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ON q-INTEGRAL TRANSFORMS AND THEIR APPLICATIONS

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Abstract. In this paper, we introduce a q-analogue of the $\mathcal{P}$-Widder transform and give a Parseval-Goldstein type theorem. Furthermore, we evaluate the q-Laplace transform of a product of q-Bessel functions. Several special cases of our results are also pointed out.

1. Introduction, Definitions and Preliminaries

In the classical analysis, a Parseval-Goldstein type theorem involving the Laplace transform, the Fourier transform, the Stieltjes transform, the Glasser transform, the Mellin transform, the Hankel transform, the Widder-Potential transform and their applications are used widely in several branches of Engineering and applied Mathematics. Some integral transforms in the classical analysis have their q-analogues in the theory of q-calculus. This has led various workers in the field of q-theory for extending all the important results involving the classical analysis to their q-analouges. With this objective in mind, this paper introduces q-analogue of the $\mathcal{P}$-Widder potential transform and establishes certain interesting properties for this integral transform.

Throughout this paper, we will assume that q satisfies the condition $0 < |q| < 1$. The q-derivative $D_q f$ of an arbitrary function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},$$

where $x \neq 0$. Clearly, if f is differentiable, then

$$\lim_{q \to 1^-} (D_q f)(x) = \frac{df(x)}{dx}.$$

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Before we continue, let us introduce some notation that is used in the remainder of the paper. For any real number α,

$$[\alpha] := \frac{q^{\alpha}-1}{q-1}.$$

In particular, if $n \in \mathbb{Z}^{+}$, we denote

$$[n] = \frac{q^{n}-1}{q-1} = q^{n-1} + \cdots + q + 1.$$

Following usual notation are very useful in the theory of q-calculus:

$$(a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right),$$

$$(a;q)_t = \frac{(a;q)_\infty}{(aq^t;q)_\infty} \quad (\ t \in \mathbb{R}).$$

It is well known that in the literature there are two types of the q-Laplace transform and studied in details by many authors. Hahn [4] defined q-analogues of the well-known classical Laplace transform

$$\phi(s) = \int_0^\infty \exp(-st)f(t)dt \qquad (\Re(s) > 0)\,, \tag{1}$$

by means of the following q-integrals:

$$L_q\{f(t);s\} = {}_qL_s\{f(t)\} = \frac{1}{1-q}\int_0^{s^{-1}} E_q(qst)f(t)d_qt \quad (\Re(s) > 0)\,, \tag{2}$$

and

$$\mathcal{L}_q\{f(t);s\} = {}_q\mathcal{L}_s\{f(t)\} = \frac{1}{1-q}\int_0^{\infty} e_q(-st)f(t)d_qt \quad (\Re(s) > 0)\,, \tag{3}$$

where the q-analogues of the classical exponential functions are defined by

$$e_q(t) = \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty} \qquad (|t| < 1)\,, \tag{4}$$

and

$$E_q(t) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}t^n}{(q;q)_n} = (t;q)_\infty \quad (t \in \mathbb{C})\,. \tag{5}$$

By virtue of the q-integral (see [6, 11, 3])

$$\int_0^\infty f(t)d_qt = (1-q)\sum_{k=-\infty}^{\infty} q^k f(q^k), \tag{6}$$

the q-Laplace operator (3) can be expressed as

$${}_q\mathcal{L}_s\{f(t)\} = \frac{1}{(-s;q)_\infty}\sum_{k\in\mathbb{Z}} q^k(-s;q)_k\ f(q^k). \tag{7}$$

Throughout this paper, we will use $\mathcal{L}_q$ instead of ${}_q\mathcal{L}_s$.

The improper integral (see [8] and [6]) is defined by

$$\int_0^{\infty/A} f(x)d_qx = (1-q)\sum_{k\in\mathbb{Z}}\frac{q^k}{A}f(\frac{q^k}{A}). \tag{8}$$

The q-analogue of the integration theorem by a change of variable can be started when $u(x) = \alpha x^{\beta}, \alpha \in \mathbb{C}$ and $\beta > 0$, as follows:

$$\int_{u(a)}^{u(b)} f(u)\, d_q u = \int_a^b f(u(x))\, D_{q^{1/\beta}} u(x)\, d_{q^{1/\beta}} x. \tag{9}$$

As a special cases of the formula (9), one has the following reciprocity relations:

$$\int_0^A f(x) d_q x = \int_{q/A}^{\infty/A} \frac{1}{x^2} f(\frac{1}{x}) d_q x,$$

$$\int_0^{\infty/A} f(x) d_q x = \int_0^{\infty\cdot A} \frac{1}{x^2} f(\frac{1}{x}) d_q x. \tag{10}$$

Furthermore, the q-hypergeometric functions and well-known q-special functions are defined by (see [9] and [7]):

$${}_r\phi_s \left[\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n,$$

$${}_r\psi_s \left[\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{matrix} ; q, z \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(b_1;q)_n \cdots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{s-r} z^n$$

and

$$\Gamma_q(\alpha) = \int_0^{1/(1-q)} x^{\alpha-1} E_q(q(1-q)x)\, d_q x \quad (\alpha > 0), \tag{11}$$

$$\Gamma_q(\alpha) = K(A;\alpha) \int_0^{\infty/A(1-q)} x^{\alpha-1} e_q(-(1-q)x)\, d_q x \quad (\alpha > 0), \tag{12}$$

$$B_q(t;s) = \frac{\Gamma_q(t)\ \Gamma_q(s)}{\Gamma_q(t+s)} \quad (t, s \in \mathbb{R}), \tag{13}$$

where last two representations are based on the following remarkable function (see [7, p.15])

$$K(A;t) = A^{t-1} \frac{(-q/A\ ;q)_\infty}{(-q^t/A\ ;q)_\infty} \frac{(-A\ ;q)_\infty}{(-Aq^{1-t}\ ;q)_\infty} \quad (t \in \mathbb{R}). \tag{14}$$

Recently Uçar and Albayrak [12] introduced q-analogues of the $\mathcal{L}_2$-transfom in terms of the following q-integrals:

$${}_qL_2\{f(t);s\} = \frac{1}{1-q^2} \int_0^{1/s} t\ E_{q^2}(q^2 s^2 t^2) f(t) d_q t \quad (\Re(s) > 0), \tag{15}$$

and

$${}_q\mathcal{L}_2\{f(t);s\} = \frac{1}{1-q^2} \int_0^{\infty} t\ e_{q^2}(-s^2 t^2) f(t) d_q t \quad (\Re(s) > 0). \tag{16}$$

In the same article [12], the author established an interesting relation between ${}_q\mathcal{L}_2$ and $\mathcal{L}_q$ transforms, namely

$${}_q\mathcal{L}_2\{f(t);s\} = \frac{1}{[2]} \mathcal{L}_{q^2} \left\{ f\left(t^{1/2}\right); s^2 \right\}. \tag{17}$$

The paper is organized in the following manner. In the next two sections we introduce a q-analogue of the $\mathcal{P}$-Widder potential transform and establish a Parseval-Goldstein type theorem and its corollaries involving q-analogue of the $\mathcal{P}$-Widder and $\mathcal{L}_2$-Laplace transforms. Whereas in Section 4, we evaluate the q-Laplace transform of a product of basic analogue of the Bessel functions. Several special cases and examples of our results are also pointed out in the concluding section.

2. q-Analogue of The $\mathcal{P}$-Widder Transform

Widder [13] has presented a theory of the integral transform

$$\mathcal{P}\left\{f(x);y\right\}=\int_0^\infty \frac{xf(x)}{x^2+y^2}\,dx,$$

in the real domain which is formally equivalent to the iterated $\mathcal{L}_2$ transform.

Definition [2] A function f is q-integrable on $[0,\infty)$ if the series $\sum_{n\in\mathbb{Z}} q^n f(q^n)$ converges absolutely.

We write $L_q^1(\mathbb{R}_q)$ for the set of all functions that are absolutely q-integrable on $[0,\infty)$, where $\mathbb{R}_q$ is the set

$$\mathbb{R}_q=\left\{q^n : n\in\mathbb{Z}\right\},$$

that is

$$\begin{aligned} L_q^1(\mathbb{R}_q) &:= \left\{f : \sum_{n\in\mathbb{Z}} q^n\left|f(q^n)\right|<\infty\right\} \\ &:= \left\{f : \frac{1}{1-q}\int_0^\infty \left|f(x)\right| d_qx<\infty\right\}. \end{aligned}$$

Now we introduce the following q-integral transform, which may be regarded as q-extension of the $\mathcal{P}$-Widder potential transform.

Definition 2.1. *A q-analogue of $\mathcal{P}$-Widder potential transform will be denoted $\mathcal{P}_q$ and defined by the following q-integral:*

$$\mathcal{P}_q\left\{f(x);s\right\}=\frac{1}{1-q^2}\int_0^\infty \frac{x}{s^2+q^2x^2}f(x)\,d_qx, \quad (\Re(s)>0). \tag{18}$$

In view of (8), (18) can be expressed as

$$\mathcal{P}_q\left\{f(x);s\right\}=\frac{1}{1+q}\sum_{n\in\mathbb{Z}}\frac{q^{2n}}{s^2+q^{2+2n}}f(q^n). \tag{19}$$

Now we prove the following theorem that provide the existence and convergence for the $\mathcal{P}_q$-transform:

Theorem 2.1. *If $f\in L_q^1(\mathbb{R}_q)$, then the improper q-integral defined by (18) is well-defined.*

Proof. From (19), we have

$$\left|\mathcal{P}_q\left\{f(x);s\right\}\right|\le\frac{1}{1+q}\sum_{n\in\mathbb{Z}}\left|\frac{q^{2n}}{s^2+q^{2+2n}}\right|\left|f(q^n)\right|.$$

Since $\left\{\frac{q^n}{s^2+q^{2+2n}} : n\in\mathbb{Z}\right\}$ is bounded, then there exists a $K\in\mathbb{R}^+$ such that $\left|\frac{q^n}{s^2+q^{2+2n}}\right|<K$. Thus, we have

$$\left|\mathcal{P}_q\left\{f(x);s\right\}\right|\le\frac{K}{1+q}\sum_{n\in\mathbb{Z}}q^n\left|f(q^n)\right|.$$

On the other hand, since $f \in L_q^1(\mathbb{R}_q)$, there is a $M \in \mathbb{R}^+$ such that

$$\frac{1}{(1-q)}\int_0^\infty |f(x)|\, d_q x = \sum_{n\in\mathbb{Z}} q^n |f(q^n)| = M < \infty.$$

Hence we have

$$|\mathcal{P}_q \{f(x);s\}| \leq \frac{MK}{1+q} < \infty.$$

This completes to proof. □

3. Main Theorems and Applications

Proposition 3.1. *For $A, t \in \mathbb{R}$, we have*

$$\lim_{q\to 1^-} K(A;t) = 1.$$

Proof. Multiplying numerator and denominator of $K(A;t)$ by $(1+1/A)$, we get

$$\begin{aligned} K(A;t) &= A^{t-1}\frac{(1+1/A)(-q/A;q)_\infty}{(1+1/A)(-q^t/A;q)_\infty}\frac{(-A;q)_\infty}{(-q^{1-t}A;q)_\infty} \\ &= A^t \frac{(-1/A;q)_\infty}{(-q^t/A;q)_\infty}\frac{(-qA;q)_\infty}{(-q^{1-t}A;q)_\infty}. \end{aligned}$$

and then using the formula as $q \to 1^-$ (see [3, p. 9, 1.3.18])

$$\frac{(aq^t;q)_\infty}{(a;q)_\infty} = {}_1\phi_0\left(q^t;-;q,a\right) \to {}_1F_0(t;-;a) = (1-a)^{-t}, \quad |a|<1, t \text{ real}$$

into last expression, we obtain

$$\begin{aligned} \lim_{q\to 1^-} K(A;t) &= \lim_{q\to 1^-} A^t \frac{(-1/A;q)_\infty}{(-q^t/A;q)_\infty}\frac{(-qA;q)_\infty}{(-q^{1-t}A;q)_\infty} \\ &= \lim_{q\to 1^-} A^t \frac{1}{{}_1\phi_0(q^t;-;q,-1/A)}\frac{1}{{}_1\phi_0(q^{-t};-;q,-qA)} \\ &= A^t\left(1+\frac{1}{A}\right)^t (1+A)^{-t} \\ &= 1. \end{aligned}$$

This completes to proof. □

Theorem 3.1. *The $\mathcal{P}_q$-Widder transform can be regarded as iterated ${}_q\mathcal{L}_2$-Laplace transforms as under:*

$${}_q\mathcal{L}_2\{{}_q\mathcal{L}_2\{f(x);s\};t\} = \frac{1}{[2]}\mathcal{P}_q\{f(x);t\}, \tag{20}$$

provided that the q-integrals involved converge absolutely.

Proof. On using the definition of the ${}_q\mathcal{L}_2$-Laplace transform (16) the left-hand side of (20) (say I) reduces to

$$I = \frac{1}{(1-q^2)^2}\int_0^\infty s e_{q^2}\left(-t^2s^2\right)\left(\int_0^\infty x e_{q^2}\left(-s^2x^2\right) f(x)\, d_q x\right) d_q s. \tag{21}$$

Interchanging the order of the q-integration in (21), which is permissible by the absolute convergence of q-integrals, we obtain

$$I = \frac{1}{\left(1-q^2\right)^2} \int_0^\infty x f(x) \left(\int_0^\infty s e_{q^2}\left(-\left(t^2+q^2x^2\right)s^2\right) d_q s \right) d_q x.$$

In view of the definition (16), we obtain

$$I = \frac{1}{1-q^2} \int_0^\infty {}_q\mathcal{L}_2 \left\{1; \left(t^2+q^2x^2\right)^{1/2}\right\} x f(x)\, d_q x.$$

On setting $f(x) = 1$ and $s = \left(t^2+q^2x^2\right)^{1/2}$ in (17) and then using the formula $\mathcal{L}_{q^2}\{1; s^2\} = \dfrac{1}{s^2}$, we get the desired result

$$\begin{aligned} I &= \frac{1}{[2]} \frac{1}{1-q^2} \int_0^\infty \frac{x f(x)}{t^2+q^2x^2} d_q x \\ &= \frac{1}{[2]} \mathcal{P}_q \left\{ f(x); t \right\}. \end{aligned}$$

□

Corollary 3.1. *If* $-2 < \alpha < 0$ *then the following formula holds:*

$$\mathcal{P}_q \left\{x^\alpha; t\right\} = \frac{1}{[2]} \frac{1}{1-q^2} \frac{B_{q^2}\left(1+\alpha/2; -\alpha/2\right)}{K\left(1; 1+\alpha/2\right) K\left(1/t^2; -\alpha/2\right)} t^\alpha, \tag{22}$$

where $K(A;t)$ *is given by (14).*

Proof. On setting $f(x) = x^\alpha$ and make use of the known result due to Uçar and Albayrak [12], namely

$${}_q\mathcal{L}_2 \left\{x^\alpha; s\right\} = \frac{1}{[2]} \frac{\Gamma_{q^2}\left(1+\alpha/2\right)}{K\left(1/s^2; 1+\alpha/2\right)} \frac{\left(1-q^2\right)^{\alpha/2}}{s^{\alpha+2}} \quad (-2 < \alpha < 0),$$

the identity (20) of Theorem 3.1 give rise to

$$\begin{aligned} \mathcal{P}_q \left\{x^\alpha; t\right\} &= [2] \left\{ {}_q\mathcal{L}_2 \left\{ \frac{1}{[2]} \frac{\Gamma_{q^2}\left(1+\alpha/2\right)}{K\left(1/s^2; 1+\alpha/2\right)} \frac{\left(1-q^2\right)^{\alpha/2}}{s^{\alpha+2}}; t \right\} \right\} \\ &= \Gamma_{q^2}\left(1+\alpha/2\right) \left(1-q^2\right)^{\alpha/2} {}_q\mathcal{L}_2 \left\{ \frac{s^{-\alpha-2}}{K\left(1/s^2; 1+\alpha/2\right)}; t \right\}. \end{aligned}$$

Using the series representation of the ${}_q\mathcal{L}_2$-transform, we obtain

$$\mathcal{P}_q \left\{x^\alpha; t\right\} = \Gamma_{q^2}\left(1+\alpha/2\right) \left(1-q^2\right)^{\alpha/2} \frac{1}{[2]} \frac{1}{\left(-t^2; q^2\right)_\infty} \sum_{n\in\mathbb{Z}} \frac{q^{2n} \left(q^n\right)^{-\alpha-2} \left(-t^2; q^2\right)_n}{K\left(1/q^{2n}; 1+\alpha/2\right)}.$$

Following Kac and Sole [7] the function of x, $K(x;t)$ is a q-constant, that is, $K(q^n x; t) = K(x;t)$ for every integer n. Hence

$$\mathcal{P}_q \left\{x^\alpha; t\right\} = \frac{\Gamma_{q^2}\left(1+\alpha/2\right) \left(1-q^2\right)^{\alpha/2}}{K\left(1; 1+\alpha/2\right)} \frac{1}{[2]} \frac{1}{\left(-t^2; q^2\right)_\infty} \sum_{n\in\mathbb{Z}} q^{2n} \left(q^n\right)^{-\alpha-2} \left(-t^2; q^2\right)_n.$$

Again, on using the series representation of the ${}_q\mathcal{L}_2$-transform, we get

$$\begin{aligned}\mathcal{P}_q\{x^\alpha;t\} &= \frac{\Gamma_{q^2}(1+\alpha/2)\left(1-q^2\right)^{\alpha/2}}{K(1;1+\alpha/2)}\ {}_q\mathcal{L}_2\left\{s^{-\alpha-2};t\right\}\\ &= \frac{\Gamma_{q^2}(1+\alpha/2)\left(1-q^2\right)^{\alpha/2}}{K(1;1+\alpha/2)}\frac{1}{[2]}\frac{\Gamma_{q^2}(-\alpha/2)\left(1-q^2\right)^{-\alpha/2-1}}{K(1/t^2;-\alpha/2)\,t^{-\alpha}}\\ &= \frac{1}{[2]}\frac{1}{1-q^2}\frac{\Gamma_{q^2}(1+\alpha/2)}{K(1;1+\alpha/2)}\frac{\Gamma_{q^2}(-\alpha/2)}{K(1/t^2;-\alpha/2)}t^\alpha\\ &= \frac{1}{[2]}\frac{1}{1-q^2}\frac{B_{q^2}(1+\alpha/2;-\alpha/2)}{K(1;1+\alpha/2)\,K(1/t^2;-\alpha/2)}t^\alpha.\end{aligned}$$

□

In the following theorem, we establish a Parseval-Goldstein type theorem that involving q-analogue of the $\mathcal{P}$-Widder and $\mathcal{L}_2$-Laplace transforms:

Theorem 3.2. *If $\mathcal{P}_q$ and ${}_q\mathcal{L}_2$ denote q-analogues of the $\mathcal{P}$-Widder and $\mathcal{L}_2$-Laplace transforms, then the following result holds true:*

$$\int_0^\infty x\ {}_q\mathcal{L}_2\{f(y);x\}\ {}_q\mathcal{L}_2\{g(z);x\}\,d_qx = \frac{1}{[2]}\int_0^\infty yf(y)\,\mathcal{P}_q\{g(z);qy\}\,d_qy, \quad (23)$$

provided that the q-integrals involved converge absolutely.

Proof. Using the definition of the ${}_q\mathcal{L}_2$-transform (16), the left-hand side of (23) (say J) yields to

$$J = \frac{1}{1-q^2}\int_0^\infty x\ {}_q\mathcal{L}_2\{g(z);x\}\left\{\int_0^\infty ye_{q^2}\left(-x^2y^2\right)f(y)\,d_qy\right\}d_qx.$$

Changing the order of the q-integration, which is permissible by the hypothesis, we find that

$$J = \frac{1}{1-q^2}\int_0^\infty yf(y)\left\{\int_0^\infty xe_{q^2}\left(-q^2y^2x^2\right)\ {}_q\mathcal{L}_2\{g(z);x\}\,d_qx\right\}d_qy.$$

In view of the definition (16) and the result (20) of Theorem 3.1, the above relation reduces to the desired right-hand side of (23). □

Corollary 3.2. *We have*

$$\int_0^\infty xh(x)\ {}_q\mathcal{L}_2\{f(y);x\}\,d_qx = \int_0^\infty yf(y)\ {}_q\mathcal{L}_2\{h(x);qy\}\,d_qy, \quad (24)$$

provided that the q-integrals involved converge absolutely.

Proof. The identity (24) follows immediately after letting $h(x) = {}_q\mathcal{L}_2\{g(z);x\}$ in the relation (23). □

Corollary 3.3. *With due regards to convergence, we have*

$$q^2\mathcal{P}_q\left\{{}_q\mathcal{L}_2\{g(u);x\};q^2z\right\} = {}_q\mathcal{L}_2\{\mathcal{P}_q\{g(u);qy\};qz\}. \quad (25)$$

Proof. To prove (25), we set $f(y) = e_{q^2}\left(-y^2z^2\right)$, then we get

$$
\begin{aligned}
{}_q\mathcal{L}_2\left\{f(y);x\right\} &= \frac{1}{1-q^2}\int_0^\infty y e_{q^2}\left(-x^2y^2\right) e_{q^2}\left(-q^2z^2y^2\right) d_q y \\
&= \frac{1}{[2]}\frac{1}{x^2+q^2z^2}. \qquad (26)
\end{aligned}
$$

Substituting (26) into the identity (23) of Theorem 3.2, we obtain

$$
\begin{aligned}
\int_0^\infty \frac{x}{x^2+q^2z^2}\ {}_q\mathcal{L}_2\left\{g(u);x\right\} d_q x &= \int_0^\infty y e_{q^2}\left(-q^2z^2y^2\right)\mathcal{P}_q\left\{g(u);qy\right\} d_q y \\
q^2\mathcal{P}_q\left\{{}_q\mathcal{L}_2\left\{g(u);x\right\};q^2z\right\} &= {}_q\mathcal{L}_2\left\{\mathcal{P}_q\left\{g(u);qy\right\};qz\right\}.
\end{aligned}
$$

Similarly, if we set $f(y) = E_{1/q^2}\left(-\dfrac{y^2z^2}{q^2}\right)$ in Theorem 3.2 and using the well-known q-exponential identity $e_{q^2}(q^2x) = E_{1/q^2}(x)$ we find that

$$
{}_q\mathcal{L}_2\left\{E_{1/q^2}\left(-\frac{y^2z^2}{q^2}\right);x\right\} = {}_q\mathcal{L}_2\left\{e_{q^2}\left(-q^2z^2y^2\right);x\right\} = \frac{1}{[2]}\frac{1}{x^2+q^2z^2} \qquad (27)
$$

Substituting (27) into (23) we obtain

$$
\int_0^\infty \frac{x}{x^2+q^2z^2}\ {}_q\mathcal{L}_2\left\{g(u);x\right\} d_q x = \int_0^\infty y E_{1/q^2}\left(-z^2y^2\right)\mathcal{P}_q\left\{g(u);qy\right\} d_q y
$$

and finally we have

$$
\begin{aligned}
q^2\mathcal{P}_q\left\{{}_q\mathcal{L}_2\left\{g(u);x\right\};q^2z\right\} &= \int_0^\infty y e_{q^2}\left(-q^2z^2y^2\right)\mathcal{P}_q\left\{g(u);qy\right\} d_q y \\
&= {}_q\mathcal{L}_2\left\{\mathcal{P}_q\left\{g(u);qy\right\};qz\right\}.
\end{aligned}
$$

□

Theorem 3.3. *We have*

$$
\begin{aligned}
&\int_0^\infty x\ {}_q\mathcal{L}_2\left\{h(y);\left(x^2+q^2z^2\right)^{1/2}\right\}\ {}_q\mathcal{L}_2\left\{g(z);x\right\} d_q x \\
&\qquad = \frac{1-q^2}{[2]}\ {}_q\mathcal{L}_2\left\{h(y)\mathcal{P}_q\left\{g(z);qy\right\};qz\right\} \qquad (28)
\end{aligned}
$$

provided that the q-integrals involved converge absolutely.

Proof. Let $f(y) = e_{q^2}\left(-y^2z^2\right)h(y)$. Using the definition of the ${}_q\mathcal{L}_2$-transform we obtain

$$
\begin{aligned}
{}_q\mathcal{L}_2\left\{e_{q^2}\left(-y^2z^2\right)h(y);x\right\} &= \frac{1}{1-q^2}\int_0^\infty y e_{q^2}\left(-\left(x^2+q^2z^2\right)y^2\right)h(y)\, d_q y \\
&= {}_q\mathcal{L}_2\left\{h(y);\left(x^2+q^2z^2\right)^{1/2}\right\}. \qquad (29)
\end{aligned}
$$

Substituting (29) into (23), we find that

$$
\begin{aligned}
&\int_0^\infty x\ {}_q\mathcal{L}_2\left\{h(y);\left(x^2+q^2z^2\right)^{1/2}\right\}\ {}_q\mathcal{L}_2\left\{g(z);x\right\} d_q x \\
&= \frac{1}{[2]}\int_0^\infty y e_{q^2}\left(-q^2z^2y^2\right)h(y)\mathcal{P}_q\left\{g(z);qy\right\} d_q y \\
&= \frac{1-q^2}{[2]}\ {}_q\mathcal{L}_2\left\{h(y)\mathcal{P}_q\left\{g(z);qy\right\};qz\right\}.
\end{aligned}
$$

□

It is interesting to observe that, if we set $h(y) = 1$ and make use of the Theorem 3.3, one can easily deduced Corollary 3.3.

Corollary 3.4. *The following result holds true:*

$$ {}_q\mathcal{L}_2\left\{\frac{1}{s^2}\ {}_q\mathcal{L}_2\left\{f(x);\frac{1}{s}\right\};t\right\} = \frac{1}{[2]}\mathcal{P}_q\left\{f(x);t\right\}, \tag{30}$$

provided that the q-integrals involved converge absolutely.

Proof. In view of the definition of the ${}_q\mathcal{L}_2$-transform, the left-hand side of (30) (say L) yields to

$$\begin{aligned} L &= \frac{1}{1-q^2}\int_0^\infty \frac{1}{s}e_{q^2}\left(-t^2\frac{1}{s^2}\right)\frac{1}{s^2}\ {}_q\mathcal{L}_2\left\{f(x);\frac{1}{s}\right\}d_qs \\ &= \frac{1}{(1-q^2)^2}\int_0^\infty \frac{1}{s^3}e_{q^2}\left(-t^2\frac{1}{s^2}\right)\left(\int_0^\infty xe_{q^2}\left(-\frac{1}{s^2}x^2\right)f(x)\,d_qx\right)d_qs \\ &= \frac{1}{(1-q^2)^2}\int_0^\infty xf(x)\left(\int_0^\infty \frac{1}{s^3}e_{q^2}\left(-t^2\frac{1}{s^2}\right)e_{q^2}\left(-\frac{1}{s^2}x^2\right)d_qs\right)d_qx. \end{aligned}$$

On making use of the identity (10) into the right-hand side, we obtain

$$\begin{aligned} L &= \frac{1}{(1-q^2)^2}\int_0^\infty xf(x)\left(\int_0^\infty se_{q^2}\left(-\left(t^2+q^2x^2\right)s^2\right)d_qs\right)d_qx \\ &= \frac{1}{1-q^2}\int_0^\infty xf(x)\ {}_q\mathcal{L}_2\left\{1;\left(t^2+q^2x^2\right)^{1/2}\right\}d_qx \\ &= \frac{1}{[2]}\frac{1}{1-q^2}\int_0^\infty \frac{xf(x)}{t^2+q^2x^2}d_qx \\ &= \frac{1}{[2]}\mathcal{P}_q\left\{f(x);t\right\}. \end{aligned}$$

□

4. q-Laplace Image of a Product of q-Bessel functions

Recently, Purohit and Kalla [10] evaluated the q-Laplace image under the L_q operator (2) for a product of basic analogue of the Bessel functions. In this section, we propose to add one more dimension to this study by introducing a theorem which give rise to q-Laplace image under the $\mathcal{L}_q$ operator (3) for a product of q-Bessel functions. The third q-Bessel function is defined by Jackson and in some literature it is called Hahn-Exton q-Bessel function. For further details see [5].

$$\begin{aligned} J_\nu^{(3)}(t;q) &= \frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty}t^\nu\ {}_1\Phi_1\left[\begin{array}{c}0\\ q^{\nu+1}\end{array};q,qt^2\right] \\ &= t^\nu\sum_{n=0}^\infty \frac{(-1)^n q^{n(n-1)/2}\left(qt^2\right)^n}{(q;q)_{\nu+n}(q;q)_n}. \end{aligned} \tag{31}$$

Theorem 4.1. *Let* $J_{2\mu_j}^{(3)}\left(\sqrt{a_j t};q\right),\ j=1,2,...,n$ *be* n *different* q*-Bessel functions. Then,* q*-Laplace transform of their product is as follow*

$$\mathcal{L}_q\left\{f\left(t\right);s\right\}=\frac{\left(1-q\right)^{\upsilon-M-1}\Gamma_q\left(\upsilon+M\right)}{\Gamma_q\left(2\mu_1+1\right)...\Gamma_q\left(2\mu_n+1\right)}\frac{a_1^{\mu_1}...a_n^{\mu_n}s^{-\upsilon-M}}{K\left(1/s;\upsilon+M\right)} \tag{32}$$
$$\sum_{m_1,...,m_n=0}^{\infty}\frac{\left(q^{\upsilon+M};q\right)_m}{\left(q^{2\mu_1+1};q\right)_{m_1}...\left(q^{2\mu_n+1};q\right)_{m_n}}...\frac{\left(-a_1/s\right)^{m_1}...\left(-a_n/s\right)^{m_n}}{\left(q;q\right)_{m_1}\cdots\left(q;q\right)_{m_n}q^{m(m-1)/2}}.$$

where $f(t)=t^{\upsilon-1}q^{\mu_1}J_{2\mu_1}^{(3)}\left(\sqrt{q^{\upsilon+M-1}a_1t};q\right)\cdots q^{\mu_n}J_{2\mu_n}^{(3)}\left(\sqrt{q^{\upsilon+M-1}a_nt};q\right),M=\mu_1+\cdots+\mu_n$*,* $\mathrm{Re}\left(s\right)>0$*,* $\mathrm{Re}\left(\upsilon+M\right)>0$.

Proof. To prove the above theorem we put

$$f\left(t\right)=t^{\upsilon-1}q^{\mu_1}J_{2\mu_1}^{(3)}\left(\sqrt{q^{\upsilon+M-1}a_1t};q\right)\cdots q^{\mu_n}J_{2\mu_n}^{(3)}\left(\sqrt{q^{\upsilon+M-1}a_nt};q\right),$$
$$M=\mu_1+\cdots+\mu_n$$

into (7) and make use of (31), to obtain

$$\mathcal{L}_q\left\{f\left(t\right);s\right\}=\frac{1}{\left(-s;q\right)_\infty}\sum_{j\in\mathbb{Z}}q^j\left(-s;q\right)_j\left(q^j\right)^{\upsilon-1}$$
$$\times\left\{\frac{\left(q^{2\mu_1+1};q\right)_\infty}{\left(q;q\right)_\infty}\left(q^{\upsilon+M}a_1q^j\right)^{\mu_1}\cdots\frac{\left(q^{2\mu_n+1};q\right)_\infty}{\left(q;q\right)_\infty}\left(q^{\upsilon+M}a_nq^j\right)^{\mu_n}\right\}$$
$$\times\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{q^{m_1(m_1-1)/2}\left(-q^{\upsilon+M}a_1q^j\right)^{m_1}}{\left(q^{2\mu_1+1};q\right)_{m_1}\left(q;q\right)_{m_1}}\cdots\frac{q^{m_n(m_n-1)/2}\left(-q^{\upsilon+M}a_nq^j\right)^{m_n}}{\left(q^{2\mu_n+1};q\right)_{m_n}\left(q;q\right)_{m_n}}.$$

On interchanging the order of summations, which is valid under the conditions given with theorem, we obtain

$$\mathcal{L}_q\left\{f\left(t\right);s\right\}=\frac{1}{\left(-s;q\right)_\infty}\frac{\left(q^{2\mu_1+1};q\right)_\infty}{\left(q;q\right)_\infty}\cdots\frac{\left(q^{2\mu_n+1};q\right)_\infty}{\left(q;q\right)_\infty}a_1^{\mu_1}\cdots a_n^{\mu_n}\left(q^{\upsilon+M}\right)^M$$
$$\times\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{q^{m_1(m_1-1)/2}\left(-q^{\upsilon+M}a_1\right)^{m_1}}{\left(q^{2\mu_1+1};q\right)_{m_1}\left(q;q\right)_{m_1}}\cdots\frac{q^{m_n(m_n-1)/2}\left(-q^{\upsilon+M}a_n\right)^{m_n}}{\left(q^{2\mu_n+1};q\right)_{m_n}\left(q;q\right)_{m_n}}.$$
$$\times\sum_{j\in\mathbb{Z}}\left(-s;q\right)_j q^{j(\upsilon+\mu_1+\cdots+\mu_n+m_1+\cdots+m_n)}$$

By using the well-known q-gamma function

$$\Gamma_q\left(t\right)=\frac{\left(q;q\right)_\infty}{\left(q^t;q\right)_\infty\left(1-q\right)^{t-1}},$$

and then summing the inner series with the help of the bilateral summation formula (see [3, p. 126, 5.2.1]), namely

$${}_1\psi_1\left(b;c;q,z\right)=\sum_{n\in\mathbb{Z}}\frac{\left(b;q\right)_n}{\left(c;q\right)_n}z^n=\frac{\left(q,c/b,bz,q/bz;q\right)_\infty}{\left(c,q/b,z,c/bz;q\right)_\infty}\ \left(\left|c/b\right|<\left|z\right|<1\right),$$

we have

$$\mathcal{L}_q\{f(t);s\} = \frac{1}{(-s;q)_\infty}\frac{(1-q)^{-2M}}{\Gamma_q(2\mu_1+1)\cdots\Gamma_q(2\mu_n+1)}a_1^{\mu_1}\cdots a_n^{\mu_n}$$
$$\times\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(-a_1)^{m_1}}{(q^{2\mu_1+1};q)_{m_1}(q;q)_{m_1}}\cdots\frac{(-a_n)^{m_n}}{(q^{2\mu_n+1};q)_{m_n}(q;q)_{m_n}}$$
$$\times\frac{\left(q,0,-sq^{\upsilon+M+m_1+\cdots+m_n},-q^{1-(\upsilon+M+m_1+\cdots+m_n)}/s;q\right)_\infty}{\left(0,-q/s,q^{\upsilon+M+m_1+\cdots+m_n},0;q\right)_\infty},$$

We may rewrite this series

$$\mathcal{L}_q\{f(t);s\} = \frac{(1-q)^{-2M}}{\Gamma_q(2\mu_1+1)\cdots\Gamma_q(2\mu_n+1)}a_1^{\mu_1}\cdots a_n^{\mu_n}\left(q^{\upsilon+M}\right)^M$$
$$\times\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{q^{m_1(m_1-1)/2}\left(-q^{\upsilon+M}a_1\right)^{m_1}}{(q^{2\mu_1+1};q)_{m_1}(q;q)_{m_1}}\cdots\frac{q^{m_n(m_n-1)/2}\left(-q^{\upsilon+M}a_n\right)^{m_n}}{(q^{2\mu_n+1};q)_{m_n}(q;q)_{m_n}}$$
$$\times\frac{(q;q)_\infty}{(q^{\upsilon+M+m};q)_\infty}\frac{\left(-sq^{\upsilon+M+m};q\right)_\infty}{(-s;q)_\infty}\frac{\left(-q^{1-(\upsilon+M+m)}/s;q\right)_\infty}{(-q/s;q)_\infty}, \tag{33}$$

where $m = m_1+\cdots+m_n$. Setting $A = s$ and $t = \upsilon+M+m$ in (14), we get

$$K(1/s;\upsilon+M+m) = \left(\frac{1}{s}\right)^{\upsilon+M+m}\frac{s}{1+s}\frac{(-s;q)_\infty}{(-q^{\upsilon+M+m}s;q)_\infty}\frac{(-1/s;q)_\infty}{\left(-q^{1-(\upsilon+M+m)}/s;q\right)_\infty}$$
$$= \left(\frac{1}{s}\right)^{\upsilon+M+m}\frac{(-s;q)_\infty}{(-q^{\upsilon+M+m}s;q)_\infty}\frac{(-q/s;q)_\infty}{\left(-q^{1-(\upsilon+M+m)}/s;q\right)_\infty}. \tag{34}$$

Substituting relation (34) into (33), we obtain

$$\mathcal{L}_q\{f(t);s\} = \frac{(1-q)^{-2M}}{\Gamma_q(2\mu_1+1)\cdots\Gamma_q(2\mu_n+1)}a_1^{\mu_1}\cdots a_n^{\mu_n}\left(q^{\upsilon+M}\right)^M$$
$$\times\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{q^{m_1(m_1-1)/2}\left(-q^{\upsilon+M}a_1\right)^{m_1}}{(q^{2\mu_1+1};q)_{m_1}(q;q)_{m_1}}\cdots\frac{q^{m_n(m_n-1)/2}\left(-q^{\upsilon+M}a_n\right)^{m_n}}{(q^{2\mu_n+1};q)_{m_n}(q;q)_{m_n}}$$
$$\times\frac{(q;q)_\infty}{(q^{\upsilon+M+m};q)_\infty}\frac{1}{K(1/s;\upsilon+M+m)}.\frac{1}{s^{\upsilon+M+m}}.$$

Finally, on considering the following remarkable identity

$$\left(q^{t+m};q\right)_\infty = \frac{(q^t;q)_\infty}{(q^t;q)_m} \qquad (m\in\mathbb{N}),$$
$$K(t;s) = q^{s-1}K(t;s-1)$$

we have

$$\mathcal{L}_q\{f(t);s\} = \frac{(1-q)^{\upsilon-M-1}\ a_1^{\mu_1}\cdots a_n^{\mu_n}\Gamma_q(\upsilon+M)}{\Gamma_q(2\mu_1+1)\cdots\Gamma_q(2\mu_n+1)K(1/s;\upsilon+M)s^{\upsilon+M}}$$
$$\times\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{\left(q^{\upsilon+M};q\right)_m}{(q^{2\mu_1+1};q)_{m_1}\cdots(q^{2\mu_n+1};q)_{m_n}}\cdots\frac{(-a_1/s)^{m_1}\ldots(-a_n/s)^{m_n}}{(q;q)_{m_1}\cdots(q;q)_{m_n}q^{m(m-1)/2}}.$$

This completes the proof. □

5. **Special Cases**

In this section, we briefly consider some consequences and special cases of the results derived in the preceding sections. If we take $n=1$, $\mu_1=\upsilon$, $\upsilon=\mu$ and $a_1=a$ in (32) , we obtain

$$\mathcal{L}_q\left\{t^{\mu-1}q^{\upsilon}J_{2\upsilon}^{(3)}\left(\sqrt{q^{\upsilon+\mu-1}at};q\right);s\right\} = \frac{\left(q^{\mu+\upsilon}\right)^{\upsilon}\Gamma_q\left(\mu+\upsilon\right)\left(1-q\right)^{\mu-\upsilon-1}}{s^{\upsilon+\mu}\Gamma_q\left(2\upsilon+1\right)K\left(1/s;\mu+\upsilon\right)}a^{\upsilon}\ {}_2\Phi_1\left(\begin{array}{c} q^{\mu+\upsilon},0 \\ q^{2\upsilon+1}\end{array};q,-\frac{a}{s}\right) \tag{35}$$

where $\operatorname{Re}(\mu+\upsilon)>0$ and $\operatorname{Re}(s)>0$.
Again, if we write $\frac{\upsilon}{2}+1$ and $\frac{\upsilon}{2}$ instead of μ and υ in (35), respectively, we obtain

$$\mathcal{L}_q\left\{(qt)^{\upsilon/2}J_{\upsilon}^{(3)}\left(\sqrt{q^{\upsilon}at};q\right);s\right\} = \frac{a^{\upsilon/2}s^{-\upsilon-1}q^{\upsilon(\upsilon+1)/2}}{K\left(1/s;\upsilon+1\right)}\ e_q\left(-a/s\right) \tag{36}$$

Now, setting $\upsilon=1$ in (36) we obtain

$$\mathcal{L}_q\left\{(qt)^{1/2}J_{1}^{(3)}\left(\sqrt{qat};q\right);s\right\} = a^{1/2}s^{-2}e_q\left(-a/s\right) \qquad \left(\operatorname{Re}(s)>0\right). \tag{37}$$

Similarly, if we set $\upsilon=0$ in (36) , then we have

$$\mathcal{L}_q\left\{J_{0}^{(3)}\left(\sqrt{at};q\right);s\right\} = s^{-1}\ e_q\left(-a/s\right) \qquad \left(\operatorname{Re}(s)>0\right). \tag{38}$$

In (35) we write $\upsilon=0$ and then $a=0$, we find that

$$\mathcal{L}_q\left\{t^{\mu-1};s\right\} = \frac{\Gamma_q\left(\mu\right)\left(1-q\right)^{\mu-1}}{s^{\mu}}\frac{1}{K\left(1/s;\mu\right)}. \tag{39}$$

If we let $q\to1^-$, and make use of the limit formulae

$$\lim_{q\to1^-}\Gamma_q\left(t\right)=\Gamma\left(t\right), \qquad \lim_{q\to1^-}K\left(A;t\right)=1$$

and

$$\lim_{q\to1^-}\frac{\left(q^a;q\right)_n}{\left(1-q\right)^n}=\left(a\right)_n$$

where $\left(a\right)_n=a\left(a+1\right)...\left(a+n-1\right)$, we observe that the identity (28) of Theorem 3.3 and (25) of Corollary 3.3 provide, respectively, the q-extensions of the known related results due to Yürekli [14, p. 97, Theorem 1 and Corollary 1]. Also, the results (32), (35), (36), (37) and (38) provide, respectively, the q-extensions of the following known results given in Erdélyi, Magnus, Oberhettinger and Tricomi [1, pp. 182-187]:

$$\begin{aligned} &L\left\{t^{\upsilon-1}J_{2\mu_1}\left(2\sqrt{a_1t}\right)J_{2\mu_2}\left(2\sqrt{a_2t}\right)\cdots J_{2\mu_n}\left(2\sqrt{a_nt}\right);s\right\} \\ &= \frac{a_1^{\mu_1}\cdots a_n^{\mu_n}}{\Gamma\left(2\mu_1+1\right)\cdots\Gamma\left(2\mu_n+1\right)}\frac{\Gamma\left(\upsilon+M\right)}{s^{\upsilon+M}}\cdot \\ &\times\Psi_2^{(n)}\left(\upsilon+M;2\mu_1+1,\cdots,2\mu_n+1;\frac{-a_1}{s},\cdots,\frac{-a_n}{s}\right)\end{aligned}$$

where $M=\mu_1+...+\mu_n, \operatorname{Re}(s)>0$ and $\operatorname{Re}(\upsilon+M)>0$.

$$L\left\{t^{\mu-1}J_{2\upsilon}\left(2\sqrt{at}\right);s\right\} = \frac{\Gamma\left(\mu+\upsilon\right)a^{\upsilon}}{s^{\mu+\upsilon}\Gamma\left(2\upsilon+1\right)}\ {}_1F_1\left[\begin{array}{c}\mu+\upsilon; \\ 2\upsilon+1;\end{array}\ -a/s\right],$$

$$L\left\{t^{\upsilon/2}J_{\upsilon}\left(2\sqrt{at}\right);s\right\} = a^{\upsilon/2}s^{-\upsilon-1}\ e^{-a/s},$$

$$L\left\{t^{1/2}J_1\left(2\sqrt{at}\right);s\right\}=a^{1/2}s^{-2}e^{-a/s},$$

and

$$L\left\{J_0\left(2\sqrt{at}\right);s\right\}=s^{-1}e^{-a/s},$$

where $\mathrm{Re}\,(s)>0$ and $\mathrm{Re}\,(\mu+\upsilon)>0$.

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