

## EXISTENCE OF POSITIVE PERIODIC SOLUTION OF AN IMPULSIVE DELAY FISHING MODEL

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ABSTRACT. In this paper, the impulsive delay fishing model is considered. By using the continuation theory for  $k$ -set contractions, the sufficient conditions of the existence of positive  $\omega$ -periodic solutions of the impulsive delay fishing model are obtained.

### 1. INTRODUCTION

Many real-world evolution processes which depend on their prehistory and are subject to short time disturbances can be modeled by impulsive delay differential equations. Such processes occur in the theory of optimal control, population dynamics, biology, economics, etc. For details, see [1,2] and references therein. In the last few years, the existence problems of positive periodic solutions of differential equations with impulsive effects and/or delay have been studied by many researchers [3-9].

In [4], the author considered the impulsive Logistic model. The sufficient conditions of the existence and asymptotic stability of  $T$ -periodic solution were obtained. In [5], the author studied an impulsive delay differential equation and sufficient conditions are obtained for the existence and global attractivity of periodic positive solutions. It is shown that under appropriate linear periodic impulsive perturbations, the impulsive delay differential equation preserves the original periodicity and global attractive properties of the non-impulsive delay differential equation. In [6], impulsive delay Logistic model was investigated. And the existence results of positive periodic solution were obtained.

The main purpose of this paper is to study the impulsive delay fishing model. By using the continuation theory for  $k$ -set contractions [10,12], the existence of positive periodic solution of this model is considered and sufficient conditions are obtained for the existence of periodic positive solutions. This paper is organized in three sections including the introduction. Section 2 formulates the problem

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and presents the preliminary results. The sufficient conditions for the existence of positive  $\omega$ -periodic solution of the model are established in section 3.

## 2. PRELIMINARIES

Consider the following impulsive delay fishing model

$$\begin{cases} x'(t) = r(t)x(t) \left(1 - \frac{x(t-\tau)}{x_m(t)}\right) - E(t)x(t), & t \neq t_k, \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k = 1, 2, \dots \end{cases} \quad (2.1)$$

where  $x(t)$  is the density of population at time  $t$ . The intrinsic growth rate of population and the carrying capacity are denoted by  $r(t), x_m(t)$  respectively.  $E(t)$  is fishing intensity and  $r(t) \geq E(t)$ .  $\tau$  is the gestation period and  $b_k$  is impulsive perturbation at the moments of time  $t_k, k = 1, 2, \dots, x(t_k^+) = \lim_{h \rightarrow 0^+} x(t+h), x(t_k) = \lim_{h \rightarrow 0^+} x(t-h)$ .

The following assumption will be needed throughout the paper.

(A<sub>1</sub>)  $0 < t_1 < t_2 < \dots$  are fixed impulsive points with  $\lim_{k \rightarrow \infty} t_k = \infty$ .

(A<sub>2</sub>)  $r(t), x_m(t)$  and  $E(t)$  are locally integrable functions on  $(0, \infty)$ .

(A<sub>3</sub>)  $\{b_k\}$  is a real sequence and  $1 + b_k > 0, k = 1, 2, \dots$

(A<sub>4</sub>)  $r(t), x_m(t), E(t), \prod_{0 < t_k < t} (1 + b_k)$  are positive continuous  $\omega$ -periodic functions and in the sequence the product equals to unity if the number of factors is zero.

We shall consider (1) with the initial condition

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0, \quad \phi(t) \in L([-\tau, 0], (0, \infty)), \quad (2.2)$$

where  $L([-\tau, 0], (0, \infty))$  denotes the set of *Lebesgue* measurable functions on  $[-\tau, 0]$ .

By a solution  $x(t)$  of (1) satisfying initial condition (2) we mean an absolutely continuous function  $x(t)$  on  $[-\tau, \infty)$ , and satisfies conditions:  $x(t_k^+)$  and  $x(t_k)$  exist for any  $t_k, k = 1, 2, \dots$ , and  $x(t)$  satisfies (1) for almost everywhere in  $[0, \infty)$  and at impulsive points  $t_k$  may have discontinuity of the first kind.

Under assumption (A<sub>1</sub>) – (A<sub>4</sub>), obviously, all solution of (1) and (2) are positive  $[0, \infty]$ . In (1), let  $x(t) = e^{y(t)}$ , then  $x'(t) = e^{y(t)}y'(t)$  and by substituting them into (1) and (2) we can obtain

$$\begin{cases} y'(t) = (r(t) - E(t)) \left(1 - \frac{r(t)}{(r(t) - E(t))x_m(t)} e^{y(t-\tau)}\right), & t \neq t_k, \\ y(t_k^+) - y(t_k) = \ln(1 + b_k), & k = 1, 2, \dots, \end{cases} \quad (2.3)$$

and the initial condition

$$y(t) = \ln \phi(t), \quad -\tau \leq t \leq 0, \quad \phi(t) \in L([-\tau, 0], (0, \infty)). \quad (2.4)$$

For investigating Eq. (3) and (4), we introduce following non-impulsive delay differential equation

$$z'(t) = r(t) - E(t) - \frac{r(t)}{N(t)} \exp \left\{ z(t - \tau) + \sum_{0 < t_j < t - \tau} \ln(1 + b_j) \right\} \quad (2.5)$$

with the initial condition

$$z(t) = \ln \phi(t), \quad -\tau \leq t \leq 0, \quad \phi(t) \in L([-\tau, 0], (0, \infty)). \quad (2.6)$$

**Theorem 2.1.** *If  $z(t)$  is a solution of (5) and (6) on  $[-\tau, \infty)$ , then  $y(t) = z(t) + \sum_{0 < t_j < t} \ln(1 + b_j)$  is a solution of (3) and (4). And if  $y(t)$  is a solution of (3) and (4) on  $[-\tau, \infty)$ , then  $z(t) = y(t) - \sum_{0 < t_j < t} \ln(1 + b_j)$  is a solution of (5) and (6).*

The proof of the Theorem 2.1 is similar to that of [Theorem 1, 11] and is omitted.

We give a brief explanation of the abstract continuation theory for  $k$ -set contractions that will be used in proof of the main results of the paper.

Let  $Z$  be a Banach space. For a bounded subset  $A \subset Z$ , let  $\Gamma_Z(A)$  denote the (Kuratovski) measure of non-compactness defined by

$$\Gamma_Z(A) = \inf\{\delta > 0 : \exists a \text{ finite number of subsets } A_i \subset A, A = \cup_i A_i, \text{diam}(A_i) \leq \delta\}.$$

Here,  $\text{diam}(A_i)$  denotes the maximum distance between the points in the set  $A_i$ . Let  $X$  and  $Y$  be Banach spaces and  $\Omega$  a bounded open subset of  $X$ . A continuous and bounded map  $N : \bar{\Omega} \rightarrow Y$  is called  $k$ -set-contractive if for any bounded  $A \subset \bar{\Omega}$  we have  $\Gamma_Y(N(A)) \leq k\Gamma_X(A)$ . Also, for a continuous and bounded map  $T : X \rightarrow Y$  we define

$$l(T) = \sup\{r \geq 0 : \forall \text{ bounded subset } A \subset X, r\Gamma_X(A) \leq \Gamma_Y T((A))\}.$$

**Theorem 2.2.** [12] *Let  $L : X \rightarrow Y$  be a Fredholm operator of index zero, and  $q(t) \in Y$  be a fixed point. Suppose that  $N : \bar{\Omega} \rightarrow Y$  is  $k$ -set contractive with  $k < l(L)$ , where  $\Omega \subset X$  is bounded, open, and symmetric about  $0 \in \Omega$ . Suppose further that:*

(i)  $Lx \neq \lambda Nx + \lambda q(t)$  for  $x \in \partial\Omega$ ,  $\lambda \in (0, 1)$  and

(ii)  $[QN(x) + Qq(t), x] \cdot [QN(-x) + Qq(t), x] < 0$ , for  $x \in \text{Ker}(L) \cap \partial\Omega$ .

where  $[\cdot, \cdot]$  is a bilinear form on  $Y \times X$  and  $Q$  is the projection of  $Y$  onto  $\text{coker}(L)$ . Then there exists  $x \in \Omega$  such that  $Lx - Nx = q(t)$

### 3. MAIN RESULTS

For the convenience of investigation, we still denote  $z(t)$  by  $x(t)$  in Eqs. (5) and (6), then the new form is obtained

$$x'(t) = q(t) - p(t)e^{x(t-\tau)} \quad (3.1)$$

with the initial condition

$$x(t) = \ln \phi(t), \quad -\tau \leq t \leq 0, \quad \phi(t) \in L([-\tau, 0], (0, \infty)), \quad (3.2)$$

where  $q(t) = r(t) - E(t)$  and  $p(t) = \frac{r(t)}{x_m(t)} \prod_{0 < t_j < t - \tau} (1 + b_j)$  are positive continuous  $\omega$ -periodic functions.

Denote  $Y = C_\omega^0$  is a linear Banach space of real-valued  $\omega$ -periodic functions on  $R$ . In  $C_\omega^0$ , for  $x \in C_\omega^0$ , the norm is defined by  $|x|_0 = \sup_{t \in R} |x(t)|$ . Let  $X = C_\omega^1$  denote the linear space of  $\omega$ -periodic functions with the first-order continuous derivative.  $C_\omega^1$  is a Banach space with norm  $|x|_1 = \max\{|x|_0, |x'|_0\}$ . let  $L : X \rightarrow Y$  be given by  $Lx = \frac{dx}{dt} = x'$ .

Since  $|Lx|_0 = |x'|_0 \leq |x|_1$ , we see that  $L$  is a bounded linear map. Next define a nonlinear map  $N : X \rightarrow Y$  by  $Nx(t) = -p(t)e^{x(t-\tau)}$ . Now, Eq. (7) has a solution  $x(t)$  if and only if  $Lx = Nx + q(t)$  for some  $x \in X$ .

**Theorem 3.1.** *Let*

$$M = \max \left\{ \left| \ln \frac{\bar{q}}{p} \right|, R_1, M_1, M_2 \right\}, \quad R_1 = \ln \frac{\bar{q}}{p^m} + 2\omega\bar{q}, \quad M_1 = |q|_0 + |p|_0 e^{R_1},$$

$$M_2 = M_1\omega + \max \left\{ \left| \ln \frac{\bar{q}}{|p|_0} \right|, \left| \ln \frac{\bar{q}}{p^m} \right| \right\}.$$

Where

$$\bar{q} = \frac{1}{\omega} \int_0^\omega q(t)dt, \bar{p} = \frac{1}{\omega} \int_0^\omega p(t)dt, p^m = \min_{t \in [0, \omega]} p(t), |q|_0 = \max_{t \in [0, \omega]} q(t), |p|_0 = \max_{t \in [0, \omega]} p(t).$$

Suppose that the condition  $|p|_0 e^M < 1$  holds, then Eq. (7) has at least one  $\omega$ -periodic solution. Therefore, the system (1) has at least one positive  $\omega$ -periodic solution.

*Proof.* Let  $\Omega = \{x(t) \in X : |x|_1 < r\}$ , where  $r > M$  such that  $k_0 = |p|_0 e^r < 1$ ,  $A \subset \bar{\Omega}$  be a bounded subset and let  $\eta = \Gamma_X(A)$ , then for any  $\varepsilon > 0$ , there is a finite family of subset  $A_i$  with  $A = \cup_i A_i$  and  $diam_1(A_i) \leq \eta + \varepsilon$ . Now it follows from the fact that  $g(t, x_1) = p(t)e^{x_1}$  is uniformly continuous on any compact subset of  $R \times R$  that, for any  $x, u \in A_i$ , may as well let  $x \leq u$ , there exists a  $\sigma \in (x, u)$ , such that

$$\begin{aligned} |Nx - Nu|_0 &= \sup_{0 \leq t \leq \omega} |g(t, x(t-\tau)) - g(t, u(t-\tau))| \leq |p|_0 \sup_{0 \leq t \leq \omega} |e^{x(t-\tau)} - e^{u(t-\tau)}| \\ &= |p|_0 \sup_{0 \leq t \leq \omega} |e^{\sigma(t-\tau)}| |x(t-\tau) - u(t-\tau)|. \end{aligned}$$

In this case,  $|\sigma|_1 < r$ , therefore,  $|Nx - Nu|_0 \leq |p|_0 e^r |x - u|_0 \leq k_0 \eta + k_0 \varepsilon$ . i.e.

$$\Gamma_Y(N(A)) \leq k_0 \Gamma_X(A).$$

Therefore, the map  $N$  is  $k_0$ -set contractive.

If  $Lx = \lambda Nx + \lambda q(t)$  for any  $x(t) \in X, \lambda \in (0, 1)$ , i.e.

$$x'(t) = \lambda \left( q(t) - p(t)e^{x(t-\tau)} \right). \quad (3.3)$$

Since  $x(t)$  is a real-valued  $\omega$ -periodic function, integrating (9) from 0 to  $\omega$ , we have

$$\int_0^\omega p(t)e^{x(t-\tau)} dt = \int_0^\omega q(t)dt. \quad (3.4)$$

Therefore

$$\int_0^\omega |x'(t)|dt \leq \lambda \left[ \int_0^\omega q(t)dt + \int_0^\omega p(t)e^{x(t-\tau)} dt \right] < 2 \int_0^\omega q(t)dt = 2\omega\bar{q}. \quad (3.5)$$

Let  $s = t - \tau$ , then

$$\int_0^\omega p(t)e^{x(t-\tau)} dt = \int_{-\tau}^{\omega-\tau} p(s+\tau)e^{x(s)} ds \geq p^m \int_{-\tau}^{\omega-\tau} e^{x(s)} ds = p^m \int_0^\omega e^{x(s)} ds.$$

It follows from (10) that

$$\int_0^\omega q(t)dt \geq p^m \int_0^\omega e^{x(s)} ds = \omega p^m e^{x(\xi_1)},$$

for some  $\xi_1 \in [0, \omega]$ . Therefore, we have

$$x(\xi_1) \leq \ln \frac{\bar{q}}{p^m}. \quad (3.6)$$

From (11) and (12), we can see that

$$x(t) \leq x(\xi_1) + \int_0^\omega |x'(t)| \leq \ln \frac{\bar{q}}{p^m} + 2\omega\bar{q} = R_1.$$

From (9), we have

$$x'(t) \leq \lambda[q(t) + p(t)e^{x(t-\tau)}] < |q|_0 + |p|_0 e^{R_1}.$$

So that

$$|x'|_0 < |q|_0 + |p|_0 e^{R_1} \equiv M_1. \quad (3.7)$$

On the other hand, there exists a  $\xi_2 \in [0, \omega]$ , such that

$$\int_0^\omega p(t)e^{x(t-\tau)} dt = \int_{-\tau}^{\omega-\tau} p(s+\tau)e^{x(s)} ds = p(\xi_2) \int_{-\tau}^{\omega-\tau} e^{x(s)} ds = p(\xi_2) \int_0^\omega e^{x(s)} ds.$$

Hence, from (10), we have

$$\int_0^\omega e^{x(t)} dt = \frac{\int_0^\omega q(t) dt}{p(\xi_2)}.$$

So, there exists a  $\xi_3 \in [0, \omega]$ , such that

$$e^{x(\xi_3)} = \frac{\bar{q}}{p(\xi_2)},$$

i.e.

$$x(\xi_3) = \ln \frac{\bar{q}}{p(\xi_2)}.$$

It is clear that

$$\ln \frac{\bar{q}}{|p|_0} \leq x(\xi_3) \leq \ln \frac{\bar{q}}{p^m}.$$

Therefore,

$$|x(\xi_3)| \leq \max \left\{ \ln \frac{\bar{q}}{|p|_0}, \ln \frac{\bar{q}}{p^m} \right\} \equiv M_0. \quad (3.8)$$

We get

$$|x|_0 \leq |x(\xi_3)| + \int_0^\omega |x'|_0 dt < M_0 + M_1 \omega \equiv M_2.$$

This implies that

$$|x|_1 \leq M \equiv \max \left\{ \left| \ln \frac{\bar{q}}{p} \right|, R_1, M_1, M_2 \right\}.$$

Therefore, for  $x(t) \in \partial\Omega$ ,  $\lambda \in (0, 1)$ ,

$$Lx \neq \lambda Nx + \lambda q(t).$$

This implies that the condition (i) of Theorem 2.2 holds.

In the following we define a bounded bilinear form  $[\cdot, \cdot]$  on  $Y \times X$  by  $[y, x] = \int_0^\omega y(t)x(t)dt$ . And define  $Q : Y \rightarrow \text{coker}(L)$  by  $y \rightarrow \int_0^\omega y(t)dt$ . For  $x \in \ker(L) \cap \partial\Omega$ , we can get  $x = r$  or  $x = -r$ . So, we have

$$\begin{aligned} & [QN(x) + Qq(t), x] \cdot [QN(-x) + Qq(t), x] = \\ & r^2 \omega^2 \left[ \int_0^\omega q(t)dt - e^r \int_0^\omega p(t)dt \right] \left[ \int_0^\omega q(t)dt - e^{-r} \int_0^\omega p(t)dt \right] = r^2 \omega^4 (\bar{q} - e^r \bar{p})(\bar{q} - e^{-r} \bar{p}). \end{aligned}$$

Since  $r > M \geq |\ln \frac{\bar{q}}{p}|$ , we have  $\bar{q} - e^r \bar{p} < 0$ ,  $\bar{q} - e^{-r} \bar{p} > 0$ . This implies that the condition (ii) of Theorem 2.2 holds.

Hence, it follows from Theorem 2.2 that there is a function  $x(t) \in \bar{\Omega} \subset X$  such that  $Lx - Nx = q(t)$ . Thus, the proof of Theorem 3.1 is completed.  $\square$

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## REFERENCES

- [1] V. Lakshmikantham, D.D. Bainov and P.S. Simenov, Theory of Impulsive Differential Equations, World Scientific Press, Singapore, 1989.
- [2] K. Gopalsamy and B.G. Zhang, On delay differential equations with impulses, J. Math. Anal. Appl., 139(1989)110-122.
- [3] D.D. Bainov and P.S. Simeonov, Periodic solutions of quasilinear impulsive differential equations with delay in the noncritical case, Int. J. Differ. Eqn. Appl., 5(2002)313-324.
- [4] D.D. Bainov and P.S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, Longman Scientific and Technical, New York, 1993.
- [5] H.F. Huo, W.T. Li and X.Z. Liu, Existence and global attractivity of positive periodic solution of an impulsive delay differential equation, Appl. Anal., 83 (2004) 1279-1290.
- [6] S.L. Sun and L.S. Chen, Existence of positive periodic solution of an impulsive delay Logistic model, Appl. Math. Comp., 184 (2007) 617-623.
- [7] Y. Kuang, Delay Differential Equations: with Applications in Population Dynamics, Academic Press, Boston, 1993.
- [8] Z.D. Liu and Y.P. Mao, Existence theorem for periodic solutions of higher order nonlinear differential equations, J. Math. Anal. Appl., 216 (1997) 481-490.
- [9] H. Fang and J.B. Li, On the existence of periodic solution of a neutral delay model of single-species population growth, J. Math. Anal. Appl., 259 (2001) 8-17.
- [10] W.V. Petryshyn and Z.S. Yu, Existence theorems for higher order nonlinear periodic boundary value problem, Nonlinear Anal., 6 (1982)943-969.
- [11] J.R. Yan and A.M. Zhao, Oscillation and stability of linear impulsive delay differential equations, J. Math. Anal. Appl., 227 (1998)187-194.
- [12] G. Hetzer, Some remarks on  $\Phi_+$  operators and on the coincidence degree for Fredholm equations with noncompact nonlinear perturbations, Ann. Soc. Sci. Bruxelles 89, No. IV(1975)

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