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COMMON FIXED POINT THEOREMS VIA w-DISTANCE

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Abstract. We prove some common fixed point theorems with the help of the notion of w-distance in a metric space. Our results will improve and supplement some results of [1] and [6].

1. Introduction

Metric fixed point theory is playing an increasing role in mathematics because of its wide range of applications in applied mathematics and sciences. Over the past two decades a considerable amount of research work for the development of fixed point theory have executed by several authors. In 1996, Kada et.al.[1] introduced the concept of w-distance in a metric space and studied some fixed point theorems. The aim of this paper is to obtain some common fixed point results by using the notion of w-distance in a metric space. Our results generalize some results of [1] and [6].

Now, we recall some basic definitions which will be needed in the sequel. Throughout this paper we denote by N the set of all positive integers.

Definition 1.1. [1] *Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following conditions are satisfied:*

(i) $p(x, z) \leq p(x, y) + p(y, z)$ *for any* $x, y, z \in X$;

(ii) for any $x \in X$, $p(x, .) : X \to [0, \infty)$ *is lower semicontinuous*;

(iii) for any $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $p(z, x) \leq \delta$ *and* $p(z, y) \leq \delta$ *imply* $d(x, y) \leq \epsilon$.

Example 1.1. [5] *If $X = \{\frac{1}{n} : n \in N\} \cup \{0\}$. For each $x, y \in X$, $d(x, y) = x + y$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$ is a metric on X. Moreover, by defining $p(x, y) = y$, p is a w-distance on (X, d).*

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Definition 1.2. *Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be expansive if there exists a real constant $c > 1$ satisfying $d(T(x), T(y)) \geq cd(x, y)$ for all $x, y \in X$.*

2. Main Results

Before presenting our results we recall the following lemma due to O. Kada et. al.[1] that will play a crucial role in this section.

Lemma 2.1. *Let (X,d) be a metric space and let p be a w-distance on X. Let (x_n) and (y_n) be sequences in X, let (α_n) and (β_n) be sequences in $[0,\infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:*
(i) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in N$, then $y = z$. In particular, if

$$p(x,y) = 0 \text{ and } p(x,z) = 0, \text{ then } y = z;$$

(ii) if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in N$, then (y_n) converges to z;
(iii) if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in N$ with $m > n$, then (x_n) is a Cauchy sequence;
(iv) if $p(y, x_n) \leq \alpha_n$ for any $n \in N$, then (x_n) is a Cauchy sequence.

Theorem 2.1. *Let p be a w-distance on a complete metric space (X,d). Let T_1, T_2 be mappings from X into itself. Suppose that there exists $r \in [0,1)$ such that*

$$max\,\{\, p(T_1(x), T_2T_1(x)),\; p(T_2(x), T_1T_2(x))\} \leq r\,min\,\{\, p(x, T_1(x)), p(x, T_2(x))\} \tag{2.1}$$

for every $x \in X$ and that

$$inf\,\{p(x,y) + min\,\{p(x,T_1(x)),\, p(x,T_2(x))\} : x \in X\} > 0 \tag{2.2}$$

for every $y \in X$ with y is not a common fixed point of T_1 and T_2. Then there exists $z \in X$ such that $z = T_1(z) = T_2(z)$. Moreover, if $v = T_1(v) = T_2(v)$, then $p(v,v) = 0$.

Proof. Let $u_0 \in X$ be arbitrary and define a sequence (u_n) by

$$\begin{aligned} u_n &= T_1(u_{n-1}),\ if\ n\ is\ odd \\ &= T_2(u_{n-1}),\ if\ n\ is\ even. \end{aligned}$$

Then if $n \in N$ is odd, we have

$$\begin{aligned} p(u_n, u_{n+1}) &= p(T_1(u_{n-1}), T_2(u_n)) \\ &= p(T_1(u_{n-1}), T_2T_1(u_{n-1})) \\ &\leq max\,\{p(T_1(u_{n-1}), T_2T_1(u_{n-1})), p(T_2(u_{n-1}), T_1T_2(u_{n-1}))\} \\ &\leq r\,min\,\{p(u_{n-1}, T_1(u_{n-1})), p(u_{n-1}, T_2(u_{n-1}))\},\ by\ (2.1) \\ &\leq r\,p(u_{n-1}, T_1(u_{n-1})) \\ &= r\,p(u_{n-1}, u_n). \end{aligned}$$

If n is even, then by (2.1), we have

$$\begin{aligned} p(u_n, u_{n+1}) &= p(T_2(u_{n-1}), T_1(u_n)) \\ &= p(T_2(u_{n-1}), T_1T_2(u_{n-1})) \\ &\leq max\,\{p(T_2(u_{n-1}), T_1T_2(u_{n-1})), p(T_1(u_{n-1}), T_2T_1(u_{n-1}))\} \\ &\leq r\, min\,\{p(u_{n-1}, T_2(u_{n-1})), p(u_{n-1}, T_1(u_{n-1}))\} \\ &\leq r\, p(u_{n-1}, T_2(u_{n-1})) \\ &= r\, p(u_{n-1}, u_n). \end{aligned}$$

Thus for any positive integer n, it must be the case that

$$p(u_n, u_{n+1}) \leq r\, p(u_{n-1}, u_n). \tag{2.3}$$

By repeated application of (2.3), we obtain

$$p(u_n, u_{n+1}) \leq r^n\, p(u_0, u_1).$$

So, if $m > n$, then

$$\begin{aligned} p(u_n, u_m) &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{m-1}, u_m) \\ &\leq \left[r^n + r^{n+1} + \cdots + r^{m-1}\right] p(u_0, u_1) \\ &\leq \frac{r^n}{1-r}\, p(u_0, u_1). \end{aligned}$$

By Lemma 2.1(iii), (u_n) is a Cauchy sequence in X. Since X is complete, (u_n) converges to some point $z \in X$. Let $n \in N$ be fixed. Then since (u_m) converges to z and $p(u_n, .)$ is lower semicontinuous, we have

$$p(u_n, z) \leq \lim_{m\to\infty} inf\, p(u_n, u_m) \leq \frac{r^n}{1-r}\, p(u_0, u_1).$$

Assume that z is not a common fixed point of T_1 and T_2. Then by hypothesis

$$\begin{aligned} 0 &< inf\ \{\, p(x, z) + min\,\{\, p(x, T_1(x)), p(x, T_2(x))\} : x \in X\} \\ &\leq inf\ \{\, p(u_n, z) + min\,\{\, p(u_n, T_1(u_n)), p(u_n, T_2(u_n))\} : n \in N\} \\ &\leq inf\ \left\{ \frac{r^n}{1-r}\, p(u_0, u_1) + p(u_n, u_{n+1}) : n \in N \right\} \\ &\leq inf\ \left\{ \frac{r^n}{1-r}\, p(u_0, u_1) + r^n\, p(u_0, u_1) : n \in N \right\} \\ &= 0 \end{aligned}$$

which is a contradiction. Therefore, $z = T_1(z) = T_2(z)$.

If $v = T_1(v) = T_2(v)$ for some $v \in X$, then

$$\begin{aligned} p(v, v) &= max\ \{\, p(T_1(v), T_2T_1(v)), p(T_2(v), T_1T_2(v))\} \\ &\leq r\, min\ \{\, p(v, T_1(v)), p(v, T_2(v))\} \\ &= r\, min\ \{\, p(v, v), p(v, v)\} \\ &= r\, p(v, v) \end{aligned}$$

which gives that, $p(v, v) = 0$. □

The following Corollary is the result [1, $Theorem\, 4$].

Corollary 2.1. *Let X be a complete metric space, let p be a w-distance on X and let T be a mapping from X into itself. Suppose that there exists $r \in [0, 1)$ such that*

$$p(T(x), T^2(x)) \le r\, p(x, T(x))$$

for every $x \in X$ and that

$$inf\ \{p(x, y) + p(x, T(x)) : x \in X\} > 0$$

for every $y \in X$ with $y \ne T(y)$. Then there exists $z \in X$ such that $z = T(z)$. Moreover, if $v = T(v)$, then $p(v, v) = 0$.

Proof. Taking $T_1 = T_2 = T$ in Theorem 2.1, the conclusion of the Corollary follows. So Corollary 2.1 can be treated as a special case of Theorem 2.1. □

We now supplement Theorem 2.1 by examination of conditions (2.1) and (2.2) in respect of their independence. We furnish Examples 2.1 and 2.2 below to show that these two conditions are independent in the sense that Theorem 2.1 shall fall through by dropping one in favour of the other.

Example 2.1. *Take $X = \{\,0\,\} \cup \left\{\frac{1}{2^n} : n \ge 1\right\}$, which is a complete metric space with usual metric d of reals. Define $T : X \to X$ by $T(0) = \frac{1}{2}$ and $T\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}$ for $n \ge 1$. Clearly, T has got no fixed point in X. It is easy to check that $d(T(x), T^2(x)) \le \frac{1}{2}\, d(x, T(x))$ for all $x \in X$. Thus condition (2.1) holds for $T_1 = T_2 = T$. However, $T(y) \ne y$ for all $y \in X$ and so*

$$\begin{aligned} & inf\,\{d(x, y) + d(x, T(x)) : x, y \in X\, with\, y \ne T(y)\} \\ = \quad & inf\,\{d(x, y) + d(x, T(x)) : x, y \in X\} \\ = \quad & 0. \end{aligned}$$

Thus, condition (2.2) is not satisfied for $T_1 = T_2 = T$. We note that Theorem 2.1 is invalid without condition (2.2).

Example 2.2. *Take $X = [2, \infty) \cup \{0, 1\}$, which is a complete metric space with usual metric d of reals. Define $T : X \to X$ where*

$$\begin{aligned} T(x) &= 0,\ for\, x \in (X \setminus \{0\}) \\ &= 1,\ for\, x = 0. \end{aligned}$$

Clearly, T possesses no fixed point in X.
Now,

$$\begin{aligned} & inf\,\{d(x, y) + d(x, T(x)) : x, y \in X\, with\, y \ne T(y)\} \\ = \quad & inf\,\{d(x, y) + d(x, T(x)) : x, y \in X\} \\ > \quad & 0. \end{aligned}$$

Thus, condition (2.2) is satisfied for $T_1 = T_2 = T$. But, for $x = 0$, we find that $d(T(x), T^2(x)) = 1 > r\, d(x, T(x))$ for any $r \in [0, 1)$. So, condition (2.1) does not hold for $T_1 = T_2 = T$. In this case we observe that Theorem 2.1 does not work without condition (2.1).

Note :*In examples above we treat d as a w-distance on X in reference to Theorem 2.1.*

Theorem 2.2. *Let p be a w-distance on a complete metric space (X, d). Let T_1, T_2 be mappings from X onto itself. Suppose that there exists $r > 1$ such that*

$$min\,\{\,p(T_2T_1(x), T_1(x)),\ p(T_1T_2(x), T_2(x))\} \geq r\,max\,\{\,p(T_1(x), x), p(T_2(x), x)\} \tag{2.4}$$

for every $x \in X$ and that

$$inf\,\{p(x, y) + min\,\{\,p(T_1(x), x), p(T_2(x), x)\} : x \in X\} > 0 \tag{2.5}$$

for every $y \in X$ with y is not a common fixed point of T_1 and T_2. Then there exists $z \in X$ such that $z = T_1(z) = T_2(z)$. Moreover, if $v = T_1(v) = T_2(v)$, then $p(v, v) = 0$.

Proof. Let $u_0 \in X$ be arbitrary. Since T_1 is onto, there is an element u_1 satisfying $u_1 \in T_1^{-1}(u_0)$. Since T_2 is also onto, there is an element u_2 satisfying $u_2 \in T_2^{-1}(u_1)$. Proceeding in the same way, we can find $u_{2n+1} \in T_1^{-1}(u_{2n})$ and $u_{2n+2} \in T_2^{-1}(u_{2n+1})$ for $n = 1, 2, 3, \cdots$.
Therefore, $u_{2n} = T_1(u_{2n+1})$ and $u_{2n+1} = T_2(u_{2n+2})$ for $n = 0, 1, 2, \cdots$.

If $n = 2m$, then using (2.4)

$$\begin{aligned}
p(u_{n-1}, u_n) &= p(u_{2m-1}, u_{2m}) \\
&= p(T_2(u_{2m}), T_1(u_{2m+1})) \\
&= p(T_2T_1(u_{2m+1}), T_1(u_{2m+1})) \\
&\geq min\,\{p(T_2T_1(u_{2m+1}), T_1(u_{2m+1})), p(T_1T_2(u_{2m+1}), T_2(u_{2m+1}))\} \\
&\geq r\,max\,\{p(T_1(u_{2m+1}), u_{2m+1}), p(T_2(u_{2m+1}), u_{2m+1})\} \\
&\geq r\,p(T_1(u_{2m+1}), u_{2m+1}) \\
&= r\,p(u_{2m}, u_{2m+1}) \\
&= r\,p(u_n, u_{n+1}).
\end{aligned}$$

If $n = 2m + 1$, then by (2.4), we have

$$\begin{aligned}
p(u_{n-1}, u_n) &= p(u_{2m}, u_{2m+1}) \\
&= p(T_1(u_{2m+1}), T_2(u_{2m+2})) \\
&= p(T_1T_2(u_{2m+2}), T_2(u_{2m+2})) \\
&\geq min\,\{p(T_2T_1(u_{2m+2}), T_1(u_{2m+2})), p(T_1T_2(u_{2m+2}), T_2(u_{2m+2}))\} \\
&\geq r\,max\,\{p(T_1(u_{2m+2}), u_{2m+2}), p(T_2(u_{2m+2}), u_{2m+2})\} \\
&\geq r\,p(T_2(u_{2m+2}), u_{2m+2}) \\
&= r\,p(u_{2m+1}, u_{2m+2}) \\
&= r\,p(u_n, u_{n+1}).
\end{aligned}$$

Thus for any positive integer n, it must be the case that

$$p(u_{n-1}, u_n) \geq r\,p(u_n, u_{n+1})$$

which implies that,

$$p(u_n, u_{n+1}) \leq \frac{1}{r}\,p(u_{n-1}, u_n) \leq \cdots \leq \left(\frac{1}{r}\right)^n p(u_0, u_1). \tag{2.6}$$

Let $\alpha = \frac{1}{r}$, then $0 < \alpha < 1$ since $r > 1$.

Now, (2.6) becomes

$$p(u_n, u_{n+1}) \leq \alpha^n \, p(u_0, u_1).$$

So, if $m > n$, then

$$\begin{aligned} p(u_n, u_m) &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{m-1}, u_m) \\ &\leq \left[\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}\right] p(u_0, u_1) \\ &\leq \frac{\alpha^n}{1-\alpha} \, p(u_0, u_1). \end{aligned}$$

By Lemma 2.1(iii), (u_n) is a Cauchy sequence in X. Since X is complete, (u_n) converges to some point $z \in X$. Let $n \in N$ be fixed. Then since (u_m) converges to z and $p(u_n, .)$ is lower semicontinuous, we have

$$p(u_n, z) \leq \lim_{m \to \infty} \, inf \, p(u_n, u_m) \leq \frac{\alpha^n}{1-\alpha} \, p(u_0, u_1).$$

Assume that z is not a common fixed point of T_1 and T_2. Then by hypothesis

$$\begin{aligned} 0 &< inf \; \{ p(x, z) + min \, \{ p(T_1(x), x), p(T_2(x), x)\} : x \in X\} \\ &\leq inf \; \{ p(u_n, z) + min \, \{ p(T_1(u_n), u_n), p(T_2(u_n), u_n)\} : n \in N\} \\ &\leq inf \; \left\{ \frac{\alpha^n}{1-\alpha} \, p(u_0, u_1) + p(u_{n-1}, u_n) : n \in N \right\} \\ &\leq inf \; \left\{ \frac{\alpha^n}{1-\alpha} \, p(u_0, u_1) + \alpha^{n-1} \, p(u_0, u_1) : n \in N \right\} \\ &= 0 \end{aligned}$$

which is a contradiction. Therefore, $z = T_1(z) = T_2(z)$.

If $v = T_1(v) = T_2(v)$ for some $v \in X$, then

$$\begin{aligned} p(v, v) &= min \; \{ p(T_2T_1(v), T_1(v)), p(T_1T_2(v), T_2(v))\} \\ &\geq r \, max \; \{ p(T_1(v), v), p(T_2(v), v)\} \\ &= r \, max \; \{ p(v, v), p(v, v)\} \\ &= r \, p(v, v) \end{aligned}$$

which gives that, $p(v, v) = 0$. □

Corollary 2.2. *Let p be a w-distance on a complete metric space (X, d) and let $T : X \to X$ be an onto mapping. Suppose that there exists $r > 1$ such that*

$$p(T^2(x), T(x)) \geq r p(T(x), x) \tag{2.7}$$

for every $x \in X$ and that

$$inf \{p(x, y) + p(T(x), x) : x \in X\} > 0 \tag{2.8}$$

for every $y \in X$ with $y \neq T(y)$. Then T has a fixed point in X. Moreover, if $v = T(v)$, then $p(v, v) = 0$.

Proof. Taking $T_1 = T_2 = T$ in Theorem 2.2, we have the desired result. □

The following Corollary is the result [6, $Theorem\,4$].

Corollary 2.3. *Let (X, d) be a complete metric space and T be a mapping of X into itself. If there is a real number r with $r > 1$ satisfying*

$$d(T^2(x), T(x)) \geq rd(T(x), x)$$

for each $x \in X$, and T is onto continuous, then T has a fixed point.

Proof. We treat d as a w-distance on X. Then d satisfies condition (2.7) of Corollary 2.2.

Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$inf\{d(x, y) + d(T(x), x) : x \in X\} = 0.$$

Then there exists a sequence (x_n) such that

$$\lim_{n\to\infty} \{d(x_n, y) + d(T(x_n), x_n)\} = 0.$$

So, we have $d(x_n, y) \to 0$ and $d(T(x_n), x_n) \to 0$ as $n \to \infty$.
Now,

$$d(T(x_n), y) \leq d(T(x_n), x_n) + d(x_n, y) \to 0 \;as\; n \to \infty.$$

Since T is continuous, we have

$$T(y) = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} T(x_n) = y.$$

This is a contradiction. Hence if $y \neq T(y)$, then

$$inf\{d(x, y) + d(T(x), x) : x \in X\} > 0,$$

which is condition (2.8) of Corollary 2.2.
By Corollary 2.2, there exists $z \in X$ such that $z = T(z)$. □

As an application of Corollary 2.2, we have the following result [6, $Theorem\,3$].

Corollary 2.4. *Let (X, d) be a complete metric space and T be a mapping of X into itself. If there is a real number r with $r > 1$ satisfying*

$$d(T(x), T(y)) \geq r\,min\{d(x, T(x)), d(T(y), y), d(x, y)\} \tag{2.9}$$

for any $x, y \in X$, and T is onto continuous, then T has a fixed point.

Proof. We treat d as a w-distance on X. Replacing y by $T(x)$ in (2.9), we obtain

$$d(T(x), T^2(x)) \geq r\,min\{d(x, T(x)), d(T^2(x), T(x)), d(x, T(x))\} \tag{2.10}$$

for all $x \in X$.
Without loss of generality, we may assume that $T(x) \neq T^2(x)$. For, otherwise, T has a fixed point.
Since $r > 1$, it follows from (2.10) that

$$d(T^2(x), T(x)) \geq rd(T(x), x)$$

for every $x \in X$.
By the argument similar to that used in Corollary 2.3, we can prove that, if $y \neq T(y)$, then

$$inf\{d(x, y) + d(T(x), x) : x \in X\} > 0,$$

which is condition (2.8) of Corollary 2.2.
So, Corollary 2.2 applies to obtain a fixed point of T. □

Remark 2.1. *The class of mappings satisfying condition* (2.9) *is strictly larger than the class of all expansive mappings.*
For, if $T : X \to X$ is expansive, then there exists $r > 1$ such that

$$d(T(x), T(y)) \geq r\, d(x, y) \geq r\ min\, \{d(x, T(x)), d(T(y), y), d(x, y)\}$$

for all $x, y \in X$. On the otherhand, the identity mapping satisfies condition (2.9) *but it is not expansive.*

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