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# REMARKS ON THE DOMINATION THEOREM FOR SUMMING OPERATORS

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Abstract. In this note we provide some applications of the general Pietsch Domination Theorem.

## 1. Introduction

Pietsch Domination Theorem plays a central role in the theory of absolutely summing linear operators (see [6]). In the last years, several Pietsch-type theorems have been presented in different nonlinear settings (we mention, for example, [1, 4, 5, 7, 8, 9, 10, 12, 15]); in [3] an abstract approach to the PDT was presented as an attempt of unification (see also [13, 14]).

From now on the Banach spaces will be considered over a fixed scalar field $\mathbb{K}$ that can be $\mathbb{R}$ or $\mathbb{C}$. The topological dual of a Banach space $X$ will be denoted by $X^*$ and its closed unit ball will be represented by $B_{X^*}$.

Let us recall the General Pietsch Domination Theorem recently presented in [3, 13]:

Let $X$, $Y$ and $E$ be (arbitrary) non-void sets, $\mathcal{H}$ be a family of mappings from $X$ to $Y$, $G$ be a Banach space and $K$ be a compact Hausdorff topological space. Let

$$S\colon \mathcal{H} \times E \times G \longrightarrow [0,\infty)$$

be an arbitrary map and

$$R\colon K \times E \times G \longrightarrow [0,\infty)$$

be such that

$$R_{x,b}\colon K \longrightarrow [0,\infty) \text{ defined by } R_{x,b}(\varphi) = R(\varphi, x, b)$$

is continuous for every $x \in E$ and $b \in G$.

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If $R$ and $S$ are as above and $0 < p < \infty$, a mapping $f \in \mathcal{H}$ is said to be $R$-$S$-abstract $p$-summing if there is a constant $C_1 > 0$ so that

$$\left(\sum_{j=1}^{m} S(f, x_j, b_j)^p\right)^{\frac{1}{p}} \leq C_1 \sup_{\varphi \in K} \left(\sum_{j=1}^{m} R(\varphi, x_j, b_j)^p\right)^{\frac{1}{p}}, \tag{1.1}$$

for all $x_1, \ldots, x_m \in E$, $b_1, \ldots, b_m \in G$ and $m \in \mathbb{N}$.

The general unified PDT reads as follows:

**Theorem 1.1** (General Pietsch Domination Theorem)**.** *Let $R$ and $S$ be as above, $0 < p < \infty$ and $f \in \mathcal{H}$. Then $f$ is $R$-$S$-abstract $p$-summing if and only if there is a constant $C > 0$ and a Borel probability measure $\mu$ on $K$ such that*

$$S(f, x, b) \leq C \left(\int_K R(\varphi, x, b)^p \, d\mu\right)^{\frac{1}{p}} \tag{1.2}$$

*for all $x \in E$ and $b \in G$.*

In [3] the following concept was introduced, as a natural adaptation of [11, Definition 3.1]:

**Definition 1.** *Let $X$ and $Y$ be Banach spaces. An arbitrary mapping $f\colon X \longrightarrow Y$ is absolutely $p$-summing at $a \in X$ if there is a $C \geq 0$ so that*

$$\sum_{j=1}^{m} \|f(a + x_j) - f(a)\|^p \leq C \sup_{\varphi \in B_{X^*}} \sum_{j=1}^{m} |\varphi(x_j)|^p$$

*for every natural number $m$ and every $x_1, \ldots, x_m \in X$.*

Also in [3], as an application of Theorem 1.1, the following Pietsch Domination type theorem is proved:

**Theorem 1.2.** *Let $X$ and $Y$ be Banach spaces. An arbitrary mapping $f\colon X \longrightarrow Y$ is absolutely $p$-summing at $a \in X$ if and only if there is a constant $C_a \geq 0$ and a Borel probability measure $\mu_a$ on $(B_{X^*}, (\sigma(X^*, X)))$ such that*

$$\|f(a + x) - f(a)\| \leq C_a \left(\int_{B_{X^*}} |\varphi(x)|^p \, d\mu_a(\varphi)\right)^{\frac{1}{p}}$$

*for all $x \in X$.*

From the theorem above, if $f\colon X \longrightarrow Y$ is absolutely $p$-summing at every $a \in X$ we have a family of constants $(C_a)_{a \in X}$ and a family of probability measures $(\mu_a)_{a \in X}$ on $(B_{X^*}, (\sigma(X^*, X)))$ so that

$$\|f(a + x) - f(a)\| \leq C_a \left(\int_{B_{X^*}} |\varphi(x)|^p \, d\mu_a(\varphi)\right)^{\frac{1}{p}}$$

for all $x \in X$.

A natural question arises:

**Problem 1.3.** *If $f\colon X \longrightarrow Y$ is absolutely $p$-summing at every $a \in X$, does there exist an universal constant $C \geq 0$ and a Borel probability measure $\mu$ on $(B_{X^*}, (\sigma(X^*, X)))$ such that*

$$\|f(a + x) - f(a)\| \leq C \left(\int_{B_{X^*}} |\varphi(x)|^p \, d\mu(\varphi)\right)^{\frac{1}{p}} \tag{1.3}$$

*for all* $(a,x) \in X \times X$ ?

In this note, among other results, we solve partially this question by characterizing the maps satisfying (1.3).

## 2. Results

We begin this section by recalling the notion of summability at a given point and introducing some concepts related to the notion of everywhere absolutely summing multilinear operators:

**Definition 2.** *Let $X, Y$ be Banach spaces.*

*(i) A map $f : X \to Y$ is absolutely $p$-summing at $a \in X$ if there is a constant $C \geq 0$ such that*

$$\left( \sum_{j=1}^{m} \| f(a + x_j) - f(a) \|^p \right)^{\frac{1}{p}} \leq C \left\| (x_j)_{j=1}^{m} \right\|_{w,p} \tag{2.1}$$

*for all $x_1, \dots, x_m \in X$ and $m \in \mathbb{N}$.*

*(ii) A map $f : X \to Y$ is strongly absolutely $p$-summing at $A \subset X$ if there is a constant $C \geq 0$ such that*

$$\left( \sum_{j=1}^{m} \| f(a_j + x_j) - f(a_j) \|^p \right)^{\frac{1}{p}} \leq C \left\| (x_j)_{j=1}^{m} \right\|_{w,p} \tag{2.2}$$

*for all $a_1, ..., a_m \in A$, $x_1, \dots, x_m \in X$ and $m \in \mathbb{N}$.*

*(iii) When $A = X$ in (ii) $f$ is called strongly everywhere absolutely $p$-summing.*

The next theorem characterizes the maps satisfying (1.3):

**Theorem 2.1.** *A map $f : X \to Y$ is strongly absolutely $p$-summing at $A$ if and only if there are a constant $C \geq 0$ and a Borel probability measure $\mu$ on $(B_{X^*}, (\sigma(X^*, X)))$ such that*

$$\| f(a + x) - f(a) \| \leq C \left( \int_{B_{X^*}} |\varphi(x)|^p \, d\mu(\varphi) \right)^{\frac{1}{p}}$$

*for all $(x, a) \in X \times A$.*

*Proof.* Let $\mathcal{H}$ be the set of all maps from $X$ to $Y$. Now choose the parameters

$$\begin{aligned} E &= A \times X \\ G &= \mathbb{K} \\ K &= (B_{X^*}, (\sigma(X^*, X))) \,. \end{aligned}$$

Define

$$\begin{aligned} S &: \mathcal{H} \times (A \times X) \times \mathbb{K} \to [0, \infty) \\ R &: (B_{X^*}, (\sigma(X^*, X))) \times (A, X) \times \mathbb{K} \to [0, \infty) \end{aligned}$$

by

$$\begin{aligned} S(f, (a, x), b) &= \| f(a + x) - f(a) \| \\ R(\varphi, (a, x), b) &= |\varphi(x)| \,. \end{aligned}$$

Note that (2.2) is equivalent to

$$\left(\sum_{j=1}^{m} S(f,(a_j,x_j),b_j)^p\right)^{\frac{1}{p}} \leq C \sup_{\varphi\in K}\left(\sum_{j=1}^{m} R\left(\varphi,(a_j,x_j),b_j\right)^p\right)^{\frac{1}{p}},$$

for all $(a_1,x_1),\ldots,(a_m,x_m)\in E$, $b_1,\ldots,b_m\in G$ and $m\in\mathbb{N}$.

From Theorem 1.1 we have

$$S(f,(a,x),b)\leq C\left(\int_K R\left(\varphi,(a,x),b\right)^p d\mu(\varphi)\right)^{1/p}$$

for all $a\in A$, $x\in X$ and $b\in\mathbb{K}$, i.e.,

$$\|f(a+x)-f(a)\|\leq C\left(\int_K |\varphi(x)|^p\, d\mu(\varphi)\right)^{\frac{1}{p}}$$

for all $(x,a)\in X\times A$. □

**Corollary 2.2.** *A map $f:X\to Y$ is strongly everywhere absolutely $p$-summing if and only if there are a constant $C\geq 0$ and a Borel probability measure $\mu$ on $B_{X^*}$ such that*

$$\|f(a+x)-f(a)\|\leq C\left(\int_{B_{X^*}} |\varphi(x)|^p\, d\mu(\varphi)\right)^{\frac{1}{p}}$$

*for all $(x,a)\in X\times X$.*

Now we note that the general PDT allows a local version.

**Definition 3.** *Let $X,Y$ be Banach spaces.*

*(i) A map $f:X\to Y$ is locally absolutely $p$-summing at $a\in X$ if there are $C\geq 0$, $\delta>0$ such that*

$$\left(\sum_{j=1}^{m}\|f(a+x_j)-f(a)\|^p\right)^{\frac{1}{p}}\leq C\left\|(x_j)_{j=1}^m\right\|_{w,p} \tag{2.3}$$

*for every $x_1,...,x_m\in X$ so that $\|x_j\|<\delta$.*

*(ii) A map $f:X\to Y$ is locally strongly absolutely $p$-summing at $A\subset X$ if there are $C\geq 0$, $\delta>0$ such that*

$$\left(\sum_{j=1}^{m}\|f(a_j+x_j)-f(a_j)\|^p\right)^{\frac{1}{p}}\leq C\left\|(x_j)_{j=1}^m\right\|_{w,p}$$

*for every $a\in A$ and every $x_1,...,x_m,a_1,...,a_m\in X$ so that $\|x_j\|<\delta$.*

*(iii) When $A=X$ in (ii) $f$ is called locally strongly everywhere absolutely $p$-summing.*

**Theorem 2.3.** *A map $f:X\to Y$ is locally strongly absolutely $p$-summing at $A$ if and only if there are $C\geq 0$, $\delta>0$ and a Borel probability measure $\mu$ on $(B_{X^*},(\sigma(X^*,X)))$ such that*

$$\|f(a+x)-f(a)\|\leq C\left(\int_{B_{X^*}} |\varphi(x)|^p\, d\mu(\varphi)\right)^{\frac{1}{p}}$$

*for all $(x,a)\in B(0,\delta)\times A$.*

*Proof.* Let $\mathcal{H}$ be the set of all maps from $X$ to $Y$. Consider also the sets

$$E = A \times B(0,\delta)$$
$$G = \mathbb{K} \text{ and } K = B_{X^*}.$$

The proof follows the lines of the proof of Theorem 2.1. □

**Corollary 2.4.** *A map $f : X \to Y$ is locally strongly everywhere absolutely $p$-summing if and only if there are $C \geq 0$, $\delta > 0$ and a Borel probability measure $\mu$ on $B_{X^*}$ such that*

$$\|f(a+x) - f(a)\| \leq C \left( \int_{B_{X^*}} |\varphi(x)|^p \, d\mu(\varphi) \right)^{\frac{1}{p}}$$

*for all $(x,a) \in B(0,\delta) \times X$.*

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