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SOME FURTHER RESULTS ON THE UNIQUE RANGE SETS OF MEROMORPHIC FUNCTIONS

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Abstract. With the aid of the notion of weighted sharing of sets we deal with the problem of Unique Range Sets for meromorphic functions and obtain a result which improve some previous results.

1. Introduction, Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. We shall use the standard notations of value distribution theory :

$$T(r,f),\ \ m(r,f),\ \ N(r,\infty;f),\ \ \overline{N}(r,\infty;f),\ldots$$

(see [7]). It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\longrightarrow\infty, r\notin E$.

For any constant a, we define

$$\Theta(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}.$$

If for some $a\in\mathbb{C}\cup\{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S)=\bigcup_{a\in S}\{z:f(z)-a=0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S)=\bigcup_{a\in S}\{z:f(z)-a=0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S)=E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S)=\overline{E}_g(S)$, we say that f and g share the set S IM.

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As a simple application of his own value distribution theory, Nevanlinna proved that a non-constant meromorphic function is uniquely determined by the inverse image of 5 distinct values (including the infinity) IM. Gross [6] extended the study by considering pre-images of a set and posed the question:

?Is there a finite set S so that an entire function is determined uniquely by the pre-image of the set S CM??

We recall that a set S is called a unique range set for meromorphic functions (URSM) if for any pair of non-constant meromorphic functions f and g, the condition $E_f(S) = E_g(S)$ implies $f \equiv g$. Similarly a set S is called a unique range set for entire functions (URSE) if for any pair of non-constant entire functions f and g, the condition $E_f(S) = E_g(S)$ implies $f \equiv g$.

We will call any set $S \subset \mathbb{C}$ a unique range set for meromorphic functions ignoring multiplicities (URSM-IM) for which $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$ for any pair of non-constant meromorphic functions.

In the last couple of years the concept of URSE, URSM and URSM-IM have caused an increasing interest among the researchers. The study is focused mainly on two problems: finding different URSM with the number of elements as small as possible, and characterizing the URSM. e.g.,[2]-[5], [14]-[16] and [18]-[22].

A recent increment to uniqueness theory has been to considering weighted sharing instead of sharing IM/CM which implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing has been introduced by I. Lahiri around 2001 in [10, 11]. Below we are giving the following definitions:

Definition 1.1. [10, 11] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [10] *Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. Let $E_f(S, k) = \bigcup_{a\in S} E_k(a; f)$.*

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

In 2003 Y. Xu [18] proved the following theorem.

Theorem A. [18] *If f and g are two non-constant meromorphic functions and $\Theta(\infty; f) > \frac{3}{4}$, $\Theta(\infty; g) > \frac{3}{4}$, then there exists a set with seven elements such that $E_f(S, \infty) = E_g(S, \infty)$ implies $f \equiv g$.*

Dealing with the question of Yi raised in [21] Lahiri and Banerjee exhibited a unique range set S with smaller cardinalities than that obtained previously other than Xu [18], imposing some restrictions on the poles of f and g. In fact, they obtained the following result:

Theorem B. [12] *Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where n (≥ 9) be an integer and a, b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If $E_f(S, 2) = E_g(S, 2)$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$ then $f \equiv g$.*

In [2] and [4] Bartels and Fang-Guo both independently proved the existence of a URSM-IM with 17 elements.

In this paper we shall continue the study and provide better results than that obtained in [2], [4], [12], [18] at the cost of consideration of a new URS.

Suppose that the polynomial $P(w)$ is defined by

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2, \tag{1.1}$$

where $n \geq 3$ is an integer, a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 1, 2$.

In fact we consider the following rational function

$$R(w) = \frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{1.2}$$

where α_1 and α_2 are two distinct roots of

$$n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0.$$

We have from (1.2) that

$$R^{'}(w) = \frac{(n-2)aw^{n-1}\,(w-b)^2}{n(n-1)\,(w-\alpha_1)^2\,(w-\alpha_2)^2}. \tag{1.3}$$

From (1.3) we know that $w = 0$ is a root with multiplicity n of the equation $R(w) = 0$ and $w = b$ is a root with multiplicity 3 of the equation $R(w) - c = 0$, where $c = \frac{ab^{n-2}}{2} \neq \frac{1}{2}, 1$. Then

$$R(w) - c = \frac{a(w-b)^3\,Q_{n-3}(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{1.4}$$

where $Q_{n-3}(w)$ is a polynomial of degree $n-3$.

Moreover from (1.1) and (1.2) we have

$$R(w) - 1 = \frac{P(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}. \tag{1.5}$$

Noting that $c = \frac{ab^{n-2}}{2} \neq 1$, from (1.3) and (1.5) we have

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2$$

has only simple zeros.

The following theorem is the main result of the paper.

Theorem 1.1. *Let $S = \{w \mid P(w) = 0\}$, where $P(w)$ is given by (1.1), where $n(\geq 6)$ is an integer. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S,m) = E_g(S,m)$. If*

(i) *$m \geq 2$ and $\Theta_f + \Theta_g + \min\{\Theta(b;f), \Theta(b;g) > 10 - n$*

(ii) *or if $m = 1$ and $\Theta_f + \Theta_g + \min\{\Theta(b;f), \Theta(b;g)\} + \frac{1}{2}\min\{\Theta(0;f) + \Theta(\infty;f), \Theta(0;g) + \Theta(\infty;g)\} > 11 - n$*

(iii) *or if $m = 0$ and $\Theta_f + \Theta_g + \Theta(0;f) + \Theta(\infty;f) + \Theta(0;g) + \Theta(\infty;g) + \min\{\Theta(0;f) + \Theta(b;f) + \Theta(\infty;f), \Theta(0;g) + \Theta(b;g) + \Theta(\infty;g) > 16 - n$*

then $f \equiv g$, where $\Theta_f = 2\Theta(0;f) + 2\,\Theta(\infty;f) + \Theta(b;f)$ and Θ_g can be similarly defined.

Corollary 1.1. *In* Theorem 1.1 *when* $m = 2$ *and* $n \geq 7$ *and* $n \geq 9$ *it is the improvements of the results of Y. Xu* [18] *and Lahiri-Banerjee* [12] *respectively. On the other hand when* $m = 0$ *and* $n \geq 17$ *it is an improvement of the results of Bartels* [2] *and Fang-Guo* [4].

It is assumed that the readers are familiar with the standard definitions and notations of the value distribution theory as those are available in [7]. Throughout this paper, we also need the following definitions:

Definition 1.3. [9] *For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $N(r, a; f \mid= 1)$ *the counting function of simple a-points of* f. *For a positive integer* m *we denote by* $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ *the counting function of those a-points of* f *whose multiplicities are not greater(less) than* m *where each a-point is counted according to its multiplicity.*

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ *are defined similarly, where in counting the a-points of* f *we ignore the multiplicities.*

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ *are defined analogously.*

Definition 1.4. *Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share* $(a, 0)$. *Let* z_0 *be an a-point of* f *with multiplicity* p, *an a-point of* g *with multiplicity* q. *We denote by* $\overline{N}_L(r, a; f)$ *the reduced counting function of those a-points of* f *and* g *where* $p > q$, *by* $N_E^{1)}(r, a; f)$ *the counting function of those a-points of* f *and* g *where* $p = q = 1$, *by* $\overline{N}_E^{(2}(r, a; f)$ *the reduced counting function of those a-points of* f *and* g *where* $p = q \geq 2$. *In the same way we can define* $\overline{N}_L(r, a; g)$, $N_E^{1)}(r, a; g)$, $\overline{N}_E^{(2}(r, a; g)$. *In a similar manner we can define* $\overline{N}_L(r, a; f)$ *and* $\overline{N}_L(r, a; g)$ *for* $a \in \mathbb{C} \cup \{\infty\}$. *When* f *and* g *share* (a, m), $m \geq 1$ *then* $N_E^{1)}(r, a; f) = N(r, a; f \mid= 1)$.

Definition 1.5. [10, 11] *Let* f, g *share* $(a, 0)$. *We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those a-points of* f *whose multiplicities differ from the multiplicities of the corresponding a-points of* g.

Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F^{''}}{F^{'}} - \frac{2F^{'}}{F-1}\right) - \left(\frac{G^{''}}{G^{'}} - \frac{2G^{'}}{G-1}\right).$$

Let f and g be two non-constant meromorphic function and

$$F = R(f), \qquad G = R(g), \tag{2.1}$$

where $R(w)$ is given as (1.2). From (1.2) and (2.1) it is clear that

$$T(r, f) = \frac{1}{n}T(r, F) + S(r, f), \quad T(r, g) = \frac{1}{n}T(r, G) + S(r, g) \tag{2.2}$$

Lemma 2.1. *Let F, G be given by (2.1) and $H \not\equiv 0$. Suppose that F, G share $(1,m)$, where $m \geq 0$ is an integer. Then*

$$\begin{aligned} N_E^{1)}(r,1;F) \leq\ & \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f) \\ & + \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \overline{N}(r.b;g) + \overline{N}_0(r,0;f^{'}) + \overline{N}_0(r,0;g^{'}), \\ & + S(r,f) + S(r,g) \end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ denotes the reduced counting function corresponding to the zeros of $f^{'}$ which are not the zeros of $f(f-b)$ and $F-1$, $\overline{N}_0(r,0;g^{'})$ is defined similarly.

Proof. We omit the proof since it can be carried out in the line of the proof of *Lemma 2.18* [1]. □

Lemma 2.2. [13] *If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\geq k) + S(r,f).$$

Lemma 2.3. [17] *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r,P(f)) = nT(r,f) + O(1)$.*

Lemma 2.4. *([22], Lemma 6) If $H \equiv 0$, then F, G share $(1,\infty)$. If further F, G share $(\infty,0)$ then F, G share (∞,∞).*

Lemma 2.5. [1] *Let F and G be given by (2.1). If F, G share $(1,m)$, where $0 \leq m < \infty$. Then*

$$(i)\overline{N}_L(r,1;F) \leq \frac{1}{m+1}[\overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\otimes}(r,0;f^{'})] + S(r,f)$$

$$(ii)\overline{N}_L(r,1;G) \leq \frac{1}{m+1}[\overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_{\otimes}(r,0;g^{'})] + S(r,g),$$

$N_{\otimes}(r,0;f^{'}) = N(r,0;f^{'} \mid f \neq 0, \omega_1, \omega_2 \ldots \omega_n)$ and $N_{\otimes}(r,0;g^{'})$ is defined similarly, where ω_i $i = 1,2,\ldots,n$ are the distinct roots of the equation $P(w) = 0$.

Lemma 2.6. *Let f, g be two non-constant meromorphic functions and suppose α_1 and α_2 are two distinct roots of the equation $n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0$. Then*

$$\frac{f^n}{(f-\alpha_1)(f-\alpha_2)}\ \frac{g^n}{(g-\alpha_1)(g-\alpha_2)} \not\equiv \frac{n^2(n-1)^2}{a^2},$$

where $n\ (\geq 5)$ is an integer.

Proof. Suppose $FG \equiv 1$. Let z_0 be a pole of f with multiplicity p. Then clearly z_0 is a zero of g with multiplicity q such that $(n-2)p = nq$ that is $q = \frac{(n-2)(p-q)}{2} \geq \frac{n-2}{2}$ and hence $p = \frac{qn}{(n-2)} \geq \frac{n}{2}$. Also it is clear that the zeros of $(f-\alpha_1)$ and $(f-\alpha_2)$ are of multiplicities at least n. Therefore, by the second fundamental theorem we obtain

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + S(r,f) \\ &\leq \frac{2}{n}\overline{N}(r,\infty;f) + \frac{1}{n}\overline{N}(r,\alpha_1;f) + \frac{1}{n}\overline{N}(r,\alpha_2;f) + S(r,f) \\ &\leq \frac{4}{n}T(r,f) + +S(r,f), \end{aligned}$$

which leads to a contradiction for $n \geq 5$. This proves the lemma. □

Lemma 2.7. *Let F, G be given by (2.1), where $n \geq 6$ is an integer. If $F \equiv G$, then $f \equiv g$.*

Proof. We omit the proof since the proof can be found out in [8]. □

Lemma 2.8. *Let F, G be given by (2.1). Also let S be given as in* Theorem 1.1, *where $n \geq 3$ is an integer. If $E_f(S,0) = E_g(S,0)$ then $S(r,f) = S(r,g)$.*

Proof. Since $E_f(S,0) = E_g(S,0)$, it follows that F and G share $(1,0)$. We denote the distinct elements of S by w_j, $j = 1, 2, \ldots n$. Since F, G share $(1,0)$ from the second fundamental theorem we have

$$\begin{aligned}
(n-2)T(r,g) &\leq \sum_{j=1}^{n} \overline{N}\left(r, w_j; g\right) + S(r,g) \\
&= \sum_{j=1}^{n} \overline{N}\left(r, w_j; f\right) + S(r,g) \\
&\leq nT(r,f) + S(r,g).
\end{aligned}$$

Similarly we can deduce

$$(n-2)T(r,f) \leq nT(r,g) + S(r,f).$$

The last inequalities imply $T(r,f) = O\left(T(r,g)\right)$ and $T(r,g) = O\left(T(r,f)\right)$ and so we have $S(r,f) = S(r,g)$. □

3. Proofs of the theorem

Proof of Theorem 1.1. Let F, G be given by (2.1). Since $E_f(S,m) = E_g(S,m)$, it follows that F, G share $(1,m)$.

Case 1. Suppose that $H \not\equiv 0$.

Subcase 1.1. $m \geq 1$. While $m \geq 2$, using *Lemma 2.2* we note that

$$\begin{aligned}
&\overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_*(r,1;F,G) \qquad (3.1)\\
\leq\ &\overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}(r,1;G \mid\geq 3) \\
\leq\ &\overline{N}_0(r,0;g^{'}) + \sum_{j=1}^{n}\{\overline{N}(r,\omega_j;g \mid= 2) + 2\overline{N}(r,\omega_j;g \mid\geq 3)\} \\
\leq\ &N(r,0;g^{'} \mid g \neq 0) + S(r,g) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,g).
\end{aligned}$$

Hence using (3.1), *Lemmas 2.1* and *2.3* we get from second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned} & (n+1)\, T(r,f) \qquad (3.2) \\ \leq\ & \overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+N(r,1;F\mid=1)+\overline{N}(r,1;F\mid\geq 2) \\ & -N_0(r,0;f^{'})+S(r,f) \\ \leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)\right\}+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,\infty;g) \\ & +\overline{N}(r,1;G\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g) \\ \leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\right\}+\overline{N}(r,b;g) \\ & +S(r,f)+S(r,g) \\ \leq\ & (11-2\Theta(0;f)-2\Theta(0;g)-2\Theta(\infty;f)-2\Theta(\infty;g)-2\Theta(b;f)-\Theta(b;g) \\ & +\varepsilon)\, T(r)+S(r). \end{aligned}$$

In a similar way we can obtain

$$\begin{aligned} & (n+1)\, T(r,g) \qquad (3.3) \\ \leq\ & (11-2\Theta(0;f)-2\Theta(0;g)-2\Theta(\infty;f)-2\Theta(\infty;g)-\Theta(b;f)-2\Theta(b;g) \\ & +\varepsilon)\, T(r)+S(r). \end{aligned}$$

Combining (3.2) and (3.3) we see that

$$\begin{aligned} & (n-10+2\Theta(0;f)+2\Theta(\infty;f)+\Theta(b;f)+2\Theta(0;g)+2\Theta(\infty;g) \qquad (3.4) \\ & +\Theta(b;g)+\min\{\Theta(b;f),\Theta(b;g)\}-\varepsilon\,)\, T(r)\leq S(r). \end{aligned}$$

Since $\varepsilon > 0$, (3.4) leads to a contradiction.
While $m = 1$, using *Lemma 2.5*, (3.1) changes to

$$\begin{aligned} & \overline{N}_0(r,0;g^{'})+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_*(r,1;F,G) \qquad (3.5) \\ \leq\ & \overline{N}_0(r,0;g^{'})+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;F) \\ \leq\ & \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\frac{1}{2}\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\}+S(r,f)+S(r,g). \end{aligned}$$

So using (3.5), *Lemmas 2.1* and *2.3* and proceeding as in (3.2) we get from second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned} & (n+1)\, T(r,f) \qquad (3.6) \\ \leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)\right\}+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,\infty;g) \\ & +\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\frac{1}{2}\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\}+S(r,f)+S(r,g) \\ \leq\ & \left\{\frac{5}{2}\overline{N}(r,0;f)+2\overline{N}(r,b;f)+\frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,0;g)+2\overline{N}(r,\infty;g)\right\} \\ & +\overline{N}(r,b;g)+S(r,f)+S(r,g) \\ \leq\ & \Bigg(12-\frac{5}{2}\Theta(0;f)-2\Theta(0;g)-\frac{5}{2}\Theta(\infty;f)-2\Theta(\infty;g)-2\Theta(b;f)-\Theta(b;g) \\ & +\varepsilon)\, T(r)+S(r). \end{aligned}$$

Similarly we can obtain

$$\begin{aligned}&(n+1)\,T(r,g) &(3.7)\\ \le\ &\Big(12-2\Theta(0;f)-\frac{5}{2}\Theta(0;g)-2\Theta(\infty;f)-\frac{5}{2}\Theta(\infty;g)-\Theta(b;f)\\ &-2\Theta(b;g)+\varepsilon\Big)\,T(r)+S(r).\end{aligned}$$

Combining (3.6) and (3.7) we see that

$$\begin{aligned}&(n-11+2\Theta(0;f)+2\Theta(\infty;f)+\Theta(b;f)+2\Theta(0;g)+2\Theta(\infty;g) &(3.8)\\ &+\Theta(b;g)+\min\{\Theta(b;f),\Theta(b;g)\}+\frac{1}{2}\min\{\Theta(0;f)+\Theta(\infty;f),\Theta(0;g)\\ &+\Theta(\infty;g)\}-\varepsilon\,)\;T(r)\le S(r).\end{aligned}$$

Since $\varepsilon>0$, (3.8) leads to a contradiction.

Subcase 1.2. $m=0$. Using *Lemma 2.5* we note that

$$\begin{aligned}&\overline{N}_0(r,0;g^{'})+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F) &(3.9)\\ \le\ &\overline{N}_0(r,0;g^{'})+\overline{N}_E^{(2}(r,1;G)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\ \le\ &\overline{N}_0(r,0;g^{'})+\overline{N}(r,1;G\mid\ge 2)+\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\ \le\ &N(r,0;g^{'}\mid g\ne 0)+\overline{N}(r,1;G\mid\ge 2)+2\overline{N}(r,1;F\mid\ge 2)\\ \le\ &2\{N(r,0;g^{'}\mid g\ne 0)+N(r,0;f^{'}\mid f\ne 0)\}\\ \le\ &2\{\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\}+S(r,f)+S(r,g).\end{aligned}$$

Hence using (3.9), *Lemmas 2.1* and *2.3* we get from second fundamental theorem for $\varepsilon>0$ that

$$\begin{aligned}&(n+1)\,T(r,f) &(3.10)\\ \le\ &\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)\\ &+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)-N_0(r,0;f^{'})+S(r,f)\\ \le\ &2\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,b;f)\right\}+\overline{N}(r,0;g)+\overline{N}(r,b;g)\\ &+\overline{N}(r,\infty;g)+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)+\overline{N}_0(r,0;g^{'})\\ &+S(r,f)+S(r,g)\\ \le\ &4\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right\}+3\{\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\}+2\overline{N}(r,b;f)\\ &+\overline{N}(r,b;g)+S(r,f)+S(r,g)\\ \le\ &(17-4\Theta(0;f)-4\Theta(\infty;f)-3\Theta(0;g)-3\Theta(\infty;g)-2\Theta(b;f)-\Theta(b;g)\\ &+\varepsilon)\,T(r)+S(r).\end{aligned}$$

In a similar manner we can obtain

$$\begin{aligned}&(n+1)\,T(r,g) &(3.11)\\ \le\ &(17-3\Theta(0;f)-3\Theta(\infty;f)-4\Theta(0;g)-4\Theta(\infty;g)-\Theta(b;f)\\ &-2\Theta(b;g)+\varepsilon)\,T(r)+S(r).\end{aligned}$$

Combining (3.10) and (3.11) we see that

$$\begin{aligned}&(n-16+3\Theta(0;f)+3\Theta(\infty;f)+\Theta(b;f)+3\Theta(0;g)+3\Theta(\infty;g)\\&+\Theta(b;g)+\min\{\Theta(0;f)+\Theta(b;f)+\Theta(\infty;f),\Theta(0;g)+\Theta(b;g)\\&+\Theta(\infty;g)\}-\varepsilon)\,T(r)\le S(r).\end{aligned}\tag{3.12}$$

Since $\varepsilon>0$, (3.12) leads to a contradiction.
Case 2. Suppose that $H\equiv 0$. Then

$$F\equiv\frac{AG+B}{CG+D},\tag{3.13}$$

where A, B, C, D are constants such that $AD-BC\neq 0$. Also

$$T(r,F)=T(r,G)+O(1),$$

and hence from *Lemma 2.3* we have

$$T(r,f)=T(r,g)+O(1).\tag{3.14}$$

From (1.4) we note that $\overline{N}(r,c;F)\le\overline{N}(r,b;f)+(n-3)T(r,f)\le(n-2)T(r,f)+S(r,f)$. Similarly $\overline{N}(r,c;G)\le(n-2)T(r,g)+S(r,g)$. We also note from *Lemma 2.4* that F and G share $(1,\infty)$. We now consider the following cases.
Subcase 2.1. Let $AC\neq 0$. Suppose $B\neq 0$. From (3.13) we get

$$\overline{N}\left(r,-\frac{B}{A};G\right)=\overline{N}(r,0;F).\tag{3.15}$$

In view of (3.14), (3.15), *Lemma 2.3* and the second fundamental theorem we get

$$\begin{aligned}nT(r,g)&\le\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}\left(r,-\frac{B}{A};G\right)+S(r,G)\\&\le\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,\alpha_1;g)+\overline{N}(r,\alpha_2;g)+\overline{N}(r,0;f)+S(r,g)\\&\le 4T(r,g)+T(r,f)+S(r,g)\le 5T(r,g)+S(r,g),\end{aligned}$$

which is a contradiction for $n\ge 6$. So we must have $B=0$ and in this case (3.13) changes to

$$F\equiv\frac{\frac{A}{C}\,G}{G+\frac{D}{C}}.\tag{3.16}$$

From (3.16) we see that

$$\overline{N}(r,\infty;F)=\overline{N}\left(r,-\frac{D}{C};G\right).\tag{3.17}$$

Suppose $c\neq-\frac{D}{C}$. So in view of (3.14), (3.17), *Lemma 2.3* and the second fundamental theorem we obtain

$$\begin{aligned}2nT(r,g)&\le\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}\left(r,-\frac{D}{C};G\right)+\overline{N}(r,c;G)+S(r,G)\\&\le\overline{N}(r,0;g)+3T(r,g)+3T(r,f)+(n-2)T(r,g)+S(r,g)\\&\le(n+5)T(r,g)+S(r,g),\end{aligned}$$

which implies a contradiction for $n\ge 6$.
Now suppose $c=-\frac{D}{C}$. Since F and G share $(1,\infty)$, from (3.16) we get $1=\frac{\frac{A}{C}}{1-c}$,

that is $\frac{A}{C} = 1 - c$. Consequently from (3.16) we get

$$G \equiv \frac{c\,F}{F - (1-c)}. \tag{3.18}$$

Clearly $c \neq 1 - c$, since according to the statement of the theorem $c \neq \frac{1}{2}$. So from the second fundamental theorem, (3.14), (3.18) and *Lemma 2.3* we see that

$$\begin{aligned}
2nT(r,f) &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}\,(r,1-c;F) + \overline{N}(r,c;F) + S(r,F)\\
&\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,\infty;g)\\
&\quad + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + (n-2)T(r,f) + S(r,f)\\
&\leq (n+5)T(r,f) + S(r,f),
\end{aligned}$$

which leads to a contradiction for $n \geq 6$.

Subcase 2.2. Let $A \neq 0$ and $C = 0$. Then $F = \alpha G + \beta$, where $\alpha = \frac{A}{D}$ and $\beta = \frac{B}{D}$.

If F has no 1-point, by the second fundamental theorem and *Lemma 2.3* we get

$$\begin{aligned}
nT(r,f) &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + S(r,f)\\
&\leq 4T(r,f) + S(r,f),
\end{aligned}$$

which implies a contradiction for $n \geq 6$.

If F and G have some 1-points then $\alpha + \beta = 1$ and so

$$F \equiv \alpha G + 1 - \alpha. \tag{3.19}$$

Suppose $\alpha \neq 1$. If $1 - \alpha \neq c$ then in view of (3.14), *Lemma 2.3* and the second fundamental theorem we get

$$\begin{aligned}
2nT(r,f) &\leq \overline{N}(r,0;F) + \overline{N}\,(r,c;F) + \overline{N}(r,1-\alpha;F) + \overline{N}(r,\infty;F) + S(r,F)\\
&\leq (n+2)T(r,f) + \overline{N}(r,0;G) + S(r,f)\\
&\leq (n+3)T(r,f) + S(r,f),
\end{aligned}$$

which implies a contradiction for $n \geq 6$. If $1 - \alpha = c$, then we have from (3.19)

$$F \equiv (1-c)G + c.$$

Since $c \neq 1$, by the second fundamental theorem we can obtain using (3.14) and *Lemma 2.3* that

$$\begin{aligned}
2nT(r,g) &\leq \overline{N}(r,0;G) + \overline{N}(r,c;G) + \overline{N}\left(r, \frac{c}{c-1}; G\right) + \overline{N}(r,\infty;G) + S(r,G)\\
&\leq (n+2)T(r,g) + \overline{N}(r,0;F) + S(r,g)\\
&\leq (n+3)T(r,g) + S(r,g),
\end{aligned}$$

which implies a contradiction since $n \geq 6$.

So $\alpha = 1$ and hence $F \equiv G$. So by *Lemma 2.7* we get $f \equiv g$.

Subcase 2.3. Let $A = 0$ and $C \neq 0$. Then $F \equiv \frac{1}{\gamma G + \delta}$, where $\gamma = \frac{C}{B}$ and $\delta = \frac{D}{B}$.

If F has no 1-point then as in *Subcase 2.2* we can deduce a contradiction.

If F and G have some 1-points then $\gamma + \delta = 1$ and so

$$F \equiv \frac{1}{\gamma G + 1 - \gamma}. \tag{3.20}$$

Suppose $\gamma \neq 1$. If $\frac{1}{1-\gamma} \neq c$, then by the second fundamental theorem, (3.14) and

Lemma 2.3 we get

$$\begin{aligned} 2nT(r,f) &\le \overline{N}(r,0;F)+\overline{N}(r,\frac{1}{1-\gamma};F)+\overline{N}(r,c;F)+\overline{N}(r,\infty;F)+S(r,f) \\ &\le (n+2)T(r,f)+\overline{N}(r,0;G)+S(r,f) \\ &\le (n+3)T(r,f)+S(r,f), \end{aligned}$$

which gives a contradiction for $n \ge 6$. If $\frac{1}{1-\gamma}=c$, from (3.20) we have

$$F \equiv \frac{c}{(c-1)\,G+1}. \tag{3.21}$$

If $c \neq \frac{1}{1-c}$, the second fundamental theorem with the help of (3.14), (3.21) and *Lemma 2.3* yields

$$\begin{aligned} 2nT(r,g) &\le \overline{N}(r,0;G)+\overline{N}(r,c;G)+\overline{N}\left(r,\frac{1}{1-c};G\right)+\overline{N}(r,\infty;G)+S(r,G) \\ &\le (n+2)T(r,g)+\overline{N}(r,\infty;F)+S(r,g) \\ &\le (n+5)T(r,g)+S(r,g), \end{aligned}$$

which implies a contradiction since $n \ge 6$. On the other hand if $c=\frac{1}{1-c}$ then from (3.21) we have

$$G \equiv \frac{c(F-c)}{F}.$$

So from the second fundamental theorem and (3.14) it follows that

$$\begin{aligned} nT(r,f) &\le \overline{N}(r,0;F)+\overline{N}(r,c;F)+\overline{N}(r,\infty;F)+S(r,F) \\ &\le 4T(r,f)+\overline{N}(r,0;G)+S(r,f) \\ &\le 5T(r,f)+S(r,f), \end{aligned}$$

which implies a contradiction since $n \ge 6$. So we must have $\gamma=1$ then $FG \equiv 1$, which is impossible by *Lemma 2.6*. This completes the proof of the theorem. □

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References

[1] A. Banerjee, *On Uniqueness Of Meromorphic Functions That Share Two Sets*, Georgian Math. J. **15 (1)** (2008) 21-38.

[2] S.Bartels, *Meromorphic functions sharing a set with 17 elements ignoring multiplicities*, Complex Var. Theory Appl. **39** (1998), 85-92.

[3] M.Fang and X.H.Hua, *Meromorphic functions that share one finite set CM*, J. Nanjing Univ. Math. Biquart. **1** (1998), 15-22.

[4] M.Fang and H.Guo, *On unique range sets for meromorphic or entire functions*, Acta Math. Sinica (New Ser.) **14(4)** (1998), 569-576.

[5] G.Frank and M. Reinders, *A unique range set for meromorphic functions with 11 elements*, Complex Var. Theory Appl., **37** (1998), 185-193.

[6] F.Gross, *Factorization of meromorphic functions and some open problems*, Proc. Conf. Univ. Kentucky, Leixngton, Ky (1976); Lecture Notes in Math., **599** (1977), 51-69, Springer(Berlin).

[7] W.K.Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[8] Q.Han and H.X.Yi, *Some further results on meromorphic functions that share two sets*, Ann. Polon. Math. **93(1)** (2008), 17-31.

[9] I.Lahiri, *Value distribution of certain differential polynomials*, Int. J. Math. Math. Sci. **28 (2)** (2001), 83-91.

[10] I.Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J. **161** (2001), 193-206.
[11] I.Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl. **46** (2001), 241-253.
[12] I.Lahiri and A.Banerjee, *Uniqueness of meromorphic functions with deficient poles*, Kyungpook Math. J. **44** (2004), 575-584.
[13] I.Lahiri and S.Dewan, *Value distribution of the product of a meromorphic function and its derivative*, Kodai Math. J. **26** (2003), 95-100.
[14] P. Li and C.C. Yang, *Some further results on the unique range sets of meromorphic functions*, Kodai Math. J. **18** (1995), 437-450.
[15] P. Li and C.C. Yang, *On the unique range sets for meromorphic functions*, Proc. Amer. Math. Soc. **124** (1996), 177-185.
[16] E. Mues and M. Reinders, *Meromorphic functions sharing one value and unique range sets*, Kodai Muth. J. **18** (1995), 515-522.
[17] C.C.Yang, *On deficiencies of differential polynomials II*, Math. Z. **125** (1972), 107-112.
[18] Y. Xu, *Meromorphic functions sharing one finite set*, Comp. Math. Appl. **45** (2003), 1489-1495.
[19] H.X.Yi, *On a question of Gross*, Sci. China (Ser.A) **38** (1995), 8-16.
[20] H.X.Yi, *A question of Gross and the uniqueness of entire functions*, Nagoya Math. J. **138** (1995) 169 - 177.
[21] H.X.Yi, *Unicity theorems for meromorphic or entire functions III*, Bull. Austral. Math. Soc. **53** (1996), 71-82.
[22] H.X.Yi, *Meromorphic functions that share one or two values II*, Kodai Math. J. **22** (1999), 264-272.

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