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ON A TYPE OF KENMOTSU MANIFOLD

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Abstract. The object of the present paper is to study some curvature conditions on Kenmotsu manifolds.

1. Introduction

In 1958, Boothby and Wong [2] studied odd dimensional manifolds with contact and almost contact structures from topological point of view. Sasaki and Hatakeyama [12] re-investigated them using tensor calculus in 1961. S. Tano [15] classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold M^n, the sectional curvature of plane sections containing ξ is a constant, say c. If $c > 0$, M^n is homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, M^n is the product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature. If $c < 0$, M^n is warped product space $\mathbb{R} \times_f \mathbb{C}^n$. In 1972, K. Kenmotsu studied a class of contact Riemannian manifold and call them Kenmotsu manifold [8]. He proved that if Kenmotsu manifold satisfies the condition $R(X, Y).R = 0$, then the manifold is of negative curvature -1, where R is the Riemannian curvature tensor of type (1, 3) and R(X, Y) denotes the derivation of the tensor algebra at each point of the tangent space. Recently, Kenmotsu manifolds have been studied by several authors such as De [5], Sinha and Shrivastava [14], Jun, De and Pathak [7], De and Pathak [4], De, Yildiz and Yaliniz [6], Özgur and De [10], Chaubey and Ojha [3], Singh, Pandey and Pandey [13] and many others. In the present paper we have studied some curvature conditions on Kenmotsu manifolds.

2. Preliminaries

If on an odd dimensional differentiable manifold M^n (where n = 2m+1) of differentiability class C^{r+1}, there exists a vector valued real linear function ϕ, a 1-form η, the associated vector field ξ and the Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

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$$\eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for arbitrary vector fields X and Y, then (M^n, g) is said to be an almost contact metric manifold [1] and the structure (ϕ, ξ, η, g) is called an almost contact metric structure to M^n. In view of equations (2.1), (2.2) and (2.3), we have

$$\eta(\xi) = 1, \tag{2.4}$$

$$g(X, \xi) = \eta(X), \tag{2.5}$$

$$\phi(\xi) = 0. \tag{2.6}$$

An almost contact metric manifold is called Kenmotsu manifold [8] if

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.7}$$

$$(\nabla_X \xi) = X - \eta(X)\xi, \tag{2.8}$$

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.9}$$

where ∇ is the Levi-Civita connection of g. Also, the following relations hold in Kenmotsu manifold [4], [6], [7]

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$R(\xi, X)Y = -R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.11}$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{2.12}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.13}$$

$$Q\xi = -(n-1)\xi, \tag{2.14}$$

where Q is the Ricci operator, i.e. $g(QX, Y) = S(X, Y)$ and

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.15}$$

for arbitrary vector fields X, Y, Z on M^n.

A Kenmotsu manifold M^n is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.16}$$

for arbitrary vector fields X and Y, where a and b are smooth functions on M^n.

3. M-projective curvature tensor of Kenmotsu manifolds

In 1971, Pokhariyal and Mishra [11] defined a tensor field W^* on a Riemannian manifold M^n as

$$\begin{aligned} W^*(X, Y)Z = R(X, Y)Z - \frac{1}{2(n-1)}[S(Y, Z)X \\ - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY], \end{aligned} \tag{3.1}$$

for arbitrary vector fields X, Y and Z, where S is the Ricci tensor of type (0, 2), Q is the Ricci operator and

$$'W^*(X, Y, Z, U) = g(W^*(X, Y)Z, U).$$

Putting $X = \xi$ in equation (3.1) and using equations (2.4), (2.5), (2.11),(2.13) and (2.14), we get

$$\begin{aligned} W^*(\xi, Y)Z = -W^*(Y, \xi)Z = \frac{1}{2}[\eta(Z)Y - g(Y, Z)\xi] \\ - \frac{1}{2(n-1)}[S(Y, Z)\xi - \eta(Z)QY]. \end{aligned} \tag{3.2}$$

Again, putting $Z = \xi$ in equation (3.1) and using equations (2.5),(2.10) and (2.13), we get

$$W^*(X,Y)\xi = \frac{1}{2}[\eta(X)Y - \eta(Y)X] - \frac{1}{2(n-1)}[\eta(Y)QX - \eta(X)QY]. \tag{3.3}$$

Now, taking the inner product of equations (3.1), (3.2) and (3.3) with ξ and using equations (2.4), (2.5) and (2.13), we get

$$\eta(W^*(X,Y)Z) = \frac{1}{2}[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] - \frac{1}{2(n-1)}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)], \tag{3.4}$$

$$\eta(W^*(\xi,Y)Z) = -\eta(W^*(Y,\xi)Z) = -\frac{1}{2}g(Y,Z) - \frac{1}{2(n-1)}S(Y,Z), \tag{3.5}$$

and

$$\eta(W^*(X,Y)\xi) = 0 \tag{3.6}$$

respectively.

Theorem 3.1. *A Kenmotsu manifold M^n satisfying the condition $R(\xi,X).W^* = 0$, is an Einstein manifold.*

Proof : Let $R(\xi,X).W^*(Y,Z)U = 0$. Then, we have

$$R(\xi,X)W^*(Y,Z)U - W^*(R(\xi,X)Y,Z)U - W^*(Y,R(\xi,X)Z)U - W^*(Y,Z)R(\xi,X)U = 0, \tag{3.7}$$

which on using equation (2.11), gives

$$\eta(W^*(Y,Z)U)X - g(X,W^*(Y,Z)U)\xi + g(X,Y)W^*(\xi,Z)U + g(X,Z)W^*(Y,\xi)U + g(X,U)W^*(Y,Z)\xi - \eta(Y)W^*(X,Z)U - \eta(Z)W^*(Y,X)U - \eta(U)W^*(Y,Z)X. \tag{3.8}$$

Now, taking the inner product of above equation with ξ and using equations (2.4)and (2.5), we get

$$\eta(W^*(Y,Z)U)\eta(X) - g(X,W^*(Y,Z)U) + g(X,Y)\eta(W^*(\xi,Z)U) + g(X,Z)\eta(W^*(Y,\xi)U + g(X,U)\eta(W^*(Y,Z)\xi) - \eta(Y)\eta(W^*(X,Z)U) - \eta(Z)\eta(W^*(Y,X)U) - \eta(U)\eta(W^*(Y,Z)X). \tag{3.9}$$

Using equations (3.1), (3.4), (3.5) and (3.6) in above equation, we get

$$'R(Y,Z,U,X) = \frac{1}{2}[g(X,Y)\eta(Z)\eta(U) - g(X,Z)\eta(Y)\eta(U) - g(X,Y)g(Z,U) + g(X,Z)g(Y,U)] + \frac{1}{2(n-1)}[S(X,Y)g(Z,U) - S(X,Z)g(Y,U) + S(X,Z)\eta(Y)\eta(U) - S(X,Y)\eta(Z)\eta(U)] + g(X,Z)\eta(Y)\eta(U) - g(X,Y)\eta(Z)\eta(U). \tag{3.10}$$

Put $Z = U = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$S(X,Y) = -(n-1)g(X,Y).$$

This shows that M^n is an Einstein manifold.

Theorem 3.2. *If a Kenmotsu manifold M^n satisfies the condition $W^*(\xi, X).R = 0$ then*

$$S(QX, Y) = -(n-1)^2 g(X, Y).$$

Proof : Let $W^*(\xi, X).R(Y, Z)U = 0$. Then, we have

$$\begin{aligned} &W^*(\xi, X)R(Y, Z)U - R(W^*(\xi, X)Y, Z)U \\ &- R(Y, W^*(\xi, X)Z)U - R(Y, Z)W^*(\xi, X)U = 0, \end{aligned} \tag{3.11}$$

which on using equation (3.2), gives

$$\begin{aligned} &\eta(R(Y, Z)U)X - g(X, R(Y, Z)U)\xi - \eta(Y)R(X, Z)U + g(X, Y)R(\xi, Z)U \\ &- \eta(Z)R(Y, X)U + g(X, Z)R(Y, \xi)U - \eta(U)R(Y, Z)X + g(X, U)R(Y, Z)\xi \\ &- \frac{1}{(n-1)}[S(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)QX - S(X, Y)R(\xi, Z)U \\ &+ \eta(Y)R(QX, Z)U - S(X, Z)R(Y, \xi)U + \eta(Z)R(Y, QX)U \\ &- S(X, U)R(Y, Z)\xi + \eta(U)R(Y, Z)QX] = 0. \end{aligned} \tag{3.12}$$

Now, taking the inner product of above equation with ξ and using equations (2.4)and (2.5), we get

$$\begin{aligned} &-'R(Y, Z, U, X) - \eta(Y)\eta(R(X, Z)U) + g(X, Y)\eta(R(\xi, Z)U) \\ &- \eta(Z)\eta(R(Y, X)U) + g(X, Z)\eta(R(Y, \xi)U) - \eta(U)\eta(R(Y, Z)X) \\ &+ g(X, U)\eta(R(Y, Z)\xi) - \frac{1}{(n-1)}['R(Y, Z, U, QX) - S(X, Y)\eta(R(\xi, Z)U) \\ &+ \eta(Y)\eta(R(QX, Z)U) - S(X, Z)\eta(R(Y, \xi)U) + \eta(Z)\eta(R(Y, QX)U) \\ &- S(X, U)\eta(R(Y, Z)\xi) + \eta(U)\eta(R(Y, Z)QX)] = 0. \end{aligned} \tag{3.13}$$

Using equations (2.10), (2.11) and (2.12) in above equation, we get

$$\begin{aligned} &'R(Y, Z, U, QX) + S(X, Y)g(Z, U) - S(X, Z)g(Y, U) \\ &= (n-1)[-'R(Y, Z, U, X) + g(X, Z)g(Y, U) - g(X, Y)g(Z, U)]. \end{aligned} \tag{3.14}$$

Put $Z = U = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$S(QX, Y) = -(n-1)^2 g(X, Y).$$

This completes the proof.

Theorem 3.3. *If a Kenmotsu manifold M^n satisfies the condition $W^*(\xi, X).S = 0$ then*

$$S(QX, Y) = 2(n-1)S(X, Y) + (n-1)^2 g(X, Y).$$

Proof : Let $W^*(\xi, X).S(Y, Z) = 0$. Then, we have

$$S(W^*(\xi, X)Y, Z) + S(Y, W^*(\xi, X)Z) = 0, \tag{3.15}$$

which on using equation (3.2), gives

$$\begin{aligned} &(n-1)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y)] + 2\eta(Y)S(X, Z) \\ &+ 2\eta(Z)S(X, Y) + \frac{1}{(n-1)}[\eta(Y)S(QX, Z) - \eta(Z)S(QX, Y)] = 0. \end{aligned} \tag{3.16}$$

Now, putting $Z = \xi$ in above equation and using equations (2.4), (2.5) and (2.13), we get

$$S(QX, Y) = 2(n-1)S(X, Y) + (n-1)^2 g(X, Y).$$

This completes the proof.

4. Kenmotsu manifolds satisfying $P(\xi, X).W^* = 0$ and $W^*(\xi, X).P = 0$

Projective curvature tensor P of the manifold M^n is given by [9]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{4.1}$$

Putting $X = \xi$ in above equation and using equations (2.11) and (2.13), we get

$$P(\xi,Y)Z = -P(Y,\xi)Z = -g(Y,Z)\xi - \frac{1}{(n-1)}S(Y,Z)\xi. \tag{4.2}$$

Again, put $Z = \xi$ in equation (4.1) and using equations (2.10) and (2.13), we get

$$P(X,Y)\xi = 0. \tag{4.3}$$

Now,taking the inner product of equations (4.1), (4.2) and (4.3) with ξ, we get

$$\begin{aligned}\eta(P(X,Y)Z) &= g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\\ &\quad - \frac{1}{(n-1)}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)],\end{aligned} \tag{4.4}$$

$$\eta(P(\xi,Y)Z) = -\eta(P(Y,\xi)Z) = -g(Y,Z) - \frac{1}{(n-1)}S(Y,Z) \tag{4.5}$$

and

$$\eta(P(X,Y)\xi) = 0 \tag{4.6}$$

respectively.

Theorem 4.1. *If a Kenmotsu manifold M^n satisfies the condition $P(\xi, X).W^* = 0$ then*

$$S(QX,Y) = -2S(X,Y) - (n-1)g(X,Y).$$

Proof : Let $P(\xi, X).W^*(Y,Z)U = 0$. Then, we have

$$\begin{aligned}&P(\xi,X)W^*(Y,Z)U - W^*(P(\xi,X)Y,Z)U\\ &- W^*(Y,P(\xi,X)Z)U - W^*(Y,Z)P(\xi,X)U = 0,\end{aligned} \tag{4.7}$$

which on using equation (4.2), gives

$$\begin{aligned}&- g(X,W^*(Y,Z)U)\xi + g(X,Y)W^*(\xi,Z)U + g(X,Z)W^*(Y,\xi)U\\ &+ g(X,U)W^*(Y,Z)\xi - \frac{1}{(n-1)}[S(X,W^*(Y,Z)U)\xi - S(X,Y)W^*(\xi,Z)U\\ &- S(X,Z)W^*(Y,\xi)U - S(X,U)W^*(Y,Z)\xi] = 0.\end{aligned} \tag{4.8}$$

Now, taking the inner product of above equation with ξ and using equations (2.4) and (2.5), we get

$$\begin{aligned}&- g(X,W^*(Y,Z)U) + g(X,Y)\eta(W^*(\xi,Z)U) + g(X,Z)\eta(W^*(Y,\xi)U)\\ &+ g(X,U)\eta(W^*(Y,Z)\xi) - \frac{1}{(n-1)}[S(X,W^*(Y,Z)U) - S(X,Y)\eta(W^*(\xi,Z)U)\\ &- S(X,Z)\eta(W^*(Y,\xi)U) - S(X,U)\eta(W^*(Y,Z)\xi)] = 0.\end{aligned} \tag{4.9}$$

Using equations (3.1), (3.4), (3.5) and (3.6) in above equation, we get

$$\frac{1}{(n-1)}\ {}'R(Y,Z,U,QX) = \frac{1}{2(n-1)^2}[g(Z,U)S(QX,Y) - g(Y,U)S(QX,Z)]$$
$$-\ {}'R(Y,Z,U,X) - \frac{1}{2}[g(Z,U)g(X,Y) - g(Y,U)g(X,Z)]. \tag{4.10}$$

Put $Z = U = e_i$ in above equation and taking summation over i,$1 \leq i \leq n$, we get

$$S(QX,Y) = -2S(X,Y) - (n-1)g(X,Y).$$

This completes the proof.

Theorem 4.2. *If a Kenmotsu manifold M^n satisfies the condition $W^*(\xi,X).P = 0$ then*

$$S(QX,Y) = (\frac{4(n-1)^2 + r}{(2-n)})S(X,Y) + (\frac{2(n-1)[n(n-1)+r]}{(2-n)})g(X,Y).$$

Proof : Let $W^*(\xi,X).P(Y,Z)U = 0$. Then, we have

$$\begin{aligned} &W^*(\xi,X)P(Y,Z)U - P(W^*(\xi,X)Y,Z)U \\ &- P(Y,W^*(\xi,X)Z)U - P(Y,Z)W^*(\xi,X)U = 0, \end{aligned} \tag{4.11}$$

which on using equation (3.2), gives

$$\begin{aligned} &\eta(P(Y,Z)U)X - g(X,P(Y,Z)U)\xi - \eta(Y)P(X,Z)U + g(X,Y)P(\xi,Z)U \\ &- \eta(Z)P(Y,X)U + g(X,Z)P(Y,\xi)U - \eta(U)P(Y,Z)X + g(X,U)P(Y,Z)\xi \\ &- \frac{1}{(n-1)}[S(X,P(Y,Z)U)\xi - \eta(P(Y,Z)U)QX - S(X,Y)P(\xi,Z)U \\ &+ \eta(Y)P(QX,Z)U - S(X,Z)P(Y,\xi)U + \eta(Z)P(Y,QX)U \\ &- S(X,U)P(Y,Z)\xi + \eta(U)P(Y,Z)QX] = 0. \end{aligned} \tag{4.12}$$

Now, taking the inner product of above equation with ξ and using equations (2.4)and (2.5), we get

$$\begin{aligned} &- g(X,P(Y,Z)U) - \eta(Y)\eta(P(X,Z)U) + g(X,Y)\eta(P(\xi,Z)U) \\ &- \eta(Z)\eta(P(Y,X)U) + g(X,Z)\eta(P(Y,\xi)U) - \eta(U)\eta(P(Y,Z)X) \\ &+ g(X,U)\eta(P(Y,Z)\xi) - \frac{1}{(n-1)}[S(X,P(Y,Z)U) - S(X,Y)\eta(P(\xi,Z)U) \\ &+ \eta(Y)\eta(P(QX,Z)U) - S(X,Z)\eta(P(Y,\xi)U) + \eta(Z)\eta(P(Y,QX)U) \\ &- S(X,U)\eta(P(Y,Z)\xi) + \eta(U)\eta(P(Y,Z)QX)] = 0. \end{aligned} \tag{4.13}$$

Using equations (4.1), (4.4), (4.5) and (4.6) in above equation, we obtain

$$\begin{aligned} &-' R(Y,Z,U,X) - g(X,Y)g(Z,U) + g(X,Z)g(Y,U) - g(X,Y)\eta(Z)\eta(U) \\ &+ g(X,Z)\eta(Y)\eta(U) + \frac{1}{(n-1)}[-'R(Y,Z,U,QX) + S(Y,U)g(X,Z) \\ &- S(Z,U)g(X,Y) + S(X,Z)g(Y,U) - S(X,Y)g(Z,U) + S(X,Z)\eta(Y)\eta(U) \\ &- S(X,Y)\eta(Z)\eta(U)] + \frac{1}{(n-1)^2}[\frac{1}{2}\{S(X,Z)S(Y,U) \\ &- S(X,Y)S(Z,U) + S(QX,Y)g(Z,U) - S(QX,Z)g(Y,U)\}]. \end{aligned} \tag{4.14}$$

Put $Z = U = e_i$ in above equation and taking summation over i,$1 \leq i \leq n$, we get

$$S(QX,Y) = (\frac{4(n-1)^2 + r}{(2-n)})S(X,Y) + (\frac{2(n-1)[n(n-1)+r]}{(2-n)})g(X,Y).$$

This completes the proof.

5. Kenmotsu manifolds satisfying $C(\xi, X).W^* = 0$ and $W^*(\xi, X).C = 0$

Conformal curvature tensor C of the manifold M^n is given by [9]

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{5.1}$$

Putting $X = \xi$ in above equation and using equations (2.11) and (2.13), we get

$$\begin{aligned} C(\xi,Y)Z = -C(Y,\xi)Z = &\frac{(1-n-r)}{(n-1)(n-2)}[\eta(Z)Y - g(Y,Z)\xi] \\ &- \frac{1}{(n-2)}[S(Y,Z)\xi - \eta(Z)QY]. \end{aligned} \tag{5.2}$$

Again, put $Z = \xi$ in equation (5.1) and using equations (2.10) and (2.13), we get

$$C(X,Y)\xi = \frac{(1-n-r)}{(n-2)}[\eta(X)Y - \eta(Y)X] - \frac{1}{(n-2)}[\eta(Y)QX - \eta(X)QY]. \tag{5.3}$$

Now, taking the inner product of equations (5.1), (5.2) and (5.3) with ξ, we get

$$\begin{aligned} \eta(C(X,Y)Z) = &\frac{(1-n-r)}{(n-1)(n-2)}[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \\ &- \frac{1}{(n-2)}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)], \end{aligned} \tag{5.4}$$

$$\begin{aligned} \eta(C(\xi,Y)Z) = -\eta(C(Y,\xi)Z) = &\frac{(1-n-r)}{(n-1)(n-2)}[\eta(Z)\eta(Y) - g(Y,Z)] \\ &- \frac{1}{(n-2)}[S(Y,Z) + \eta(Y)\eta(Z)] \end{aligned} \tag{5.5}$$

and

$$\eta(C(X,Y)\xi) = 0 \tag{5.6}$$

respectively.

Theorem 5.1. *If a Kenmotsu manifold M^n satisfies the condition $C(\xi, X).W^* = 0$ then*

$$S(QX,Y) = \frac{(-n^2+3n+2r-2)}{n}S(X,Y) + (n+r-1)g(X,Y) + (n-r+1)\eta(X)\eta(Y).$$

Proof : Let $C(\xi, X).W^*(Y, Z)U = 0$. Then, we have

$$\begin{aligned} &C(\xi, X)W^*(Y, Z)U - W^*(C(\xi, X)Y, Z)U \\ &- W^*(Y, C(\xi, X)Z)U - W^*(Y, Z)C(\xi, X)U = 0, \end{aligned} \tag{5.7}$$

which on using equation (5.2), gives

$$\begin{aligned} &\frac{(1-n-r)}{(n-1)(n-2)}[\eta(W^*(Y, Z)U)X - g(X, W^*(Y, Z)U)\xi - \eta(Y)W^*(X, Z)U \\ &+ g(X, Y)W^*(\xi, Z)U - \eta(Z)W^*(Y, X)U + g(X, Z)W^*(Y, \xi)U - \eta(U)W^*(Y, Z)X \\ &+ g(X, U)W^*(Y, Z)\xi] - \frac{1}{(n-2)}[S(X, W^*(Y, Z)U)\xi - \eta(W^*(Y, Z)U)QX \\ &- S(X, Y)W^*(\xi, Z)U + \eta(Y)W^*(QX, Z)U - S(X, Z)W^*(Y, \xi)U \\ &+ \eta(Z)W^*(Y, QX)U - S(X, U)W^*(Y, Z)\xi + \eta(U)W^*(Y, Z)QX] = 0. \end{aligned} \tag{5.8}$$

Now, taking the inner product of above equation with ξ and using equations (2.4) and (2.5), we get

$$\begin{aligned} &\frac{(1-n-r)}{(n-1)(n-2)}[\eta(W^*(Y, Z)U)\eta(X) - g(X, W^*(Y, Z)U) - \eta(Y)\eta(W^*(X, Z)U) \\ &+ g(X, Y)\eta(W^*(\xi, Z)U) - \eta(Z)\eta(W^*(Y, X)U) + g(X, Z)\eta(W^*(Y, \xi)U) \\ &- \eta(U)\eta(W^*(Y, Z)X) + g(X, U)\eta(W^*(Y, Z)\xi)] - \frac{1}{(n-2)}[S(X, W^*(Y, Z)U) \\ &+ (n-1)\eta(W^*(Y, Z)U)\eta(X) - S(X, Y)\eta(W^*(\xi, Z)U) + \eta(Y)\eta(W^*(QX, Z)U) \\ &- S(X, Z)\eta(W^*(Y, \xi)U) + \eta(Z)\eta(W^*(Y, QX)U) - S(X, U)\eta(W^*(Y, Z)\xi) \\ &+ \eta(U)\eta(W^*(Y, Z)QX)] = 0. \end{aligned} \tag{5.9}$$

Using equations (3.1),(3.4), (3.5) and (3.6) in above equation, we get

$$\begin{aligned} &\frac{(1-n-r)}{(n-1)(n-2)}[-\,'R(Y, Z, U, X) + \frac{1}{2(n-1)}\{g(Z, U)S(X, Y) - g(Y, U)S(X, Z)\} \\ &- \frac{1}{2}\{g(X, Y)g(Z, U) - g(X, Z)g(Y, U) + g(X, Y)\eta(Z)\eta(U) - g(X, Z)\eta(Y)\eta(U)\} \\ &+ \frac{1}{2}\{S(X, Z)\eta(Y)\eta(U) - S(X, Y)\eta(Z)\eta(U)\}] - \frac{1}{(n-2)}[\,'R(Y, Z, U, QX) \\ &- \frac{1}{2(n-1)}\{g(Z, U)S(QX, Y) - g(Y, U)S(QX, Z)\} + \frac{1}{2}\{S(X, Y)g(Z, U) \\ &- S(X, Z)g(Y, U) + S(X, Y)\eta(Z)\eta(U) - S(X, Z)\eta(Y)\eta(U)\} \\ &- \frac{1}{2(n-1)}\{S(QX, Z)\eta(Y)\eta(U) - S(QX, Y)\eta(Z)\eta(U)\}] = 0. \end{aligned} \tag{5.10}$$

Put $Z = U = e_i$ in above equation and taking summation over i, we get

$$S(QX, Y) = \frac{(-n^2+3n+2r-2)}{n}S(X, Y) + (n+r-1)g(X, Y) + (n-r+1)\eta(X)\eta(Y).$$

This completes the proof.

Theorem 5.2. *If a Kenmotsu manifold M^n satisfies the condition $W^*(\xi, X).C = 0$ then*

$$S(QX, Y) = -\frac{rn}{2(n-1)} S(X, Y).$$

Proof : Let $W^*(\xi, X).C(Y, Z)U = 0$. Then, we have

$$\begin{aligned} &W^*(\xi, X)C(Y, Z)U - C(W^*(\xi, X)Y, Z)U \\ &- C(Y, W^*(\xi, X)Z)U - C(Y, Z)W^*(\xi, X)U = 0, \end{aligned} \tag{5.11}$$

which on using equation (3.2), gives

$$\begin{aligned} &[\eta(C(Y, Z)U)X - g(X, C(Y, Z)U)\xi - \eta(Y)C(X, Z)U + g(X, Y)C(\xi, Z)U \\ &- \eta(Z)C(Y, X)U + g(X, Z)C(Y, \xi)U - \eta(U)C(Y, Z)X + g(X, U)C(Y, Z)\xi] \\ &- \frac{1}{(n-1)}[-\eta(C(Y, Z)U)QX + S(X, C(Y, Z)U)\xi + S(X, Y)C(\xi, Z)U \\ &- \eta(Y)C(QX, Z)U + S(X, Z)C(Y, \xi)U - \eta(Z)C(Y, QX)U \\ &+ S(X, U)C(Y, Z)\xi - \eta(U)C(Y, Z)QX] = 0. \end{aligned} \tag{5.12}$$

Now, taking the inner product of above equation with ξ and using equations (2.4)and (2.5), we get

$$\begin{aligned} &- g(X, C(Y, Z)U) - \eta(Y)\eta(C(X, Z)U) + g(X, Y)\eta(C(\xi, Z)U) \\ &- \eta(Z)\eta(C(Y, X)U) + g(X, Z)\eta(C(Y, \xi)U) - \eta(U)\eta(C(Y, Z)X) \\ &+ g(X, U)\eta(C(Y, Z)\xi) - \frac{1}{(n-1)}[S(X, C(Y, Z)U) + S(X, Y)\eta(C(\xi, Z)U) \\ &- \eta(Y)\eta(C(QX, Z)U) + S(X, Z)\eta(C(Y, \xi)U) - \eta(Z)\eta(C(Y, QX)U) \\ &+ S(X, U)\eta(C(Y, Z)\xi) - \eta(U)\eta(C(Y, Z)QX)] = 0. \end{aligned} \tag{5.13}$$

Using equations (5.1), (5.4), (5.5) and (5.6) in above equation, we get

$$\begin{aligned} &\frac{1}{(n-1)}\ {}'R(Y, Z, U, QX) = -\ {}'R(Y, Z, U, X) + \frac{1}{(n-2)}[S(X, Y)g(Z, U) \\ &- S(X, Z)g(Y, U) + (n-1)\{g(X, Z)\eta(Y)\eta(U) - g(X, Y)\eta(Z)\eta(U)\} \\ &+ 2\{S(Z, U)\eta(X)\eta(Y) - S(Y, U)\eta(X)\eta(Z)\} + \frac{1}{(n-1)}\{S(QX, Y)g(Z, U) \\ &- S(QX, Z)g(Y, U) + S(QX, Y)\eta(Z)\eta(U) - S(QX, Z)\eta(Y)\eta(U)\}] \\ &- \frac{r}{(n-1)(n-2)}[g(X, Y)g(Z, U) - g(X, Z)g(Y, U) + S(X, Y)g(Z, U) \\ &- S(X, Z)g(Y, U)] + \frac{1-n-r}{(n-1)(n-2)}[2\{g(Z, U)\eta(X)\eta(Y) \\ &- g(Y, U)\eta(X)\eta(Z)\} - g(X, Y)g(Z, U) + g(X, Z)g(Y, U) \\ &+ \frac{1}{(n-1)}\{S(X, Y)g(Z, U) - S(X, Z)g(Y, U)\}]. \end{aligned} \tag{5.14}$$

Put $Z = U = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$S(QX, Y) = -\frac{rn}{2(n-1)} S(X, Y).$$

This completes the proof.

6. Kenmotsu manifolds satisfying $\tilde{C}(\xi,X).W^*=0$ and $W^*(\xi,X).\tilde{C}=0$

The notion of the quasi-conformal curvature tensor $\tilde{C}$ was introduced by Yano and Sawaki [16]. They defined the quasi- conformal curvature tensor by

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY] - \frac{r}{n}\{\frac{a}{n-1} + 2b\}[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{6.1}$$

where a and b are constants such that $ab \neq 0$, R is the Riemannian curvature tensor, S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature of the manifold. If $a = 1$ and $b = -\frac{1}{n-2}$, then above equation takes the form

$$\begin{aligned}\tilde{C}(X,Y)Z =& R(X,Y)Z - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \\ =& C(X,Y)Z,\end{aligned}$$

where C is the conformal curvature tensor [16]. Thus the conformal curvature tensor C is a particular case of the quasi -conformal curvature tensor $\tilde{C}$.
Putting $X = \xi$ in equation (6.1) and using equations (2.11) and (2.13), we get

$$\begin{aligned}\tilde{C}(\xi,Y)Z = -\tilde{C}(Y,\xi)Z =& [a + b(n-1) + \frac{r}{n}(\frac{a}{n-1} + 2b)][\eta(Z)Y - g(Y,Z)\xi] \\ &+ b[S(Y,Z)\xi - \eta(Z)QY].\end{aligned} \tag{6.2}$$

Again, put $Z = \xi$ in equation (6.1) and using equations (2.10) and (2.13), we get

$$\begin{aligned}\tilde{C}(X,Y)\xi =& [a + b(n-1) + \frac{r}{n}(\frac{a}{n-1} + 2b)][\eta(X)Y - \eta(Y)X] \\ &+ b[\eta(Y)QX - \eta(X)QY].\end{aligned} \tag{6.3}$$

Now, taking the inner product of equations (6.1), (6.2) and (6.3) with ξ, we get

$$\begin{aligned}\eta(\tilde{C}(X,Y)Z) =& [a + b(n-1) + \frac{r}{n}(\frac{a}{n-1} + 2b)][g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \\ &+ b[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)],\end{aligned} \tag{6.4}$$

$$\begin{aligned}\eta(\tilde{C}(\xi,Y)Z) = -\eta(\tilde{C}(Y,\xi)Z) =& [a + b(n-1) + \frac{r}{n}(\frac{a}{n-1} + 2b)][\eta(Z)\eta(Y) - g(Y,Z)] \\ &+ b[S(Y,Z) + \eta(Y)\eta(Z)]\end{aligned} \tag{6.5}$$

and

$$\eta(\tilde{C}(X,Y)\xi) = 0 \tag{6.6}$$

respectively.

Theorem 6.1. *If a Kenmotsu manifold M^n satisfies the condition $\tilde{C}(\xi,X).W^* = 0$, then*

$$S(QX,Y) = \frac{A}{b}S(X,Y) + \frac{A}{b}(n-1)g(X,Y) + \frac{(n-1)}{n}(n^2 - 3n + nr - 2r + 2)\eta(X)\eta(Y),$$

where

$$A = [a + b(n-1) + \frac{r}{n}(\frac{a}{n-1} + 2b)].$$

Proof : Let $\tilde{C}(\xi, X).W^*(Y,Z)U = 0$. Then, we have

$$\begin{aligned} &\tilde{C}(\xi, X)W^*(Y,Z)U - W^*(\tilde{C}(\xi, X)Y, Z)U \\ &- W^*(Y, \tilde{C}(\xi, X)Z)U - W^*(Y,Z)\tilde{C}(\xi, X)U = 0, \end{aligned} \tag{6.7}$$

which on using equation (6.2), gives

$$\begin{aligned} &A[\eta(W^*(Y,Z)U)X - g(X, W^*(Y,Z)U)\xi - \eta(Y)W^*(X,Z)U \\ &+ g(X,Y)W^*(\xi, Z)U - \eta(Z)W^*(Y,X)U + g(X,Z)W^*(Y,\xi)U - \eta(U)W^*(Y,Z)X \\ &+ g(X,U)W^*(Y,Z)\xi] + b[S(X, W^*(Y,Z)U)\xi - \eta(W^*(Y,Z)U)QX \\ &- S(X,Y)W^*(\xi, Z)U + \eta(Y)W^*(QX, Z)U - S(X,Z)W^*(Y,\xi)U \\ &+ \eta(Z)W^*(Y, QX)U - S(X,U)W^*(Y,Z)\xi + \eta(U)W^*(Y,Z)QX] = 0, \end{aligned} \tag{6.8}$$

where

$$A = [a + b(n-1) + \frac{r}{n}(\frac{a}{n-1} + 2b)].$$

Now, taking the inner product of above equation with ξ and using equations (2.4) and (2.5), we get

$$\begin{aligned} &A[\eta(W^*(Y,Z)U)\eta(X) - g(X, W^*(Y,Z)U) - \eta(Y)\eta(W^*(X,Z)U) \\ &+ g(X,Y)\eta(W^*(\xi,Z)U) - \eta(Z)\eta(W^*(Y,X)U) + g(X,Z)\eta(W^*(Y,\xi)U) \\ &- \eta(U)\eta(W^*(Y,Z)X) + g(X,U)\eta(W^*(Y,Z)\xi)] + b[S(X, W^*(Y,Z)U) \\ &+ (n-1)\eta(W^*(Y,Z)U)\eta(X) - S(X,Y)\eta(W^*(\xi,Z)U) - \eta(Y)\eta(W^*(QX,Z)U) \\ &- S(X,Z)\eta(W^*(Y,\xi)U) + \eta(Z)\eta(W^*(Y,QX)U) - S(X,U)\eta(W^*(Y,Z)\xi) \\ &+ \eta(U)\eta(W^*(Y,Z)QX)] = 0. \end{aligned} \tag{6.9}$$

Using equations (3.1),(3.4), (3.5) and (3.6) in above equation, we get

$$\begin{aligned} &A[-\,'R(Y,Z,U,X) + \frac{1}{2(n-1)}\{g(Z,U)S(X,Y) - g(Y,U)S(X,Z)\} \\ &- \frac{1}{2}g(X,Y)g(Z,U) + \frac{1}{2}g(X,Z)g(Y,U) - \frac{1}{2}\{g(X,Y)\eta(Z)\eta(U) - g(X,Z)\eta(Y)\eta(U)\} \\ &+ \frac{1}{2(n-1)}\{S(X,Z)\eta(Y)\eta(U) - S(X,Y)\eta(Z)\eta(U)\}] + b[\,'R(Y,Z,U,QX) \\ &- \frac{1}{2(n-1)}\{g(Z,U)S(QX,Y) - g(Y,U)S(QX,Z)\} - (\frac{n-1}{2})\{S(Z,U)\eta(Y)\eta(X) \\ &- S(Y,U)\eta(X)\eta(Z)\} + \frac{1}{2}S(X,Y)g(Z,U) - \frac{1}{2}S(X,Z)g(Y,U) \\ &+ \frac{1}{2}\{S(X,Y)\eta(Z)\eta(U) - S(X,Z)\eta(Y)\eta(U)\} \\ &- \frac{1}{2(n-1)}\{S(QX,Z)\eta(Y)\eta(U) - S(QX,Y)\eta(Z)\eta(U)\} \\ &- \frac{1}{2(n-1)}\{S(QX,U)\eta(Y)\eta(Z) + (n-1)S(Y,U)\eta(X)\eta(Z)\} \\ &+ \frac{1}{2(n-1)}\{(n-1)S(Z,U)\eta(X)\eta(Y) + S(QX,U)\eta(Y)\eta(Z)\}] = 0. \end{aligned} \tag{6.10}$$

Put $Z = U = e_i$ in above equation and taking summation over i , $1 \leq i \leq n$, we get

$$S(QX,Y) = \frac{A}{b}S(X,Y)+\frac{A}{b}(n-1)g(X,Y)+\frac{(n-1)}{n}(n^2-3n+nr-2r+2)\eta(X)\eta(Y),$$

where

$$A = [a + b(n-1) + \frac{r}{n}(\frac{a}{n-1} + 2b)].$$

This completes the proof.

Theorem 6.2. *If a Kenmotsu manifold M^n satisfies $W^*(\xi,X).\tilde{C} = 0$ then*

$$\begin{aligned} S(QX,Y) = &-2(n-1)S(X,Y) - (n-1)\frac{[a(n+1) + b(n-1)(n-2)]}{a + b(n-2)}g(X,Y) \\ &- \frac{n(n-1)b}{a + b(n-2)}\eta(X)\eta(Y). \end{aligned}$$

Proof : Let $W^*(\xi,X).\tilde{C}(Y,Z)U = 0$. Then, we have

$$\begin{aligned} &W^*(\xi,X)\tilde{C}(Y,Z)U - \tilde{C}(W^*(\xi,X)Y,Z)U \\ &- \tilde{C}(Y,W^*(\xi,X)Z)U - \tilde{C}(Y,Z)W^*(\xi,X)U = 0, \end{aligned} \tag{6.11}$$

which on using equation (3.2), gives

$$\begin{aligned} &\frac{1}{2}[\eta(\tilde{C}(Y,Z)U)X - g(X,\tilde{C}(Y,Z)U)\xi - \eta(Y)\tilde{C}(X,Z)U + g(X,Y)\tilde{C}(\xi,Z)U \\ &- \eta(Z)\tilde{C}(Y,X)U + g(X,Z)\tilde{C}(Y,\xi)U - \eta(U)\tilde{C}(Y,Z)X + g(X,U)\tilde{C}(Y,Z)\xi] \\ &- \frac{1}{2(n-1)}[S(X,\tilde{C}(Y,Z)U)\xi - \eta(\tilde{C}(Y,Z)U)QX - S(X,Y)\tilde{C}(\xi,Z)U \\ &+ \eta(Y)\tilde{C}(QX,Z)U - S(X,Z)\tilde{C}(Y,\xi)U + \eta(Z)\tilde{C}(Y,QX)U \\ &- S(X,U)\tilde{C}(Y,Z)\xi + \eta(U)\tilde{C}(Y,Z)QX] = 0. \end{aligned} \tag{6.12}$$

Now, taking the inner product of above equation with ξ and using equations (2.4) and (2.5), we get

$$\begin{aligned} &- g(X,\tilde{C}(Y,Z)U) - \eta(Y)\eta(\tilde{C}(X,Z)U) + g(X,Y)\eta(\tilde{C}(\xi,Z)U) \\ &- \eta(Z)\eta(\tilde{C}(Y,X)U) + g(X,Z)\eta(\tilde{C}(Y,\xi)U) - \eta(U)\eta(\tilde{C}(Y,Z)X) \\ &+ g(X,U)\eta(\tilde{C}(Y,Z)\xi) - \frac{1}{(n-1)}[S(X,\tilde{C}(Y,Z)U) - S(X,Y)\eta(\tilde{C}(\xi,Z)U) \\ &+ \eta(Y)\eta(\tilde{C}(QX,Z)U) - S(X,Z)\eta(\tilde{C}(Y,\xi)U) + \eta(Z)\eta(\tilde{C}(Y,QX)U) \\ &- S(X,U)\eta(\tilde{C}(Y,Z)\xi) + \eta(U)\eta(\tilde{C}(Y,Z)QX)] = 0. \end{aligned} \tag{6.13}$$

Using equations (6.1), (6.4), (6.5) and (6.6) in above equation, we get

$$\begin{aligned}
&-a[\,'R(Y,Z,U,X)+\frac{1}{(n-1)}\,'R(Y,Z,U,QX)]\\
&+\frac{r}{n}(\frac{a}{n-1}+2b)[g(X,Y)g(Z,U)-g(Y,U)g(X,Z)\\
&+\frac{1}{(n-1)}\{S(X,Y)g(Z,U)-S(X,Z)g(Y,U)\}]+A[g(X,Z)g(Y,U)\\
&-g(X,Y)g(Z,U)-\frac{1}{(n-1)}\{S(X,Y)g(Z,U)-S(X,Z)g(Y,U)\}] \qquad (6.14)\\
&+b[g(Y,U)S(X,Z)-g(Z,U)S(X,Y)+(n-1)\{g(X,Y)\eta(Z)\eta(U)\\
&-g(X,Z)\eta(Y)\eta(U)\}+2\{S(X,Y)\eta(Z)\eta(U)-S(X,Z)\eta(Y)\eta(U)\}\\
&-\frac{1}{(n-1)}\{S(QX,Y)g(Z,U)-S(QX,Z)g(Y,U)+S(QX,Y)\eta(Z)\eta(U)\\
&-S(QX,Z)\eta(Y)\eta(U)\}]=0,
\end{aligned}$$

where

$$A=a+b(n-1)+\frac{r}{n}(\frac{a}{n-1}+2b).$$

Put $Z=U=e_i$ in above equation and taking summation over i, $1\leq i\leq n$, we get

$$\begin{aligned}
S(QX,Y)=&-2(n-1)S(X,Y)-(n-1)\frac{[a(n+1)+b(n-1)(n-2)]}{a+b(n-2)}g(X,Y)\\
&-\frac{n(n-1)b}{a+b(n-2)}\eta(X)\eta(Y).
\end{aligned}$$

This completes the proof.

7. Kenmotsu manifolds satisfying $V(\xi,X).W^*=0$ and $W^*(\xi,X).V=0$

Concircular curvature tensor V of the manifold M^n is defined by [9]

$$V(X,Y)Z=R(X,Y)Z-\frac{r}{n(n-1)}[g(Y,Z)X-g(X,Z)Y]. \qquad (7.1)$$

Putting $X=\xi$ in above equation and using equations (2.11) and (2.13), we get

$$V(\xi,Y)Z=-V(Y,\xi)Z=\frac{(n(n-1)+r)}{n(n-1)}[\eta(Z)Y-g(Y,Z)\xi]. \qquad (7.2)$$

Again, put $Z=\xi$ in equation (7.1) and using equations (2.10) and (2.13), we get

$$V(X,Y)\xi=\frac{(n(n-1)+r)}{n(n-1)}[\eta(X)Y-\eta(Y)X]-\frac{1}{(n-2)}[\eta(Y)QX-\eta(X)QY]. \qquad (7.3)$$

Now, taking the inner product of equations (7.1), (7.2) and (7.3) with ξ, we get

$$\eta(V(X,Y)Z)=\frac{(n(n-1)+r)}{n(n-1)}[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)], \qquad (7.4)$$

$$\eta(V(\xi,Y)Z)=-\eta(V(Y,\xi)Z)=\frac{(n(n-1)+r)}{n(n-1)}[\eta(Z)\eta(Y)-g(Y,Z)] \qquad (7.5)$$

and

$$\eta(V(X,Y)\xi)=0 \qquad (7.6)$$

respectively.

Theorem 7.1. *If a Kenmotsu manifold M^n satisfies the condition $V(\xi, X).W^* = 0$ then either $r = -n(n-1)$ or M^n is an $\eta-$Einstein manifold.*

Proof : Let $V(\xi, X).W^*(Y, Z)U = 0$. Then, we have

$$\begin{aligned} &V(\xi, X)W^*(Y, Z)U - W^*(V(\xi, X)Y, Z)U \\ &- W^*(Y, V(\xi, X)Z)U - W^*(Y, Z)V(\xi, X)U = 0, \end{aligned} \tag{7.7}$$

which on using equation (7.2), gives

$$\begin{aligned} &\frac{(n(n-1)+r)}{n(n-1)}[\eta(W^*(Y, Z)U)X - g(X, W^*(Y, Z)U)\xi - \eta(Y)W^*(X, Z)U \\ &+ g(X, Y)W^*(\xi, Z)U - \eta(Z)W^*(Y, X)U + g(X, Z)W^*(Y, \xi)U - \eta(U)W^*(Y, Z)X \\ &+ g(X, U)W^*(Y, Z)\xi] = 0. \end{aligned} \tag{7.8}$$

Now, taking the inner product of above equation with ξ and using equations (2.4) and (2.5), we get

$$\begin{aligned} &\frac{(n(n-1)+r)}{n(n-1)}[\eta(W^*(Y, Z)U)\eta(X) - g(X, W^*(Y, Z)U) - \eta(Y)\eta(W^*(X, Z)U) \\ &+ g(X, Y)\eta(W^*(\xi, Z)U) - \eta(Z)\eta(W^*(Y, X)U) + g(X, Z)\eta(W^*(Y, \xi)U) \\ &- \eta(U)\eta(W^*(Y, Z)X) + g(X, U)\eta(W^*(Y, Z)\xi)] = 0. \end{aligned} \tag{7.9}$$

Using equations (3.1),(3.4), (3.5) and (3.6) in above equation, we get

$$\begin{aligned} &[\frac{(n(n-1)+r)}{n(n-1)}][g(X, Z)g(Y, U) - g(X, Y)g(Z, U) + g(X, Z)S(Y, U) \\ &- g(X, Y)\eta(Z)\eta(U) + g(X, Z)\eta(Y)\eta(U) + S(X, Z)\eta(Y)\eta(U) - S(X, Y)\eta(Z)\eta(U) \\ &+ \frac{1}{(n-1)}\{S(X, Y)g(Z, U) - S(Y, U)g(X, Z) - g(Y, U)S(X, Z)\}] = 0. \end{aligned} \tag{7.10}$$

Put $Z = U = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$[\frac{(n(n-1)+r)}{n(n-1)}][\frac{-n}{(n-1)}S(X, Y) - ng(X, Y) - (n-2)\eta(X)\eta(Y)] = 0,$$

which gives either $r = -n(n-1)$ or

$$S(X, Y) = -(n-1)g(X, Y) - \frac{(n-1)(n-2)}{n}\eta(X)\eta(Y),$$

This shows that either $r = -n(n-1)$ or M^n is an $\eta-$Einstein manifold. This completes the proof.

Theorem 7.2. *If a Kenmotsu manifold M^n satisfies the condition $W^*(\xi, X).V = 0$ then*

$$S(QX, Y) = [\frac{n(n-1)(3n-2)+r}{n(n-1)}]S(X, Y) - (n-1)^2 g(X, Y) + [\frac{n(n-1)+r}{n}]\eta(X)\eta(Y).$$

Proof : Let $W^*(\xi, X).V(Y,Z)U = 0$. Then, we have

$$\begin{aligned} &W^*(\xi, X)V(Y,Z)U - V(W^*(\xi, X)Y, Z)U \\ &- V(Y, W^*(\xi, X)Z)U - V(Y,Z)W^*(\xi, X)U = 0, \end{aligned} \tag{7.11}$$

which on using equation (3.2), gives

$$\begin{aligned} &\eta(V(Y,Z)U)X - g(X, V(Y,Z)U)\xi - \eta(Y)V(X,Z)U + g(X,Y)V(\xi,Z)U \\ &- \eta(Z)V(Y,X)U + g(X,Z)V(Y,\xi)U - \eta(U)V(Y,Z)X + g(X,U)V(Y,Z)\xi \\ &- \frac{1}{(n-1)}[S(X,V(Y,Z)U)\xi - \eta(V(Y,Z)U)QX - S(X,Y)V(\xi,Z)U \\ &+ \eta(Y)V(QX,Z)U - S(X,Z)V(Y,\xi)U + \eta(Z)V(Y,QX)U \\ &- S(X,U)V(Y,Z)\xi + \eta(U)V(Y,Z)QX] = 0. \end{aligned} \tag{7.12}$$

Now, taking the inner product of above equation with ξ and using equations (2.4)and (2.5), we get

$$\begin{aligned} &- g(X, V(Y,Z)U) - \eta(Y)\eta(V(X,Z)U) + g(X,Y)\eta(V(\xi,Z)U) \\ &- \eta(Z)\eta(V(Y,X)U) + g(X,Z)\eta(V(Y,\xi)U) - \eta(U)\eta(V(Y,Z)X) + g(X,U)\eta(V(Y,Z)\xi) \\ &- \frac{1}{(n-1)}[S(X,V(Y,Z)U) - S(X,Y)\eta(V(\xi,Z)U) \\ &+ \eta(Y)\eta(V(QX,Z)U) - S(X,Z)\eta(V(Y,\xi)U) + \eta(Z)\eta(V(Y,QX)U) \\ &- S(X,U)\eta(V(Y,Z)\xi) + \eta(U)\eta(V(Y,Z)QX)] = 0. \end{aligned} \tag{7.13}$$

Using equations (7.1), (7.4), (7.5) and (7.6) in above equation, we get

$$\begin{aligned} &- {}'R(Y,Z,U,X) - [g(X,Y)g(Z,U) - g(Y,U)g(X,Z)] \\ &- \frac{1}{(n-1)}[{}'R(Y,Z,U,QX) + \{S(X,Y)g(Z,U) - S(X,Z)g(Y,U)\}] = 0. \end{aligned} \tag{7.14}$$

Put $Z = U = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$S(QX,Y) = [\frac{n(n-1)(3n-2)+r}{n(n-1)}]S(X,Y) - (n-1)^2 g(X,Y) + [\frac{n(n-1)+r}{n}]\eta(X)\eta(Y).$$

This completes the proof.

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