varphi\Delta(f_N\circ\varphi) \\
&-(n-2)^2 f_\varphi^2 |grad^M \ln\lambda|^2 + |grad^M f_\varphi|^2 \\
&-ng(grad^M f_\varphi, grad^M (f_N\circ\varphi)) + \frac{n^2}{4}|grad^M (f_N\circ\varphi)|^2,
\end{aligned}$$

$$\begin{aligned}
F_4 \;=\;& F_2 + ng(grad^M f_\varphi, grad^M \ln\lambda) - \frac{n(n-2)}{2}(f_\varphi+1)|grad^M \ln\lambda|^2 \\
&-\frac{n^2}{2} g(grad^M (f_N\circ\varphi), grad^M \ln\lambda),
\end{aligned}$$

and

$$\begin{aligned}
F_5 \;=\;& 4(n-2)f_\varphi |grad^M \ln\lambda|^2 - 2g(grad^M f_\varphi, grad^M \ln\lambda) \\
&+ng(grad^M \ln\lambda, grad^M (f_N\circ\varphi)).
\end{aligned}$$

Proof. The Theorem B follows from Theorem A and the following formulae

$$\begin{aligned}
h(<\nabla d\varphi, \nabla d\ln\gamma>, d\varphi(X)) \;=\;& g(2\lambda^2 \nabla_{grad^M \ln\lambda} grad^M \ln\gamma \\
&-\lambda^2\Delta(\ln\gamma) grad^M \ln\lambda, X),
\end{aligned} \tag{13}$$

$$\begin{aligned}
h(\nabla^\varphi_X d\varphi(grad^M \ln\gamma), d\varphi(Y)) \;=\;& g(\lambda^2 X(\ln\lambda) grad^M \ln\gamma - \lambda^2 X(\ln\gamma) grad^M \ln\lambda \\
&+\; \lambda^2 d\ln\lambda(grad^M \ln\gamma)X + \lambda^2\nabla^M_X grad^M \ln\gamma, Y),
\end{aligned} \tag{14}$$

for all smooth function $\gamma : M \longrightarrow \mathbb{R}$ and $X, Y \in \Gamma(TM)$. □

Corollary A. *Let $\varphi : (M^n, g) \longrightarrow (N^n, h)$ be a conformal map with constant dilation λ. Then φ is f-bi-harmonic if and only if the function f satisfies the equation*

$$\begin{aligned} 0 &= \Big(\Delta(f_\varphi) - \frac{n}{2}\Delta(f_N \circ \varphi)\Big) grad^M(f_N \circ \varphi) - 2f_\varphi Ricci^M(grad^M f_\varphi) \\ &\quad + nf_\varphi Ricci^M(grad^M(f_N \circ \varphi)) - f_\varphi grad^M \Delta(f_\varphi) + \frac{n}{2} f_\varphi grad^M \Delta(f_N \circ \varphi) \\ &\quad - \nabla^M_{grad^M(f_N\circ\varphi)} grad^M f_\varphi - \frac{1}{2} grad^M(|grad^M f_\varphi|^2) \\ &\quad + (n-1)\nabla^M_{grad^M f_\varphi} grad^M(f_N \circ \varphi) - \frac{n(n-2)}{4} grad^M(|grad^M(f_N \circ \varphi)|^2). \end{aligned}$$

Corollary B. *Let $\varphi : (M^n, g) \longrightarrow (N^n, h)$ be a conformal map with constant dilation λ. If $f_\varphi = f_N \circ \varphi = \gamma$, then φ is f-bi-harmonic if and only if the function γ satisfies the equation*

$$\begin{aligned} 0 &= \frac{2-n}{2}\Delta(\gamma) grad^M \gamma - (2-n)\gamma\, Ricci^M(grad^M \gamma) \\ &\quad - \frac{2-n}{2}\gamma\, grad^M \Delta(\gamma) - \Big(\frac{(2-n)^2}{4} + \frac{1}{2}\Big) grad^M(|grad^M \gamma|^2). \end{aligned}$$

Example A. *Let $\varphi : \mathbb{R}^2 \longrightarrow (N^2, h)$ be a conformal map with constant dilation λ. If $f_\varphi = f_N \circ \varphi = \gamma$, then φ is f-bi-harmonic if and only if*

$$\begin{cases} \frac{\partial\gamma}{\partial x}\frac{\partial^2\gamma}{\partial x^2} + \frac{\partial\gamma}{\partial y}\frac{\partial^2\gamma}{\partial x\partial y} = 0, \\ \frac{\partial\gamma}{\partial y}\frac{\partial^2\gamma}{\partial y^2} + \frac{\partial\gamma}{\partial x}\frac{\partial^2\gamma}{\partial x\partial y} = 0. \end{cases}$$

Example B. *Let $\varphi : (M^2, g) \longrightarrow (N^2, h)$ be a conformal map with dilation λ. If $f_\varphi = f_N \circ \varphi = \ln\lambda$, then φ is f-bi-harmonic if and only if the function λ satisfies the equation*

$$grad^M(|grad^M \ln\lambda|^2) = 0.$$

Moreover, if M is related, then φ is f-bi-harmonic if and only if the function $|grad^M \ln\lambda|$ is constant in M.

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Ahmed Cherif Mohamed
Laboratory of Geometry, Analysis, Contrle and Applications.
Saida University,BP 138, 200000, Algeria

E-mail address: Ahmedcherif29@hotmail.fr

Elhendi Hichem
Laboratory of Geometry, Analysis, Controle and Applications.
Saida University,BP 138, 200000, Algeria.
E-mail address: elhendihichem@yahoo.fr

Terbeche Mekki
Department of Mathematics, University of Oran Es-Senia, Algeria.
E-mail address: Lgaca_Saida2009@hotmail.com