Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 4 (2011), Pages 116-122.

ITERATIVE SEQUENCE OF MAPS IN A METRIC SPACE WITH SOME CHAOTIC PROPERTIES

(communicated by uday chand de)

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Abstract. In this paper we have introduced the concept of 'sequence of strongly transitive maps in the iterative way' and 'sequence of turbulent maps in the iterative way' and proved some related results on their dynamical behaviors. Some examples are also given. The work is a continuation of a recently introduced new concept of chaos in metric spaces for sequence of maps.

1. Introduction

A general discrete dynamical system is sometimes defined as a pair (X, f) consisting of a set X together with a continuous map f from X into itself. The subject dynamical system has its roots in classical mechanics, where X is taken as the set of all possible states of a system and the transformation f is the time evolution map. Chaotic dynamical systems constitute a special class of dynamical systems. During the last four decades discrete dynamical systems, in particular chaotic dynamical systems, have been studied extensively. Although there is no universally accepted mathematical definition of chaos, it is generally believed that if for a system the distance between two nearby points increases and the distance between two far away points decreases with time, the system is said to be chaotic. The first mathematical definition of chaos was given by Li and Yorke [8] in 1975. Robinson's chaos [9] is another type of chaos. Later, Devaney [4] characterized chaos in a somewhat different way. Devaney's definition of chaos is one of the most popular and most widely known definitions of chaos for the discrete dynamical systems. The three conditions of Devaney's definition are i) topological transitivity, ii) denseness properties of the set of periodic points and iii) sensitive dependence on initial conditions. Later it was shown by Banks et al [1] that conditions i) and ii) together imply condition iii). Although chaotic behaviors of continuous maps in general metric spaces are difficult to study, some progress has been made in this direction during the last three decades. Most of these research papers are concerned with compact metric

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2000 *Mathematics Subject Classification.* 74H62, 37C15, 37E05.

Key words and phrases. Chaos; Sequence of maps; Strongly transitive in the iterative way; Strictly topologically transitive in the iterative way; Turbulent in the iterative way.

Submitted July 4, 2011. Published October 17, 2011.

spaces. In this paper we have considered turbulent maps and strongly transitive maps in general metric spaces which are not necessarily compact. In 1992, Block and Copple [3] introduced the concept of turbulent maps in an interval. In 2004, Yang and Tang [12] obtained a criterion for strictly turbulent maps in a compact metric space. Shi and Yu [10] investigated some properties of turbulent maps on non-compact subsets of a metric space. There are also some more interesting research works on turbulent maps which we have noted in [5] and [6]. It has been shown that a turbulent interval map is chaotic in the sense of Devaney, but in a general metric space it is not always true. Strong transitivity is another important property for discrete dynamical systems. In [7], Kameyama discussed a relation between strong transitivity and topological transitivity in a metric space.

Recently, Tian and Chen introduced a new concept of chaos in [11], where they defined two types of chaos, namely, 'in the iterative way' and 'in the successive way' for sequence of maps. They also established that these chaotic behaviors are preserved under h-conjugacy. Precisely, the following result was established in [11].

Theorem 1.1 (Theorem 3.1, [11]) *Let (X,d) and $(Y,\widetilde{d})$ be two metric spaces, and $F=\{f_k\}_{k=1}^{\infty}$ and $G=\{g_k\}_{k=1}^{\infty}$ be two sequences of maps in X and Y, respectively. If there exists an uniform homeomorphism $h:X\to Y$ (that is, when both h and h^{-1} are uniformly continuous) such that $F=\{f_k\}_{k=1}^{\infty}$ and $G=\{g_k\}_{k=1}^{\infty}$ are h-conjugate, then $F=\{f_k\}_{k=1}^{\infty}$ is chaotic on X in the iterative (or successive) way if and only if $G=\{g_k\}_{k=1}^{\infty}$ is chaotic on Y in the iterative (or successive) way.*

The present authors also published a paper [2] in this new line of research by considering the the sequence of maps 'in the successive way'.

Here, in furtherance of this line of research, we have introduced the concepts of the 'sequence of turbulent maps in the iterative way' and 'sequence of strongly transitive maps in the iterative way'. We have established that dynamical behaviors of the sequence of maps of the above two types are preserved under h-conjugacy. Some other results for such sequence of mappings are also proved. In Section 4, some examples are provided relating to our new concepts.

2. Mathematical Preliminaries

In this section we give some existing definitions, results and notations which are essential for the discussion in next sections. We also introduce some new definitions.

Throughout this paper we have used the following notations.

R denotes the set of real numbers and $d^\star(x,y)=|x-y|$, for all $x,y\in R$. Then $(R,d^\star)$ is a metric space called the usual metric space and is denoted by R_u. The metric $d^\star$ defined above is called the usual metric on R.

Let a be any non-negative real number. Then we define, $\langle a\rangle$ is the decimal part of the real number a, that is, for example $\langle 2.56\rangle=0.56$, $\langle 19\rangle=0$ etc.

For any set $A\subseteq X$, where (X,d) is a metric space, we denote the interior of A by $Int(A)$. Let $f:X\to X$ be a map on a metric space (X,d), then we define $f^0(x)=x$ and $f^n(x)=f(f^{n-1}(x))$, for all $n\geq 1$.

For any two non-empty subsets A and B of a metric space (X,d) we define $A\backslash B=\{x:x\in A,\text{ but } x\notin B\}$. Let $f:X\to X$ and $g:X\to X$ be two maps, then the product of f and g is denoted by fg or by $f(g(x))$, for all $x\in X$, where (X,d) is a metric space.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions from X to X, where (X, d) is a metric space. Then we denote $f_k \circ f_{k-1} \circ .... \circ f_1(x)$ by $F_k(x)$, for all $k \geq 1$ and $x \in X$.

We now give some definitions.

Definition 2.1 (Topologically conjugate [4]**):** Let $f : A \to A$ and $g : B \to B$ be two continuous mappings. Then f and g are said to be topologically conjugate if there exists a homeomorphism $h : A \to B$ such that $hf = gh$. The homeomorphism h is called a topological conjugacy between f and g.

Definition 2.2 (h-conjugate [11]**):** Let (X, d) and (Y, ρ) be two metric spaces. Also let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be two sequences of continuous maps in X and Y respectively. If there exists a homeomorphism $h : X \to Y$ such that for any positive integer k, $g_k(h(x)) = h(f_k(x))$, for all $x \in X$, the sequence of maps is called h-conjugate.

Definition 2.3 (Topologically transitive [4]**):** A continuous map $f : X \to X$ is called topologically transitive if for any pair of non-empty open sets $U, V \subseteq X$ there exists $k \geq 0$ such that $f^k(U) \cap V \neq \phi$, where (X, d) is a metric space.

Definition 2.4 (Orbit in the iterative way [11]**):** Let (X, d) be a metric space and $x \in X$ be any point. Also let $f_n : X \to X, n \geq 1$, be a sequence of continuous maps. Then $\{x, f_1(x), f_2 \circ f_1(x), f_3 \circ f_2 \circ f_1(x), ...\}$ is called orbit of the sequence $\{f_n\}_{n=1}^{\infty}$ (starting at x) in the iterative way.

Definition 2.5 (Topologically transitive in the iterative way [11]**):** Let (X, d) be a metric space and $f_n : X \to X, n \geq 1$, be a sequence of continuous maps. If, for any two open subsets U and V of X, there exists a positive integer k such that $F_k(U) \cap V \neq \phi$, the sequence $\{f_n\}_{n=1}^{\infty}$ is called topologically transitive in the iterative way.

We now extend the idea of topologically transitive in the iterative way to strictly topologically transitive in the iterative way.

Definition 2.6 (Strictly topologically transitive in the iterative way): Let (X, d) be a metric space and $f_n : X \to X, n \geq 1$, be a sequence of continuous maps. If, for any two disjoint non-empty connected open subsets U and V of X, there exists a positive integer k such that $F_k(U) \cap V \neq \phi$, the sequence $\{f_n\}_{n=1}^{\infty}$ is called strictly topologically transitive in the iterative way.

Definition 2.7 (Sensitive dependence on initial conditions [4]**):** A continuous map $f : X \to X$, where (X, d) is a metric space, is said to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood $N(x)$ of x, there exist $y \in N(x)$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.

Definition 2.8 (Sensitive dependence on initial conditions in the iterative way [11]**):** Let (X, d) be a metric space and $f_n : X \to X, n \geq 1$, be a sequence of continuous maps. If there exists a constant $\delta > 0$ such that for any point $x \in X$ and any neighborhood $N(x)$ of x, there exist a point $y \in N(x)$ and a positive integer k such that $d(F_k(x), F_k(y)) > \delta$, the sequence of maps $\{f_n\}_{n=1}^{\infty}$ is said to have sensitive dependence on initial conditions in the iterative way.

Definition 2.9 (Strongly transitive [7]**):** Let (X, d) be a metric space and $f : X \to X$ be a continuous map. Then f is called strongly transitive if for any non-empty open set

$$U \subseteq X \text{ we get } X = \bigcup_{k=0}^{s} f^k(U), \text{ for some } s > 0.$$

In the following we introduce the notion of a sequence of strongly transitive maps in a metric space in the iterative way.

Definition 2.10 (Strongly transitive in the iterative way): Let (X, d) be a metric space and $f_n : X \to X, n \geq 1$, be a sequence of continuous maps. Then $\{f_n\}_{n=1}^{\infty}$ is called strongly transitive in the iterative way if for any non-empty open set $U \subseteq X$ we get

$$X = U \bigcup_{k=1}^{s} F_k(U), \text{ for some } s > 0.$$

Definition 2.11 (Turbulent in the interval [3]**):** Let R be the set of real numbers. A continuous map $f : R \to R$ is called turbulent if there exist compact subintervals J and K with at most one common point such that $f(J) \supseteq J \cup K$ and $f(K) \supseteq J \cup K$. The map f is called strictly turbulent if J and K are disjoint.

Next we introduce the definition of a sequence of turbulent maps in the iterative way in a metric space.

Definition 2.12 (Turbulent in the iterative way): Let (X, d) be a metric space and $f_n : X \to X, n \geq 1$, be a sequence of continuous maps. If, for any two compact subsets M and N of X, where $Int(M) \cap Int(N) = \phi$, there exists a positive integer k such that $F_k(M) \supseteq M \cup N$ and $F_k(N) \supseteq M \cup N$, we say that $\{f_n\}_{n=1}^{\infty}$ is a sequence of turbulent maps in the iterative way. The sequence of maps $\{f_n\}_{n=1}^{\infty}$ is called strictly turbulent in the iterative way if M and N are disjoint.

We shall require the following lemma to establish our theorems in the next section.

Lemma 2.1 *Let (X, d) and (Y, ρ) be two metric spaces. Also let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be two sequences of continuous maps in X and Y respectively. Then $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are h-conjugate if and only if $\{g_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ are h^{-1}-conjugate.*

Proof. The proof is straight forward.

3. The Main Results

Theorem 3.1 *Let R_u be the usual metric space and $f_n : R \to R$, $n \geq 1$, be a sequence of continuous turbulent maps in the iterative way. Then the sequence $\{f_n\}_{n=1}^{\infty}$ is strictly topologically transitive in the iterative way on R.*

Proof. Since only open intervals are both connected and open subsets of real line, we consider here the cases of open sub-intervals. Let (x_1, y_1) and (x_2, y_2) be any two non-empty disjoint open sub-intervals of R. We now consider the compact sub-intervals $M = [x_1, y_1]$ and $N = [x_2, y_2]$ of R. Hence $M \cap N$ contains at most one point and also $Int(M) \cap Int(N) = \phi$. So we get $F_k(M) \supseteq M \cup N$ and $F_k(N) \supseteq M \cup N$ for some positive integer k, since $\{f_n\}_{n=1}^{\infty}$ is a sequence of turbulent maps in the iterative way. By the continuity of $f_1, f_2, ..., f_k$ we get that $f_k \circ f_{k-1} \circ .... \circ f_1(M)$ and $f_k \circ f_{k-1} \circ .... \circ f_1(N)$ are also compact sub-intervals of R. That is, we get that $F_k(M)$ and $F_k(N)$ are also compact sub-intervals of R.

Then the following two cases are possible.

Case I: $M \cap N \neq \phi$.

Without any loss of generality we may assume that $y_1 = x_2$ is the only common point. Then $F_k([x_1, y_1]) \supseteq [x_1, y_1] \cup [x_2, y_2] = [x_1, y_2]$. Since F_k is a continuous map, this gives $F_k((x_1, y_1)) \cap (x_2, y_2) \neq \phi$, for some positive integer k.

Hence the sequence $\{f_n\}_{n=1}^{\infty}$ is topologically transitive in the iterative way on R.

Case II: $M \cap N = \phi$.

In this case $F_k([x_1, y_1]) \supseteq [x_1, y_1] \cup [x_2, y_2]$ with $[x_1, y_1] \cap [x_2, y_2] = \phi$. Here also by the continuity of F_k we get $F_k((x_1, y_1)) \cap (x_2, y_2) \neq \phi$, for some positive integer k.

Hence the sequence $\{f_n\}_{n=1}^{\infty}$ is topologically transitive in the iterative way on R.

Combining Case I and Case II as in the above, the theorem is proved.

Theorem 3.2 *Let (X, d) and (Y, ρ) be two metric spaces and $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be two sequences of continuous maps in X and Y respectively. Let the homeomorphism $h : X \to Y$ be such that $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are h-conjugate then $\{f_n\}_{n=1}^{\infty}$ is strictly turbulent in the iterative way if and only if $\{g_n\}_{n=1}^{\infty}$ is strictly turbulent in the iterative way.*

Proof. The proof of this theorem can be easily done by applying Lemma 2.1. Hence it is left as elementary exercise to the reader.

Theorem 3.3 *Let (X, d) and (Y, ρ) be two metric spaces and $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be two sequences of continuous maps in X and Y respectively. Let the homeomorphism $h : X \to Y$ be such that $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are h-conjugate, then $\{f_n\}_{n=1}^{\infty}$ is strongly transitive in the iterative way if and only if $\{g_n\}_{n=1}^{\infty}$ is strongly transitive in the iterative way.*

Proof. The proof of this theorem can be similarly done by applying Lemma 2.1. Hence it is left as elementary exercise to the reader.

Theorem 3.4 *Let $f_n : X \to X$, $n \geq 1$, be a sequence of continuous maps in a metric space (X, d). Also let U and V be any two non-empty open subsets of X such that none is a subset of the other, then $F_m(U) \cap V \neq \phi$ for some positive integer m, if $\{f_n\}_{n=1}^{\infty}$ is strongly transitive in the iterative way.*

Proof. Let us assume $f_n : X \to X$, $n \geq 1$, be a sequence of continuous strongly transitive maps in the iterative way on (X, d) and U, V are any two non-empty open subsets of X such that none is a subset of the other.

Then the following two cases are possible.

Case I: U and V are disjoint.

Now by the given condition we get that $X = U \bigcup_{k=1}^{s} F_k(U)$, for some $s > 0$.

So there exists at least one positive integer $m_1 \leq s$ such that either $F_{m_1}(U)$ intersects V or $F_{m_1}(U) \supseteq V$. Hence $F_{m_1}(U) \cap V \neq \phi$.

Case II: U and V are distinct but not disjoint.

Here we also get that $X = U \bigcup_{k=0}^{s} F_k(U)$, for some $s > 0$. We now consider the set $V \backslash U$. For reasons similar to Case I, there exists a positive integer $m_2 \leq s$ such that either $F_{m_2}(U)$ intersects $V \backslash U$ or $F_{m_2}(U) \supseteq V \backslash U$. Hence $F_{m_2}(U) \cap V \neq \phi$.

Combining these two cases we have $F_m(U) \cap V \neq \phi$ for some positive integer m.

This proves the theorem.

4. Few Examples

In this section we give some examples for illustrating some of the notions we have introduced in this paper.

In the following we show that there exists a strongly transitive sequence of maps in the iterative way.

Example 4.1 Let us consider the metric space $(X, d^\star)$, where $X = [0, 1)$ and $d^\star$ is the usual metric.

We now consider the sequence of maps $f_n : X \to X, n \geq 1$, where $f_n(x)$ is defined by $f_n(x) = \langle 10^{n-1}x \rangle$, for all $n \geq 1$.

Then for any non-empty open interval $U \subseteq X,$ we get $X = U \bigcup\limits_{k=1}^{s} F_k(U),$ for some $s > 0$. Hence the sequence of maps as defined above is strongly transitive in the iterative way.

Our next example proves that there also exists sequence of strictly turbulent maps in the iterative way.

Example 4.2 We now consider the sequence of maps $f_n : X \to X, n \geq 1$, where $f_n(x)$ is defined by $f_n(x) = \langle (1 + \frac{1+(-1)^n}{2})x \rangle$, for all $n \geq 1$.

Then we get $F_{2k-1}(x) = \langle 2^{k-1}x \rangle$ and $F_{2k}(x) = \langle 2^k x \rangle$, for all $k \geq 1$. Hence for any two non-empty compact disjoint sub-intervals $M, N \subset X$, it is seen that there always exists $k \geq 1$ such that $F_k(M) \supseteq M \cup N$ and $F_k(N) \supseteq M \cup N$. Hence the sequence of maps as defined above is strictly turbulent in the iterative way.

5. Conclusions

We have worked out our results in general metric spaces while most of the results in this line of research were proved in compact metric spaces. Further most of the notions we have introduced here are illustrated through examples. It is our perception that the explorations of dynamical properties of sequences of maps (of several types) will be an interesting area of research.

Acknowledgments. Indranil Bhaumik acknowledges his father Sadhan Chandra Bhaumik for his help and encouragement in preparing the manuscript.

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