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INCLUSION PROPERTIES FOR CERTAIN K-UNIFORMLY SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH CERTAIN INTEGRAL OPERATOR

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Abstract. In this paper, we introduce several new k-uniformly classes of analytic functions defined by using the integral operator and investigate various inclusion relationships for these classes. Some interesting applications involving certain classes of integral operators are also considered.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbf{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic in $\mathbf{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function ω, analytic in $\mathbf{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbf{U})$, such that $f(z) = g(\omega(z))$ $(z \in \mathbf{U})$. In particular, if the function g is univalent in $\mathbf{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbf{U}) \subset g(\mathbf{U})$ (see [9] and [10]). For $0 \leq \gamma, \beta < 1$, we denote by $S^*(\gamma)$, $C(\gamma)$, $K(\gamma, \beta)$ and $K^*(\gamma, \beta)$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike of order γ, convex of order γ, close-to-convex of order γ, and type β and quasi-convex of order γ, and type β in $\mathbf{U}$.

Now, we introduce the subclasses $US^*(k;\gamma)$, $UC(k;\gamma)$, $UK(k;\gamma,\beta)$ and $UK^*(k;\gamma,\beta)$ of the class $\mathcal{A}$ for $0 \leq \gamma, \beta < 1$, and $k \geq 0$, which are defined by

$$US^*(k;\gamma) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)} - \gamma \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\}, \tag{1.2}$$

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$$UC(k;\gamma)=\left\{f\in\mathcal{A}:\Re\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right)>k\left|\frac{zf''(z)}{f'(z)}\right|+\gamma\right\},\tag{1.3}$$

$$UK(k;\gamma,\beta)=\left\{f\in\mathcal{A}:\exists\ g\in US^{*}(k;\beta)\ \ s.t.\ \Re\left(\frac{zf'(z)}{g(z)}-\gamma\right)>k\left|\frac{zf'(z)}{g(z)}-1\right|\right\},\tag{1.4}$$

$$UK^{*}(k;\gamma,\beta)=\left\{f\in\mathcal{A}:\exists\ g\in UC(k;\gamma)\ \ s.t.\Re\left(\frac{\left(zf'(z)\right)'}{g'(z)}-\gamma\right)>k\left|\frac{\left(zf'(z)\right)'}{g'(z)}-1\right|\right\}.\tag{1.5}$$

We note that

$$US^{*}(0;\gamma)=S^{*}(k;\gamma),\ UC(0;\gamma)=C(\gamma),$$
$$UK(0;\gamma,\beta)=K(\gamma,\beta),\ UK^{*}(0;\gamma,\beta)=K^{*}(\gamma,\beta)\quad(0\le\gamma,\beta<1).$$

Corresponding to a conic domain $\Omega_{k,\gamma}$ defined by

$$\Omega_{k,\gamma}=\left\{u+iv:u>k\sqrt{(u-1)^{2}+v^{2}}+\gamma\right\},\tag{1.6}$$

we define the function $q_{k,\gamma}(z)$ which maps $\mathbf{U}$ onto the conic domain $\Omega_{k,\gamma}$ such that $1\in\Omega_{k,\gamma}$ as the following:

$$q_{k,\gamma}(z)=\begin{cases}\frac{1+(1-2\gamma)z}{1-z} & (k=0),\\ \frac{1-\gamma}{1-k^{2}}\cos\left\{\frac{2}{\pi}\left(\cos^{-1}k\right)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\frac{k^{2}-\gamma}{1-k^{2}} & (0<k<1),\\ 1+\frac{2(1-\gamma)}{\pi^{2}}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{2} & (k=1),\\ \frac{1-\gamma}{k^{2}-1}\sin\left\{\frac{\pi}{2\zeta(k)}\int_{0}^{\frac{u(z)}{\sqrt{k}}}\frac{dt}{\sqrt{1-t^{2}}\sqrt{1-k^{2}t^{2}}}\right\}+\frac{k^{2}-\gamma}{k^{2}-1} & (k>1).\end{cases}\tag{1.7}$$

where $u(z)=\dfrac{z-\sqrt{k}}{1-\sqrt{k}z}$ and $\zeta(k)$ is such that $k=\cosh\dfrac{\pi\zeta'(z)}{4\zeta(z)}$. By virture of the properties of the conic domain $\Omega_{k,\gamma}$, we have

$$\Re\left\{q_{k,\gamma}(z)\right\}>\frac{k+\gamma}{k+1}.\tag{1.8}$$

Making use of the principal of subordination and the definition of $q_{k,\gamma}(z)$, we may rewrite the subclasses $US^{*}(k;\gamma)$, $UC(k;\gamma)$, $UK(k;\gamma,\beta)$ and $UK^{*}(k;\gamma,\beta)$ as the following:

$$US^{*}(k;\gamma)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec q_{k,\gamma}(z)\right\},\tag{1.9}$$

$$UC(k;\gamma)=\left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec q_{k,\gamma}(z)\right\},\tag{1.10}$$

$$UK(k;\gamma,\beta)=\left\{f\in\mathcal{A}:\exists\ g\in US^{*}(k;\beta)\ \ s.t.\ \frac{zf'(z)}{g(z)}\prec q_{k,\gamma}(z)\right\},\tag{1.11}$$

$$UK^{*}(k;\gamma,\beta)=\left\{f\in\mathcal{A}:\exists\ g\in UC(k;\gamma)\ \ s.t.\frac{\left(zf'(z)\right)'}{g'(z)}\prec q_{k,\gamma}(z)\right\}.\tag{1.12}$$

Recently, Komatu [5] introduced a certain integral operator $\mathcal{L}_a^\lambda : \mathcal{A} \to \mathcal{A}$ $(a > 0; \lambda \geq 0)$ as follows:

$$\mathcal{L}_a^0 f(z) = f(z) \quad (a > 0; \lambda = 0) \tag{1.13}$$

and

$$\mathcal{L}_a^\lambda f(z) = \frac{(1+a)^\lambda}{\Gamma(\lambda)} \int_0^1 t^{a-1} \left(\log \frac{1}{t}\right)^{\lambda-1} f(tz)\ dt \quad (a > 0; \lambda > 0). \tag{1.14}$$

Thus, if $f \in \mathcal{A}$ is of the form (1.1), it is easily seen from (1.13) and (1.14) that

$$\mathcal{L}_a^\lambda f(z) = z + \sum_{n=2}^{\infty} \left(\frac{a+1}{a+n}\right)^\lambda a_n z^n \quad (a > 0; \lambda \geq 0). \tag{1.15}$$

It is easily to deduce from (1.15) that

$$z\left(\mathcal{L}_a^{\lambda+1} f(z)\right)' = (a+1)\, \mathcal{L}_a^\lambda f(z) - a\mathcal{L}_a^{\lambda+1} f(z). \tag{1.16}$$

The special case $a = 1$ of the inegral operator $\mathcal{L}_a^\lambda$ is essentially the operator which considered by Jung et al. [4].

Next, by using the operator $\mathcal{L}_a^\lambda$, we introduce the following classes of analytic functions for $a > 0, \lambda \geq 0$, $k \geq 0$ and $0 \leq \gamma, \beta < 1$:

$$US^*(\lambda; k; \gamma) = \left\{f \in \mathcal{A} : \mathcal{L}_a^\lambda f(z) \in US^*(k; \gamma)\right\}, \tag{1.17}$$

$$UC(\lambda; k; \gamma) = \left\{f \in \mathcal{A} : \mathcal{L}_a^\lambda f(z) \in UC(k; \gamma)\right\}, \tag{1.18}$$

$$UK(\lambda; k; \gamma, \beta) = \left\{f \in \mathcal{A} : \mathcal{L}_a^\lambda f(z) \in UK(k; \gamma, \beta)\right\}, \tag{1.19}$$

$$UK^*(\lambda; k; \gamma, \beta) = \left\{f \in \mathcal{A} : \mathcal{L}_a^\lambda f(z) \in UK^*(k; \gamma, \beta)\right\}. \tag{1.20}$$

We also note that

$$f(z) \in US^*(\lambda; k; \gamma) \Leftrightarrow zf'(z) \in UC(\lambda; k; \gamma), \tag{1.21}$$

and

$$f(z) \in UK(\lambda; k; \gamma, \beta) \Leftrightarrow zf'(z) \in UK^*(\lambda; k; \gamma, \beta). \tag{1.22}$$

In this paper, we investgate several inclusion properties of the classes $US^*(\lambda; k; \gamma)$, $UC(\lambda; k; \gamma)$, $UK(\lambda; k; \gamma, \beta)$ and $UK^*(\lambda; k; \gamma, \beta)$ associated with the operator $\mathcal{L}_a^\lambda$. Some applications involving integral operators are also considered.

2. Inclusion Properties Involving the Operator $\mathcal{L}_a^\lambda$

In order to prove the main results, we shall need The following lemmas.

Lemma 1[3]. *Let $h(z)$ be convex univalent in* $\mathbf{U}$ *with $h(0) = 1$ and $\Re\{\eta h(z) + \gamma\} > 0$ $(\eta, \gamma \in \mathbb{C})$. If $p(z)$ is analytic in* $\mathbf{U}$ *with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\eta p(z) + \gamma} \prec h(z) \tag{2.1}$$

implies

$$p(z) \prec h(z)\ . \tag{2.2}$$

Lemma 2[8]. *Let $h(z)$ be convex univalent in* $\mathbf{U}$ *and let w be analytic in* $\mathbf{U}$ *with $\Re\{w(z)\} \geq 0$. If $p(z)$ is analytic in* $\mathbf{U}$ *and $p(0) = h(0)$, then*

$$p(z) + w(z)\, zp'(z) \prec h(z) \tag{2.3}$$

implies

$$p(z) \prec h(z). \tag{2.4}$$

Theorem 1. $US^*(\lambda;k;\gamma) \subset US^*(\lambda+1;k;\gamma)$.

Proof. Let $f \in US^*(\lambda;k;\gamma)$ and set

$$p(z) = \frac{z\left(\mathcal{L}_a^{\lambda+1} f(z)\right)'}{\mathcal{L}_a^{\lambda+1} f(z)} \quad (z \in \mathbf{U}), \tag{2.5}$$

where $p(z)$ is analytic in $\mathbf{U}$ with $p(0) = 1$. From (1.16) and (2.5), we have

$$\frac{\mathcal{L}_a^{\lambda} f(z)}{\mathcal{L}_a^{\lambda+1} f(z)} = \frac{1}{a+1}\{p(z) + a\}. \tag{2.6}$$

Differentiating (2.6) with respect to z and multiplying the result equation by z, we obtain

$$\frac{z\left(\mathcal{L}_a^{\lambda} f(z)\right)'}{\mathcal{L}_a^{\lambda} f(z)} = p(z) + \frac{zp'(z)}{p(z)+a}. \tag{2.7}$$

From this and the argument given in Section 1, we may write

$$p(z) + \frac{zp'(z)}{p(z)+a} \prec q_{k,\gamma}(z) \quad (z \in \mathbf{U}). \tag{2.8}$$

Since $a > 0$ and $\Re\{q_{k,\gamma}(z)\} > \dfrac{k+\gamma}{k+1}$, we see that

$$\Re\{q_{k,\gamma}(z) + a\} > 0 \quad (z \in \mathbf{U}). \tag{2.9}$$

Applying Lemma 1 to (2.8), it follows that $p(z) \prec q_{k,\gamma}(z)$, that is, $f \in US^*(\lambda+1;k;\gamma)$. ∎

Theorem 2. $UC(\lambda;k;\gamma) \subset UC(\lambda+1;k;\gamma)$.

Proof. Applying (1.21) and Theorem 1, we observe that

$$\begin{aligned} f(z) \quad \in \quad & UC(\lambda;k;\gamma) \Longleftrightarrow zf'(z) \in US^*(\lambda;k;\gamma) \\ \Longrightarrow \quad & zf'(z) \in US^*(\lambda+1;k;\gamma) \\ \Longleftrightarrow \quad & f(z) \in UC(\lambda+1;k;\gamma), \end{aligned}$$

which evidently proves Theorem 2. ∎

Theorem 3. $UK(\lambda;k;\gamma,\beta) \subset UK(\lambda+1;k;\gamma,\beta)$.

Proof. Let $f \in UK(\lambda;k;\gamma,\beta)$. Then, from the definition of $UK(\lambda;k;\gamma,\beta)$, there exists a function $r(z) \in US^*(k;\gamma)$ such that

$$\frac{z\left(\mathcal{L}_a^{\lambda} f(z)\right)'}{r(z)} \prec q_{k,\gamma}(z). \tag{2.10}$$

Choose the function $g(z)$ such that $\mathcal{L}_a^{\lambda} g(z) = r(z)$. Then, $g \in US^*(\lambda;k;\gamma)$ and

$$\frac{z\left(\mathcal{L}_a^{\lambda} f(z)\right)'}{\mathcal{L}_a^{\lambda} g(z)} \prec q_{k,\gamma}(z). \tag{2.11}$$

Now let

$$p(z) = \frac{z\left(\mathcal{L}_a^{\lambda+1} f(z)\right)'}{\mathcal{L}_a^{\lambda+1} g(z)}, \tag{2.12}$$

where $p(z)$ is analytic in $\mathbf{U}$ with $p(0)=1$. Since $g \in US^*(\lambda; k; \gamma)$, by Theorem 1, we know that $g \in US^*(\lambda+1; k; \gamma)$. Let

$$t(z) = \frac{z\left(\mathcal{L}_a^{\lambda+1} g(z)\right)'}{\mathcal{L}_a^{\lambda+1} g(z)} \quad (z \in \mathbf{U}), \tag{2.13}$$

where $t(z)$ is analytic in $\mathbf{U}$ with $\Re\{t(z)\} > \frac{k+\gamma}{k+1}$. Also, from (2.13), we note that

$$z\left(\mathcal{L}_a^{\lambda+1} f(z)\right)' = \mathcal{L}_a^{\lambda+1} z f'(z) = \left(\mathcal{L}_a^{\lambda+1} g(z)\right)\ p(z). \tag{2.14}$$

Differentiating both sides of (2.14) with respect to z and multiplying the result equation by z, we obtain

$$\frac{z\left(\mathcal{L}_a^{\lambda+1} z f'(z)\right)'}{\mathcal{L}_a^{\lambda+1} g(z)} = \frac{z\left(\mathcal{L}_a^{\lambda+1} g(z)\right)'}{\mathcal{L}_a^{\lambda+1} g(z)} p(z) + z p'(z) = t(z)\, p(z) + z p'(z). \tag{2.15}$$

Now using the identity (1.16) and (2.15), we obtain

$$\begin{aligned}
\frac{z\left(\mathcal{L}_a^{\lambda} f(z)\right)'}{\mathcal{L}_a^{\lambda} g(z)} &= \frac{\mathcal{L}_a^{\lambda} z f'(z)}{\mathcal{L}_a^{\lambda} g(z)} = \frac{z\left(\mathcal{L}_a^{\lambda+1} z f'(z)\right)' + a\mathcal{L}_a^{\lambda+1} z f'(z)}{z\left(\mathcal{L}_a^{\lambda+1} g(z)\right)' + a\mathcal{L}_a^{\lambda+1} g(z)} \\
&= \frac{\frac{z\left(\mathcal{L}_a^{\lambda+1} z f'(z)\right)'}{\mathcal{L}_a^{\lambda+1} g(z)} + a\frac{z\left(\mathcal{L}_a^{\lambda+1} f(z)\right)'}{\mathcal{L}_a^{\lambda+1} g(z)}}{\frac{z\left(\mathcal{L}_a^{\lambda+1} g(z)\right)'}{\mathcal{L}_a^{\lambda+1} g(z)} + a} \\
&= \frac{t(z)\, p(z) + z p'(z) + a p(z)}{t(z) + a} \\
&= p(z) + \frac{z p'(z)}{t(z) + a}.
\end{aligned} \tag{2.16}$$

Since $a > 0$ and $\Re\{t(z)\} > \dfrac{k+\gamma}{k+1}$, we see that

$$\Re\{t(z) + a\} > 0 \quad (z \in \mathbf{U}). \tag{2.17}$$

Hence, applying Lemma 2, we can show that $p(z) \prec q_{k,\gamma}(z)$ so that $f \in UK(\lambda+1; k; \gamma, \beta)$. This completes the proof of Theorem 3. ∎

Theorem 4. $UK^*(\lambda; k; \gamma, \beta) \subset UK^*(\lambda+1; k; \gamma, \beta)$.

Proof. Just as we derived Theorem 2 as consequence of Theorem 1 by using the equivalence (1.21), we can also prove Theorem 4 by using Theorem 3 and the equivalence (1.22). ∎

3. Inclusion Properties Involving the Integral Operator F_c

In this section, we consider the generalized Libera integral operator F_c (see [2], [6] and [7]) defined by

$$F_c(f)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \qquad (f \in \mathcal{A}; c > -1). \tag{3.1}$$

Theorem 5. *Let* $c > -\frac{k+\gamma}{k+1}$. *If* $f \in US^*(\lambda; k; \gamma)$, *then* $F_c(f) \in US^*(\lambda; k; \gamma)$.

Proof. Let $f \in US^*(\lambda;k;\gamma)$ and set

$$p(z) = \frac{z\left(\mathcal{L}_a^\lambda F_c(f)(z)\right)'}{\mathcal{L}_a^\lambda F_c(f)(z)} \quad (z \in \mathbf{U}), \tag{3.2}$$

where $p(z)$ is analytic in $\mathbf{U}$ with $p(0) = 1$. From (3.1), we have

$$z\left(\mathcal{L}_a^\lambda F_c(f)(z)\right)' = (c+1)\mathcal{L}_a^\lambda f(z) - c\mathcal{L}_a^\lambda F_c(f)(z). \tag{3.3}$$

Then, by using (3.2) and (3.3), we obtain

$$(c+1)\frac{\mathcal{L}_a^\lambda f(z)}{\mathcal{L}_a^\lambda F_c(f)(z)} = p(z) + c. \tag{3.4}$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by z, we have

$$p(z) + \frac{zp'(z)}{p(z)+c} = \frac{z\left(\mathcal{L}_a^\lambda f(z)\right)'}{\mathcal{L}_a^\lambda f(z)} \prec q_{k,\gamma}(z). \tag{3.5}$$

Hence, by virtue of Lemma 1, we conclude that $p(z) \prec q_{k,\gamma}(z)$ in $\mathbf{U}$, which implies that $F_c(f) \in US^*(\lambda;k;\gamma)$. ∎

Theorem 6. *Let* $c > -\frac{k+\gamma}{k+1}$*. If* $f \in UC(\lambda;k;\gamma)$*, then* $F_c(f) \in UC(\lambda;k;\gamma)$*.*

Proof. By applying Theorem 5, it follows that

$$\begin{aligned} f(z) \quad &\in \quad UC(\lambda;k;\gamma) \Longleftrightarrow zf'(z) \in US^*(\lambda;k;\gamma) \\ &\Longrightarrow \quad F_c\left(zf'\right)(z) \in US^*(\lambda;k;\gamma) \quad (\text{by Theorem } 5) \\ &\Longleftrightarrow \quad z\left(F_c(f)(z)\right)' \in US^*(\lambda;k;\gamma) \\ &\Longleftrightarrow \quad F_c(f)(z) \in UC(\lambda;k;\gamma), \end{aligned} \tag{3.6}$$

which proves Theorem 6. ∎

Theorem 7. *Let* $c > -\frac{k+\gamma}{k+1}$*. If* $f \in UK(\lambda;k;\gamma,\beta)$*, then* $F_c(f) \in UK(\lambda;k;\gamma,\beta)$*.*

Proof. Let $f \in UK(\lambda;k;\gamma,\beta)$. Then, in view of the definition of the class $UK(\lambda;k;\gamma,\beta)$, there exists a function $g \in US^*(\lambda;k;\gamma)$ such that

$$\frac{z\left(\mathcal{L}_a^\lambda f(z)\right)'}{\mathcal{L}_a^\lambda g(z)} \prec q_{k,\gamma}(z). \tag{3.7}$$

Thus, we set

$$p(z) = \frac{z\left(\mathcal{L}_a^\lambda F_c(f)(z)\right)'}{\mathcal{L}_a^\lambda F_c(g)(z)} \quad (z \in \mathbf{U}), \tag{3.8}$$

where $p(z)$ is analytic in $\mathbf{U}$ with $p(0) = 1$. Since $g \in US^*(\lambda;k;\gamma)$, we see from Theorem 5 that $F_c(g) \in US^*(\lambda;k;\gamma)$. Using (3.3) and let

$$t(z) = \frac{z\left(\mathcal{L}_a^\lambda F_c(g)(z)\right)'}{\mathcal{L}_a^\lambda F_c(g)(z)}, \tag{3.9}$$

where $t(z)$ is analytic in $\mathbf{U}$ with $\Re\{t(z)\} > \dfrac{k+\gamma}{k+1}$. Using (3.8), we have

$$\mathcal{L}_a^\lambda zF_c'(f)(z) = \left(\mathcal{L}_a^\lambda F_c(g)(z)\right)\, p(z). \tag{3.10}$$

Differentiating both sides of (3.10) with respect to z and multiplying by z, we obtain

$$\begin{aligned} \frac{z\left(\mathcal{L}_a^{\lambda} zF_c^{'}(f)(z)\right)^{'}}{\mathcal{L}_a^{\lambda} F_c(g)(z)} &= \frac{z\left(\mathcal{L}_a^{\lambda} F_c(f)(z)\right)^{'}}{\mathcal{L}_a^{\lambda} F_c(g)(z)} p(z) + zp^{'}(z) \\ &= t(z)\, p(z) + zp^{'}(z). \end{aligned} \tag{3.11}$$

Now using the identity (3.3) and (3.11), we obtain

$$\begin{aligned} \frac{z\left(\mathcal{L}_a^{\lambda} f(z)\right)^{'}}{\mathcal{L}_a^{\lambda} g(z)} &= \frac{\mathcal{L}_a^{\lambda} zf^{'}(z)}{\mathcal{L}_a^{\lambda} g(z)} = \frac{z\left(\mathcal{L}_a^{\lambda} zF_c^{'}(f)(z)\right)^{'} + c\mathcal{L}_a^{\lambda} zF_c^{'}(f)(z)}{z\left(\mathcal{L}_a^{\lambda} F_c(g)(z)\right)^{'} + c\mathcal{L}_a^{\lambda} F_c(g)(z)} \\ &= \frac{\dfrac{z\left(\mathcal{L}_a^{\lambda} zF_c^{'}(f)(z)\right)^{'}}{\mathcal{L}_a^{\lambda} F_c(g)(z)} + c\dfrac{z\left(\mathcal{L}_a^{\lambda} F_c(f)(z)\right)^{'}}{\mathcal{L}_a^{\lambda} F_c(g)(z)}}{\dfrac{z\left(\mathcal{L}_a^{\lambda} F_c(g)(z)\right)^{'}}{\mathcal{L}_a^{\lambda} F_c(g)(z)} + c} \\ &= \frac{t(z)\, p(z) + zp^{'}(z) + cp(z)}{t(z) + c} \\ &= p(z) + \frac{zp^{'}(z)}{t(z) + c}. \end{aligned} \tag{3.12}$$

Since $c > -\dfrac{k+\gamma}{k+1}$ and $\Re\{t(z)\} > \dfrac{k+\gamma}{k+1}$, we see that

$$\Re\{t(z) + c\} > 0 \quad (z \in \mathbf{U}). \tag{3.13}$$

Applying Lemma 2 to (3.12), it follows that $p(z) \prec q_{k,\gamma}(z)$, that is $F_c(f) \in UK(\lambda; k; \gamma, \beta)$. ∎

Theorem 8. *Let $c > -\frac{k+\gamma}{k+1}$. If $f \in UK^*(\lambda; k; \gamma, \beta)$, then $F_c(f) \in UK^*(\lambda; k; \gamma, \beta)$.*

Proof. Just as we derived Theorem 6 as consequence of Theorem 5 and (1.21), we easily deduce the integral-preserving property asserted by Theorem 8 by using Theorem 7 and (1.22). ∎

Remark. Putting $a = 1$ in the above results,we obtain the results of Aghalary and Jahangiri [1].

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