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ON SUMMABILITY FACTORS FOR $|R, p_n|_k$

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Abstract. Necessary and sufficient conditions are obtained for $\sum a_n\lambda_n$ to be $|T|_k$ −summable, $k \geq 1$, whenever $\sum a_n$ is (N, p, q)-bounded.

1. Introduction

Let T be a lower triangular matrix, (s_n) a sequence, then

$$T_n := \sum_{v=0}^{n} t_{nv} s_v. \tag{1.1}$$

A series $\sum a_n$ is said to be $|T|_k$ −summable, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\Delta T_{n-1}\right|^k < \infty, \left(\Delta T_{n-1} = T_{n-1} - T_n\right). \tag{1.2}$$

Given any lower triangular matrix T one can associate the matrices $\overline{T}$ and $\widehat{T}$, with entries defined by

$$\bar{t}_{nv} = \sum_{i=v}^{n} t_{ni}, \quad n, i = 0, 1, 2, \ldots, \qquad \widehat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v}$$

respectively. With $s_n = \sum_{i=0}^{n} a_i\lambda_i$,

$$t_n = \sum_{v=0}^{n} t_{nv} s_v = \sum_{v=0}^{n} t_{nv} \sum_{i=0}^{v} a_i\lambda_i = \sum_{i=0}^{n} a_i\lambda_i \sum_{v=i}^{n} t_{nv} = \sum_{i=0}^{n} \hat{t}_{ni} a_i\lambda_i \tag{1.3}$$

$$Y_n := t_n - t_{n-1} = \sum_{i=0}^{n} \bar{t}_{ni} a_i\lambda_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i} a_i\lambda_i = \sum_{i=0}^{n} \hat{t}_{ni} a_i\lambda_i \text{ as } \bar{t}_{n-1,n} = 0. \tag{1.4}$$

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We call T a triangle if T is lower triangular and $t_{nn} \neq 0$ for all n. A triangle A is called factorable if its nonzero entries a_{mn} can be written in the form $b_m c_n$ for each m and n. We assume that (p_n) is a positive sequences of numbers such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty, \text{ as } n \to \infty,$$

The series $\sum a_n$ is said to be summable $|R, p_n|_k$, (or $\left|\overline{N}, p_n\right|_k$) $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|T_n - T_{n-1}\right|^k < \infty \quad ,$$

where

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v. \tag{1.5}$$

Let (t_n) be a sequence of the $(N, p, q)-$mean of the sequence (s_n) is defined by

$$t_n = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_v \qquad (p_{-1} = q_{-1} = R_{-1} = 0). \tag{1.6}$$

The sequence of constants (c_n) is formally defined by the identity

$$\left(\sum_{v=0}^{n} p_n x^n\right)^{-1} = \sum_{n=0}^{\infty} c_n x^n, \quad c_{-j} = 0, \quad j \geq 1. \tag{1.7}$$

By M, we denote the set of sequences (p_n) satisfying

$$\frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1, \quad p_n > 0, \quad n = 0, 1, \ldots \ . \tag{1.8}$$

The series $\sum a_n$ is said to be (N, p, q) bounded or $\sum a_n = O(1)(N, p, q)$ if

$$\sum_{v=1}^{n} p_{n-v} q_v s_v = O(R_n) \text{ as } n \to \infty. \tag{1.9}$$

The sequence (λ_n) is said to belong to the class $[X, Y]$, where X and Y are two methods of summability if $\sum a_n \lambda_n$ is $Y-$summable whenever $\sum a_n$ is $X-$summable.

Das , in 1966, proved the following result

Theorem 1.1. [3]. *Let $(p_n) \in M$, $q_n \geq 0$. Then if $\sum a_n$ is $|N, p, q|-$summable it is $\left|\overline{N}, q_n\right|-$summable.*

Recently Singh and Sharma proved the following theorem

Theorem 1.2. [4]. *Let $(p_n) \in M$, $q_n > 0$ and let (q_n) be a monotonic non-decreasing sequence for $n \geq 0$. The necessary and sufficient condition that $\sum a_n \lambda_n$ is $\left|\overline{N}, q_n\right|-$summable whenever*

$$\sum a_n = O(1)(N, p, q),$$

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} \left|\lambda_n\right| < \infty,$$

$$\sum_{n=0}^{\infty} |\Delta\lambda_n| < \infty,$$

$$\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} \left|\Delta^2\lambda_n\right| < \infty,$$

is that

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |s_n| |\lambda_n| < \infty.$$

Very recently Rhoades and Savas generalized the above result by proving the following

Theorem 1.3. [1]. *Let* $(p_n) \in M$, $q_n > 0$ *and let* (q_n) *be a monotonic non-decreasing sequence and* $nq_n = O(Q_n)$. *A necessary and sufficient condition that* $\lambda \in \left(|N, p, q|_k, \left|\overline{N}, q_n\right|_k\right)$ *whenever*

(i) $\sum a_n = O(1) [N, p, q]_k$,

(ii) $\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\lambda_n|^k < \infty,$

(iii) $\sum_{n=0}^{\infty} \left(\frac{Q_n}{q_n}\right)^{k-1} |\Delta\lambda_n|^k < \infty,$

(iv) $\sum_{n=0}^{\infty} \left(\frac{Q_{n+1}\Delta q_n}{q_n q_{n+1}}\right)^k \left(\frac{Q_n}{q_n}\right)^{k-1} |\Delta\lambda_n|^k < \infty,$

(v) $\sum_{n=0}^{\infty} \left(\frac{Q_{n+1}}{q_{n+1}}\right)^k \left(\frac{Q_n}{q_n}\right)^{k-1} \left|\Delta^2\lambda_n\right|^k < \infty,$

is that

(vi) $\sum_{n=1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n}\right)^k |s_n|^k |\lambda_n|^k < \infty.$

2. Results

We prove the following result

Theorem 2.1. *Let* $(p_n) \in M$, $q_n > 0$, (q_n) *be a monotonic non-increasing sequence. Let*

(a) $\sum_{v=0}^{n-1} \left|\Delta_v \hat{t}_{n,v+1}\right| = O(|t_{nn}|)$,

(b) $\sum_{n=v+1}^{m+1} (n|t_{nn}|)^{k-1} \left|\hat{t}_{n,v+1}\right| = O(v|t_{vv}|)^{k-1}$, *as* $m \to \infty$.

(c) $\sum_{v=0}^{n-1} |t_{vv}| \left|\hat{t}_{n,v+1}\right| = O(|t_{nn}|)$,

(d) $\sum_{n=v+1}^{m+1} (n|t_{nn}|)^{k-1} \left|\Delta_v \hat{t}_{n,v+1}\right| = O\left(v^{k-1}|t_{vv}|^k\right)$, *as* $m \to \infty$.

$(e) \sum\limits_{v=0}^{n-1} \left|\Delta_v^2 t_{nv}\right| = O\left(|\Delta t_{nn}|\right),$

$(f) \sum\limits_{n=v+1}^{\infty} n^{k-1} \left|\Delta t_{nn}\right|^k \left|\Delta_v^2 t_{nv}\right| = O\left(\left(v^{k-1} \left|\Delta t_{vv}\right|\right)^k\right),$

Then the necessary and sufficient conditions that $\lambda \in \left(\left|N, p, q\right|_k, \left|T\right|_k\right)$ *whenever*

$(i) \sum a_n = O\left(1\right) \left[N, p, q\right]_k$

$(ii) \ \sum\limits_{v=0}^{\infty} v^{k-1} \left|t_{vv}\right|^k \dfrac{\left|\lambda_v\right|^k Q_v^k}{q_v^k} = O(1)$

$(iii) \sum\limits_{v=1}^{\infty} v^{k-1} \left|t_{vv}\right|^k \left|\Delta\left(\dfrac{1}{q_v}\right)\right|^k \left|\lambda_v\right|^k Q_v^k = O(1)$

$(iv) \sum\limits_{v=0}^{\infty} v^{k-1} \left|t_{vv}\right|^k \dfrac{\left|\Delta\lambda_v\right|^k}{q_{v+1}^k} Q_v^k = O(1)$

$(v) \sum\limits_{v=1}^{\infty} v^{k-1} \left|\Delta t_{vv}\right|^k \ \dfrac{\left|\lambda_v\right|^k}{q_v^k} Q_{v-1}^k = O(1)$

$(vi) \ \sum\limits_{v=0}^{\infty} v^{k-1} \left|\Delta\left(\dfrac{1}{q_v}\right)\right|^k \left|\Delta\lambda_v\right|^k Q_v^k = O(1)$

$(vii) \sum\limits_{v=1}^{\infty} v^{k-1} \dfrac{1}{q_{v+1}^k} \left|\Delta^2\lambda_v\right|^k Q_v^k = O(1)$

is that

$(viii) \sum\limits_{n=1}^{\infty} n^{k-1} \left|t_{nn}\right|^k \left|s_n\right|^k \left|\lambda_n\right|^k < \infty.$

3. Lemmas

Lemma 3.1. [2]. *Let* $p_n \in M$. *Then*

(i) $c_0 > 0,\ c_n \le 0,\ n = 1, 2, \ldots,$

(ii) $\sum\limits_{n=0}^{\infty} c_n x^n$ *is absolutely convergent for* $|x| < 1$, *and*

(iii) $\sum\limits_{n=0}^{\infty} c_n > 0$, *except when* $\sum\limits_{n=0}^{\infty} p_n = \infty$, *in which case*

(iv) $\sum\limits_{n=0}^{\infty} c_n = 0.$

Lemma 3.2. [3]. *If*

$$t_n = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_v \ ,$$

then

$$s_n = \frac{1}{q_n} \sum_{v=0}^{n} c_{n-v} R_v t_v \ .$$

Lemma 3.3. [3]. *We have*

$$\sum_{v=0}^{n} c_{n-v}^{(1)} R_v = Q_n \ ,$$

where $c_n^{(1)} = c_0 + c_1 + \ldots + c_n$.

4. Proof Of Theorem 2.1.

Let T_n denote the nth term of A-transform of the series $\sum a_n \lambda_n$, then

$$\begin{aligned} Y_n := T_n - T_{n-1} &= \sum_{v=0}^{n} a_v \hat{t}_{nv} \lambda_v \\ &= \sum_{v=0}^{n-1} s_v \Delta_v \left(\hat{t}_{nv} \lambda_v\right) + t_{nn} \lambda_n s_n \\ &= \sum_{v=0}^{n-1} \left(\Delta_v \hat{t}_{nv} \lambda_v + \hat{t}_{n,v+1} \Delta \lambda_v\right) s_v + t_{nn} \lambda_n s_n \\ &= Y_{n1} + Y_{n2} + Y_{n3}. \end{aligned}$$

In order to prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} \left|Y_{nr}\right|^k < \infty, \qquad r = 1, 2, 3.$$

We have

$$\begin{aligned} Y_{n1} = & \sum_{v=0}^{n-1} \Delta_v \hat{t}_{nv} \lambda_v s_v \\ = & \sum_{v=0}^{n-1} \Delta_v \hat{t}_{nv} \lambda_v \frac{1}{q_v} \sum_{\mu=0}^{v} c_{v-\mu} R_\mu t_\mu \\ = & \sum_{\mu=0}^{n-1} R_\mu t_\mu \sum_{v=\mu}^{n-1} c_{v-\mu} \Delta_v \hat{t}_{nv} \frac{\lambda_v}{q_v} \\ = & \sum_{\mu=0}^{n-1} R_\mu t_\mu \left(\sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \left(\Delta_v^2 \hat{t}_{nv} \frac{\lambda_v}{q_v} + \Delta \hat{t}_{n,v+1} \Delta \left(\frac{1}{q_v}\right) \lambda_v + \Delta \hat{t}_{n,v+1} \frac{\Delta \lambda_v}{q_{v+1}} \right) \right. \\ & \left. + c_{n-\mu-1}^{(1)} \Delta t_{nn} \frac{\lambda_n}{q_n} \right) \\ = & Y_{n11} + Y_{n12} + Y_{n13} + Y_{n14}. \end{aligned}$$

Applying Holder's inequality

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1} \left|Y_{n11}\right|^k = & \sum_{n=1}^{\infty} n^{k-1} \left| \sum_{\mu=0}^{n-1} R_\mu t_\mu \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \Delta_v^2 \hat{t}_{nv} \frac{\lambda_v}{q_v} \right| \\
\leq & \sum_{n=1}^{\infty} n^{k-1} \left(\sum_{\mu=0}^{n-1} R_\mu t_\mu \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \left| \Delta_v^2 \hat{t}_{nv} \right| \frac{|\lambda_v|}{q_v} \right)^k \\
= & O(1) \sum_{n=1}^{\infty} n^{k-1} \left(\sum_{v=0}^{n-1} \left| \Delta_v^2 \hat{t}_{nv} \right| \frac{|\lambda_v|}{q_v} \sum_{\mu=0}^{v} c_{v-\mu}^{(1)} R_\mu \right)^k \qquad (t_\mu = O(1)) \\
= & O(1) \sum_{n=1}^{\infty} n^{k-1} \left(\sum_{v=0}^{n-1} \left| \Delta_v^2 \hat{t}_{nv} \right| \frac{|\lambda_v|}{q_v} Q_v \right)^k \\
= & O(1) \sum_{n=1}^{\infty} n^{k-1} \sum_{v=0}^{n-1} \left| \Delta_v^2 \hat{t}_{nv} \right| \frac{|\lambda_v|^k}{q_v^k} Q_v^k \left(\sum_{v=0}^{n-1} \left| \Delta_v^2 \hat{t}_{nv} \right| \right)^{k-1} \\
= & O(1) \sum_{n=0}^{\infty} \left(n \left| \Delta t_{nn} \right| \right)^{k-1} \sum_{v=0}^{n-1} \left| \Delta_v^2 \hat{t}_{nv} \right| \frac{|\lambda_v|^k}{q_v^k} Q_v^k \\
= & O(1) \sum_{v=0}^{\infty} \frac{|\lambda_v|^k}{q_v^k} Q_v^k \sum_{n=v+1}^{\infty} \left(n \left| \Delta t_{nn} \right| \right)^{k-1} \left| \Delta_v \hat{t}_{nv} \right|^k \\
= & O(1) \sum_{v=0}^{\infty} v^{k-1} \left| t_{vv} \right|^k \frac{|\lambda_v|^k}{q_v^k} Q_v^k \\
= & O(1).
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1}\left|Y_{n12}\right|^{k} &= \sum_{n=1}^{\infty} n^{k-1}\left|\sum_{\mu=0}^{n-1} R_{\mu} t_{\mu} \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \Delta_{v} \hat{t}_{n,v+1} \Delta\left(\frac{1}{q_{v}}\right) \lambda_{v}\right|^{k} \\
&\leq \sum_{n=1}^{\infty} n^{k-1}\left(\sum_{\mu=0}^{n-1} R_{\mu} t_{\mu} \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)}\left|\Delta_{v} \hat{t}_{n,v+1}\right|\left|\Delta\left(\frac{1}{q_{v}}\right)\right|\left|\lambda_{v}\right|\right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1}\left(\sum_{v=0}^{n-1}\left|\Delta_{v} \hat{t}_{n,v+1}\right|\left|\Delta\left(\frac{1}{q_{v}}\right)\right|^{k}\left|\lambda_{v}\right| \sum_{\mu=0}^{v} c_{v-\mu}^{(1)} R_{\mu}\right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1}\left(\sum_{v=0}^{n-1}\left|\Delta_{v} \hat{t}_{n,v+1}\right|\left|\Delta\left(\frac{1}{q_{v}}\right)\right|^{k}\left|\lambda_{v}\right| Q_{v}\right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \sum_{v=0}^{n-1}\left|\Delta_{v} \hat{t}_{n,v+1}\right|\left|\Delta\left(\frac{1}{q_{v}}\right)\right|^{k}\left|\lambda_{v}\right|^{k} Q_{v}^{k}\left(\sum_{v=0}^{n-1}\left|\Delta_{v} \hat{t}_{n,v+1}\right|\right)^{k-1} \\
&= O(1) \sum_{n=1}^{\infty}\left(n\left|t_{nn}\right|\right)^{k-1} \sum_{v=0}^{n-1}\left|\Delta_{v} \hat{t}_{n,v+1}\right|\left|\Delta\left(\frac{1}{q_{v}}\right)\right|^{k}\left|\lambda_{v}\right|^{k} Q_{v}^{k} \\
&= O(1) \sum_{v=0}^{\infty}\left|\Delta\left(\frac{1}{q_{v}}\right)\right|^{k}\left|\lambda_{v}\right|^{k} Q_{v}^{k} \sum_{n=v+1}^{\infty}\left(n\left|t_{nn}\right|\right)^{k-1}\left|\Delta_{v} \hat{t}_{n,v+1}\right| \\
&= O(1) \sum_{v=1}^{\infty}\left|\Delta\left(\frac{1}{q_{v}}\right)\right|^{k}\left|\lambda_{v}\right|^{k} Q_{v}^{k}\left(v^{k-1}\left|t_{vv}\right|^{k}\right) \\
&= O(1).
\end{aligned}
$$

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1}\left|Y_{n13}\right|^k &= \sum_{n=1}^{\infty} n^{k-1}\left|\sum_{\mu=0}^{n-1} R_\mu t_\mu \sum_{v=\mu}^{n-1} c_{n-\mu-1}^{(1)} \Delta_v \hat{t}_{n,v+1} \frac{\Delta\lambda_v}{q_{v+1}}\right|^k \\
&\leq \sum_{n=1}^{\infty} n^{k-1}\left(\sum_{\mu=0}^{n-1} R_\mu t_\mu \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \left|\Delta_v \hat{t}_{n,v+1}\right| \frac{\left|\Delta\lambda_v\right|}{q_{v+1}}\right)^k \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\left(\sum_{v=0}^{n-1}\left|\Delta_v \hat{t}_{n,v+1}\right| \frac{\left|\Delta\lambda_v\right|}{q_{v+1}} \sum_{\mu=0}^{v} c_{v-\mu}^{(1)} R_\mu\right)^k \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\left(\sum_{v=0}^{n-1}\left|\Delta_v \hat{t}_{n,v+1}\right| \frac{\left|\Delta\lambda_v\right|}{q_{v+1}} Q_v\right)^k \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\sum_{v=0}^{n-1}\left|\Delta_v \hat{t}_{n,v+1}\right| \frac{\left|\Delta\lambda_v\right|^k}{q_{v+1}^k} Q_v^k \left(\sum_{v=0}^{n-1}\left|\Delta_v \hat{t}_{n,v+1}\right|\right)^{k-1} \\
&= O(1)\sum_{v=0}^{\infty} \frac{\left|\Delta\lambda_v\right|^k}{q_{v+1}^k} Q_v^k \sum_{n=v+1}^{\infty}\left(n\left|t_{nn}\right|\right)^{k-1}\left|\Delta_v \hat{t}_{n,v+1}\right| \\
&= O(1)\sum_{v=1}^{\infty} v^{k-1}\left|t_{vv}\right|^k \frac{\left|\Delta\lambda_v\right|^k}{q_{v+1}^k} Q_v^k \\
&= O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1}\left|Y_{n14}\right|^k &= \sum_{n=1}^{\infty} n^{k-1}\left|\sum_{\mu=0}^{n-1} R_\mu t_\mu c_{n-\mu-1}^{(1)} \Delta_n t_{nn} \frac{\lambda_n}{q_n}\right|^k \\
&\leq O(1)\sum_{n=1}^{\infty} n^{k-1}\left|\Delta_n t_{nn}\right|^k \frac{\left|\lambda_n\right|^k}{q_n^k}\left(\sum_{\mu=0}^{n-1} R_\mu c_{n-\mu-1}^{(1)}\right)^k \\
&= O(1)\sum_{n=1}^{\infty} n^{k-1}\left|\Delta_n t_{nn}\right|^k \frac{\left|\lambda_n\right|^k}{q_n^k} Q_{n-1}^k \\
&= O(1).
\end{aligned}$$

$$\begin{aligned}
Y_{n2} = & \sum_{v=0}^{n-1} \hat{t}_{n,v+1} \Delta\lambda_v s_v \\
= & \sum_{v=0}^{n-1} \hat{t}_{n,v+1} \Delta\lambda_v \frac{1}{q_v} \sum_{\mu=0}^{v} c_{v-\mu} R_\mu t_\mu \\
= & \sum_{\mu=0}^{n-1} R_\mu t_\mu \sum_{v=\mu}^{n-1} c_{v-\mu} \hat{t}_{n,v+1} \frac{\Delta\lambda_v}{q_v} \\
= & \sum_{\mu=0}^{n-1} R_\mu t_\mu \left(\sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \left(\Delta_v \hat{t}_{n,v+1} \frac{\Delta\lambda_v}{q_v} + \hat{t}_{n,v+2} \Delta \left(\frac{1}{q_v} \right) \Delta\lambda_v + \hat{t}_{n,v+2} \frac{\Delta^2 \lambda_v}{q_{v+1}} \right) + c_{n-\mu-1}^{(1)} t_{n,n+1} \frac{\Delta\lambda_n}{q_n} \right) \\
= & Y_{n21} + Y_{n22} + Y_{n23} + Y_{n24}.
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1} |Y_{n21}|^k = & \sum_{n=1}^{\infty} n^{k-1} \left| \sum_{\mu=0}^{n-1} R_\mu t_\mu \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \Delta_v \hat{t}_{n,v+1} \frac{\Delta\lambda_v}{q_v} \right|^k \\
\leq & \sum_{n=1}^{\infty} n^{k-1} \left(\sum_{\mu=0}^{n-1} R_\mu t_\mu \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \left| \Delta_v \hat{t}_{n,v+1} \right| \frac{|\Delta\lambda_v|}{q_v} \right)^k \\
= & O(1) \sum_{n=1}^{\infty} n^{k-1} \left(\sum_{v=0}^{n-1} \left| \Delta_v \hat{t}_{n,v+1} \right| \frac{|\Delta\lambda_v|}{q_v} \sum_{\mu=0}^{v} c_{v-\mu}^{(1)} R_\mu \right)^k \\
= & O(1) \sum_{n=1}^{\infty} n^{k-1} \left(\sum_{v=0}^{n-1} \left| \Delta_v \hat{t}_{n,v+1} \right| \frac{|\Delta\lambda_v|}{q_v} Q_v \right)^k \\
= & O(1) \sum_{n=1}^{\infty} n^{k-1} \sum_{v=0}^{n-1} \left| \Delta_v \hat{t}_{n,v+1} \right| \frac{|\Delta\lambda_v|^k}{q_v^k} Q_v^k \left(\sum_{v=0}^{n-1} \left| \Delta_v \hat{t}_{n,v+1} \right| \right)^{k-1} \\
= & O(1) \sum_{n=1}^{\infty} (n |t_{nn}|)^{k-1} \sum_{v=0}^{n-1} \left| \Delta_v \hat{t}_{n,v+1} \right| \frac{|\Delta\lambda_v|^k}{q_{v+1}^k} Q_v^k \\
= & O(1) \sum_{v=1}^{\infty} \frac{|\Delta\lambda_v|^k}{q_{v+1}^k} Q_v^k \sum_{n=v+1}^{\infty} (n |t_{nn}|)^{k-1} \left| \Delta_v \hat{t}_{n,v+1} \right| \\
= & O(1) \sum_{v=1}^{\infty} \frac{|\Delta\lambda_v|^k}{q_{v+1}^k} Q_v^k (v |t_{vv}|)^{k-1} \\
= & O(1).
\end{aligned}$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1}\left|Y_{n22}\right|^{k} &= \sum_{n=1}^{\infty} n^{k-1}\left|\sum_{\mu=0}^{n-1} R_{\mu} t_{\mu} \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \hat{t}_{n,v+2} \Delta\left(\frac{1}{q_v}\right) \Delta\lambda_v\right|^{k} \\
&\leq \sum_{n=1}^{\infty} n^{k-1}\left(\sum_{\mu=0}^{n-1} R_{\mu} t_{\mu} \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \left|\hat{t}_{n,v+2}\right| \left|\Delta\left(\frac{1}{q_v}\right)\right| \left|\Delta\lambda_v\right|\right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1}\left(\sum_{v=0}^{n-1} \left|\hat{t}_{n,v+2}\right| \left|\Delta\left(\frac{1}{q_v}\right)\right| \left|\Delta\lambda_v\right| \sum_{\mu=0}^{v} c_{v-\mu}^{(1)} R_{\mu}\right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1}\left(\sum_{v=0}^{n-1} \left|\hat{t}_{n,v+2}\right| \left|\Delta\left(\frac{1}{q_v}\right)\right| \left|\Delta\lambda_v\right| Q_v\right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1}\left(\sum_{v=0}^{n-1} \frac{\left|t_{vv}\right|}{\left|t_{vv}\right|} \left|\hat{t}_{n,v+2}\right| \left|\Delta\left(\frac{1}{q_v}\right)\right| \Delta\left|\lambda_v\right| Q_v\right)^{k} \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \sum_{v=0}^{n-1} \frac{\left|\hat{t}_{n,v+2}\right|}{\left|t_{vv}\right|^{k-1}} \left|\Delta\left(\frac{1}{q_v}\right)\right|^{k} \left|\Delta\lambda_v\right|^{k} Q_v^{k} \left(\sum_{v=0}^{n-1} \left|t_{vv}\right| \left|\hat{t}_{n,v+2}\right|\right)^{k-1} \\
&= O(1) \sum_{n=1}^{\infty} \left(n\left|t_{nn}\right|\right)^{k-1} \sum_{v=0}^{n-1} \frac{\left|\hat{t}_{n,v+2}\right|}{\left|t_{vv}\right|^{k-1}} \left|\Delta\left(\frac{1}{q_v}\right)\right|^{k} \left|\Delta\lambda_v\right|^{k} Q_v^{k} \\
&= O(1) \sum_{v=0}^{\infty} \frac{1}{\left|t_{vv}\right|^{k-1}} \left|\Delta\left(\frac{1}{q_v}\right)\right|^{k} \left|\Delta\lambda_v\right|^{k} Q_v^{k} \sum_{n=v+1}^{\infty} \left(n\left|t_{nn}\right|\right)^{k-1} \left|\hat{t}_{n,v+2}\right| \\
&= O(1) \sum_{v=0}^{\infty} v^{k-1} \left|\Delta\left(\frac{1}{q_v}\right)\right|^{k} \left|\Delta\lambda_v\right|^{k} Q_v^{k} \\
&= O(1).
\end{aligned}
$$

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1} \left|Y_{n23}\right|^{k} &= \sum_{n=1}^{\infty} n^{k-1} \left|\sum_{\mu=0}^{n-1} R_{\mu} t_{\mu} \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \hat{t}_{n,v+2} \frac{1}{q_v} \Delta^2 \lambda_v\right|^k \\
&\leq \sum_{n=1}^{\infty} n^{k-1} \left(\sum_{\mu=0}^{n-1} R_{\mu} t_{\mu} \sum_{v=\mu}^{n-1} c_{v-\mu}^{(1)} \left|\hat{t}_{n,v+2}\right| \frac{1}{q_{v+1}} \left|\Delta^2 \lambda_v\right|\right)^k \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \left(\sum_{v=0}^{n-1} \left|\hat{t}_{n,v+2}\right| \frac{1}{q_{v+1}} \left|\Delta^2 \lambda_v\right| \sum_{\mu=0}^{v} c_{v-\mu}^{(1)} R_{\mu}\right)^k \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \left(\sum_{v=0}^{n-1} \left|\hat{t}_{n,v+2}\right| \frac{1}{q_{v+1}} \left|\Delta^2 \lambda_v\right| Q_v\right)^k \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \sum_{v=0}^{n-1} \frac{\left|\hat{t}_{n,v+2}\right|}{\left|t_{vv}\right|^{k-1}} \frac{1}{q_{v+1}^k} \left|\Delta^2 \lambda_v\right|^k Q_v^k \left(\sum_{v=0}^{n-1} \left|t_{vv}\right| \left|\hat{t}_{n,v+2}\right|\right)^{k-1} \\
&= O(1) \sum_{n=1}^{\infty} \left(n \left|t_{nn}\right|\right)^{k-1} \sum_{v=0}^{n-1} \frac{\left|\hat{t}_{n,v+2}\right|}{\left|t_{vv}\right|^{k-1}} \frac{1}{q_{v+1}^k} \left|\Delta^2 \lambda_v\right|^k Q_v^k \\
&= O(1) \sum_{v=0}^{\infty} \frac{1}{\left|t_{vv}\right|^{k-1}} \frac{1}{q_{v+1}^k} \left|\Delta^2 \lambda_v\right|^k Q_v^k \sum_{n=v+1}^{\infty} \left(n \left|t_{nn}\right|\right)^{k-1} \left|\hat{t}_{n,v+2}\right| \\
&= O(1) \sum_{v=0}^{\infty} v^{k-1} \frac{1}{q_{v+1}^k} \left|\Delta^2 \lambda_v\right|^k Q_v^k \\
&= O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1} \left|Y_{n24}\right|^{k} &= \sum_{n=1}^{\infty} n^{k-1} \left|\sum_{\mu=0}^{n-1} R_{\mu} t_{\mu} c_{n-\mu-1}^{(1)} t_{n,n+1} \frac{\Delta \lambda_n}{q_n}\right|^k \\
&\leq O(1) \sum_{n=1}^{\infty} n^{k-1} \left|t_{n,n+1}\right|^k \frac{\left|\Delta \lambda_n\right|^k}{q_n^k} \left|\sum_{\mu=0}^{n-1} R_{\mu} c_{n-\mu-1}^{(1)}\right|^k \\
&= O(1) \sum_{n=1}^{\infty} n^{k-1} \left|t_{n,n+1}\right|^k \frac{\left|\Delta \lambda_n\right|^k}{q_n^k} Q_{n-1}^k \\
&= O(1).
\end{aligned}$$

The proof is complete.

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