Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 2(2011), Pages 238-245.

MODIFIED VARIATIONAL ITERATION METHOD FOR HEAT EQUATION USING HE'S POLYNOMIALS

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Abstract. In this paper, we apply the modified variational iteration method (MVIM) for solving heat transfer problems. The proposed modification is made by introducing He's polynomials in the correction functional. The suggested algorithm is quite efficient and is practically well suited for use in these problems. The proposed iterative scheme finds the solution without any discretization, linearization, or restrictive assumptions. Several examples are given to verify the reliability and efficiency of the method. The fact that the proposed technique solves nonlinear problems without using the Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method.

1. Introduction

This paper is devoted to the study of heat transfer problems which are known to arise in a variety of physical phenomenon and applied sciences [1, 4]. He [5-20] developed the variational iteration and homotopy perturbation methods for solving linear, nonlinear, initial and boundary value problems. Moreover, He realized the physical significance of the variational iteration method, its compatibility with the physical problems and applied this promising technique to a wide class of linear and nonlinear, ordinary, partial, deterministic or stochastic differential equation [5-8,23]. The homotopy perturbation method was also developed by He by merging two techniques, the standard homotopy and the perturbation. The homotopy perturbation method was formulated by taking full advantage of the standard homotopy and perturbation methods. In these methods the solution is given in an infinite series usually converging to an accurate solution. In a later work Ghorbani et al. [2, 3] split the nonlinear term into a series of polynomials calling them as the He's polynomials. Most recently, Noor and Mohyud-Din used this concept for solving nonlinear boundary value problems [24-28]. The basic motivation of this paper is the extension of the Modified Variational Iteration Method (MVIM) which is formulated by the coupling of variational iteration method and He's polynomials

2000 *Mathematics Subject Classification.* 35A07, 35Q53.
Key words and phrases. Bourgain spaces; KdV equation; local smoothing effect.

Submitted August 12, 2010. Accepted March 2, 2011.

for solving the heat transfer problems. The general heat equation with variable coefficient, with the indicated initial condition has the form [4]

$$\frac{\partial u}{\partial t} = A(x,y,z,t)\frac{\partial^2 u}{\partial x^2} + B(x,y,z,t)\frac{\partial^2 u}{\partial y^2} + C(x,y,z,t)\frac{\partial^2 u}{\partial z^2} + D(x,y,z,t)$$
$$u(x,y,z,0) = f(x,y,z). \tag{1.1}$$

The proposed MVIM provides the solution in a rapid convergent series which may lead the solution to a closed form. In this technique, the correction functional is developed [5-8] and the Lagrange multipliers are calculated optimally via variational theory. The use of Lagrange multipliers reduces the successive application of the integral operator and the cumbersome of huge computational work while still maintaining a very high level of accuracy. Finally, He's polynomials are introduced in the correction functional and the comparison of like powers of p gives solutions of various orders. The proposed iterative scheme takes full advantage of variational iteration and the homotopy perturbation methods and absorbs all the positive features of the coupled techniques. It is worth mentioning that the suggested method is applied without any discretization, restrictive assumption or transformation and is free from round off errors.

2. Variational iteration method

For the purpose of illustration of the methodology to the proposed method, using variational iteration method, we begin by considering a differential equation in the formal form,

$$L[u(x,t)] + N[u(x,t)] = g(x,t), \tag{2.1}$$

where L is a linear operator, N a nonlinear operator and $g(x,t)$ is the source inhomogeneous term. According to the variational iteration method, we can construct a correction functional for (2.1) as follows;

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{Lu_n(x,\tau) + N\tilde{u_n}(x,\tau) - g(x,\tau)\right\} d\tau, \qquad n \geq 0,$$

where λ is a general Lagrangian multiplier [21], which can be identified optimally via the variational theory, the subscript n denotes the nth order approximation, and $\tilde{u_n}$ is considered as a restricted variation [7, 21] i.e., $\delta\tilde{u_n} = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_n(x,t), n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

3. Homotopy perturbation method

In this section to illustrate the basic ideas of this method, we consider the following equation :

$$L[u(x,t)] + N[u(x,t)] = g(x,t), \ r \in \Omega, \tag{3.1}$$

with the boundary condition of:

$$B(u, \frac{\partial u}{\partial n}) = 0, \ r \in \Gamma, \tag{3.2}$$

where L is a linear operator, N a nonlinear operator and $g(x,t)$ is the source inhomogeneous term, B is a boundary operator and Γ is the boundary of the domain Ω. Homotopy perturbation structure is shown as follows:

$$H(v,p) = (1-p) * [L(v) - L(u_0)] + p\ [L(u) + N(u) - g(x,t)] = 0, \tag{3.3}$$

In Eq.(3.3), $p \in [0,1]$ is an embedding parameter and is the first approximation that satisfies the boundary conditions. We can assume that the solution of Eq. (3.3) can be written as a power series in p, as following:

$$v = v_0 + p\ v_1 + p^2 v_2 + \ \ldots, \tag{3.4}$$

The comparisons of like powers of p give solutions of various orders and the best approximation is:

$$u = \lim_{p \longrightarrow 1}\ v = v_0 +\ v_1 +\ v_2 +\ \ldots. \tag{3.5}$$

The convergence of series (3.5) is discussed in [19]. The method considers the nonlinear term $N[u]$ as

$$N[u] = \sum_{i=0}^{+\infty} p^i H_i = H_0 + pH_1 + p^2 H_2 + \cdots$$

where H_n's are the so-called He's polynomials [2], which can be calculated by using the formula

$$H_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i u_i\right)\right)_{p=0},\ \ n = 0,1,2,\cdots$$

4. Modified Variational Iteration Method (MVIM)

To illustrate the basic idea of the MVIM, we consider the following general differential equation:

$$L[u(x,t)] + N[u(x,t)] = g(x,t), \tag{4.1}$$

where L is a linear operator, N a nonlinear operator and $g(x,t)$ is the source inhomogeneous term. According to section (2) we can construct a correct functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{ Lu_n(x,\tau) + N\tilde{u_n}(x,\tau) - g(x,\tau) \right\} d\tau, \qquad n \geq 0, \tag{4.2}$$

Now, we apply the homotopy perturbation method,

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = u_0(x,t) + p\int_0^t \lambda \left[\sum_{n=0}^{\infty} p^n \left(L(v_n(x,\tau) + N(v_n(x,\tau)) - g(x,\tau)\right)\right] d\tau.$$

which is the modified variational iteration method (MVIM) and is formulated by the coupling of variational iteration method and He's polynomials. The comparison of like powers of p gives solutions of various orders.

5. MVIM for Heat equation

In order to assess the advantages and the accuracy of MVIM for solving linear equations, we will consider the following three examples.

Example 1: Consider the following heat equation with the indicated initial conditions:

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial y^2} + C\frac{\partial^2 u}{\partial z^2} + D, \tag{5.1}$$

with the initial condition

$$u(x, y, z, 0) = a_1 x + a_2 x^2 + b_1 y + b_2 y^2 + c_1 z + c_2 z^2,$$

where A,B,C and D are constants. To solve Eq. (5.1) By means of MVIM, we choose

$$L(u) = u_t, \qquad N(u) = -Au_{xx} - Bu_{yy} - Cu_{zz},$$

where L is a linear and N is a nonlinear operators. The correction functional for the above problem is given by

$$u_{n+1} = u_n + \int_0^t \lambda(\tau)(u_{n_\tau} - A\tilde{u}_{n_{xx}} - B\tilde{u}_{n_{yy}} - C\tilde{u}_{n_{zz}} - D)d\tau,$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(\tau) = -1$, consequently

$$u_{n+1} = u_n - \int_0^t (u_{n_\tau} - Au_{n_{xx}} - Bu_{n_{yy}} - Cu_{n_{zz}} - D)d\tau.$$

Applying the modified variational iteration method, we have

$$\begin{aligned} u_0 + pu_1 + p^2u_2 + \cdots = u(x, y, z, 0) \quad & +p\int_0^t A(u_{0_{xx}} + pu_{1_{xx}} + p^2u_{2_{xx}} + ...)d\tau \\ & +p\int_0^t B(u_{0_{yy}} + pu_{1_{yy}} + p^2u_{2_{yy}} + ...)d\tau \\ & +p\int_0^t C(u_{0_{zz}} + pu_{1_{zz}} + p^2u_{2_{zz}} + ...)d\tau \\ & +\int_0^t D d\tau. \end{aligned}$$

Comparing the coefficient of like powers of p

$$\begin{aligned} p^{(0)} &: u_0(x, y, z, t) = u(x, y, z, 0) + \int_0^t D d\tau, \\ p^{(1)} &: u_1(x, y, z, t) = \int_0^t (Au_{0_{xx}} + Bu_{0_{yy}} + Cu_{0_{zz}})d\tau, \\ p^{(2)} &: u_2(x, y, z, t) = \int_0^t (Au_{1_{xx}} + Bu_{1_{yy}} + Cu_{1_{zz}})d\tau, \\ &\vdots \end{aligned}$$

Therefore

$$\begin{aligned} u_0(x,y,z,t) =& \ a_1x + a_2x^2 + b_1y + b_2y^2 + c_1z + c_2z^2 + Dt, \\ u_1(x,y,z,t) =& \ 2Aa_2t + 2Bb_2t + 2Cc_2t, \\ u_n(x,y,z,t) =& \ 0, \ n \geq 2 \end{aligned}$$

In the same manner the rest of the components of the iteration formula can be obtained, which is the exact solution.

Example 2: Let us solve the following partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{1}{6}(x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}), \tag{5.2}$$

with the initial condition

$$u(x,y,z,0) = x^2y^2z^2.$$

The correction functional for the above problem is given by

$$u_{n+1} = u_n + \int_0^t \lambda(\tau)(u_{n_\tau} - \frac{1}{6}x^2\tilde{u}_{n_{xx}} - \frac{1}{6}y^2\tilde{u}_{n_{yy}} - \frac{1}{6}z^2\tilde{u}_{n_{zz}})d\tau.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(\tau) = -1$, consequently

$$u_{n+1} = u_n - \int_0^t (u_{n_\tau} - \frac{1}{6}x^2u_{n_{xx}} - \frac{1}{6}y^2u_{n_{yy}} - \frac{1}{6}z^2u_{n_{zz}})d\tau.$$

Applying the modified variational iteration method, we have

$$\begin{aligned} u_0 + pu_1 + p^2u_2 + \cdots = u(x,y,z,0) \qquad & +p\int_0^t \frac{1}{6}x^2(u_{0_{xx}} + pu_{1_{xx}} + p^2u_{2_{xx}} + ...)d\tau \\ & +p\int_0^t \frac{1}{6}y^2(u_{0_{yy}} + pu_{1_{yy}} + p^2u_{2_{yy}} + ...)d\tau \\ & +p\int_0^t \frac{1}{6}z^2(u_{0_{zz}} + pu_{1_{zz}} + p^2u_{2_{zz}} + ...)d\tau. \end{aligned}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} & p^{(0)} : u_0(x,y,z,t) = u(x,y,z,0), \\ & p^{(1)} : u_1(x,y,z,t) = \int_0^t \frac{1}{6}(x^2u_{0_{xx}} + y^2u_{0_{yy}} + z^2u_{0_{zz}})d\tau, \\ & p^{(2)} : u_2(x,y,z,t) = \int_0^t \frac{1}{6}(x^2u_{1_{xx}} + y^2u_{1_{yy}} + z^2u_{1_{zz}})d\tau, \\ & \qquad \vdots \end{aligned}$$

Therefore

$$\begin{aligned}
u_0(x,y,z,t) &= x^2y^2z^2,\\
u_1(x,y,z,t) &= x^2y^2z^2t,\\
u_2(x,y,z,t) &= x^2y^2z^2\frac{t^2}{2!},\\
u_3(x,y,z,t) &= x^2y^2z^2\frac{t^3}{3!},\\
&\vdots
\end{aligned}$$

and so on. Thus the series solution is given by

$$u(x,y,z,t) = x^2y^2z^2\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right) = x^2y^2z^2e^t,$$

which is the exact solution [1].

Example 3: Consider the following partial differential equation, with specified initial conditions:

$$\frac{\partial u}{\partial t} - e^{(-xt)}\frac{\partial^2 u}{\partial x^2} - e^{(-yt)}\frac{\partial^2 u}{\partial y^2} - e^{(-zt)}\frac{\partial^2 u}{\partial z^2} = 0, \tag{5.3}$$

with the initial condition

$$u(x,y,z,0) = x^2y^2z^2.$$

The correction functional for the above problem is given by

$$u_{n+1} = u_n + \int_0^t \lambda(\tau)(u_{n_\tau} - e^{(-x\tau)}\tilde{u}_{n_{xx}} - e^{(-y\tau)}\tilde{u}_{n_{yy}} - e^{(-z\tau)}\tilde{u}_{n_{zz}})d\tau.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(\tau) = -1$, consequently

$$u_{n+1} = u_n - \int_0^t (u_{n_\tau} - e^{(-x\tau)}u_{n_{xx}} - e^{(-y\tau)}u_{n_{yy}} - e^{(-z\tau)}u_{n_{zz}})d\tau.$$

Applying the modified variational iteration method, we have

$$\begin{aligned}
u_0 + pu_1 + p^2u_2 + \cdots = u(x,y,z,0) \qquad &+p\int_0^t e^{(-x\tau)}(u_{0_{xx}} + pu_{1_{xx}} + p^2u_{2_{xx}} + ...)d\tau\\
&+p\int_0^t e^{(-y\tau)}(u_{0_{yy}} + pu_{1_{yy}} + p^2u_{2_{yy}} + ...)d\tau\\
&+p\int_0^t e^{(-z\tau)}(u_{0_{zz}} + pu_{1_{zz}} + p^2u_{2_{zz}} + ...)d\tau.
\end{aligned}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned}
p^{(0)} : u_0(x,y,z,t) &= u(x,y,z,0),\\
p^{(1)} : u_1(x,y,z,t) &= \int_0^t (e^{(-x\tau)}u_{0_{xx}} + e^{(-y\tau)}u_{0_{yy}} + e^{(-z\tau)}u_{0_{zz}})d\tau,\\
p^{(2)} : u_2(x,y,z,t) &= \int_0^t (e^{(-x\tau)}u_{1_{xx}} + e^{(-y\tau)}u_{1_{yy}} + e^{(-z\tau)}u_{1_{zz}})d\tau,\\
&\vdots
\end{aligned}$$

Therefore

$$
\begin{aligned}
u_0(x,y,z,t) &= x^2y^2z^2,\\
u_1(x,y,z,t) &= \frac{1}{xyz}(2y^3z^3+2x^3z^3+2x^3y^3-2e^{(-xt)}y^3z^3-2e^{(-yt)}x^3z^3\\
&- 2e^{(-zt)}x^3y^3),\\
u_2(x,y,z,t) &= \frac{-4z^2e^{(-xt)}}{yx}-\frac{4y^2e^{(-xt)}}{zx}-\frac{4z^2y^2e^{(-xt)}}{x^4}+\frac{7z^2y^2e^{(-2xt)}}{2x^4}\\
&- \frac{4x^2e^{(-yt)}}{zy}-\frac{4z^2x^2e^{(-yt)}}{y^4}+\frac{7z^2x^2e^{(-2yt)}}{2y^4}-\frac{4z^2e^{(-yt)}}{yx}\\
&- \frac{4x^2e^{(-zt)}}{zy}-\frac{4y^2x^2e^{(-zt)}}{z^4}+\frac{7y^2x^2e^{(-2zt)}}{2z^4}-\frac{4y^2e^{(-zt)}}{zx}\\
&+ \frac{z^2y^2t^2e^{(-2xt)}}{x^2}+\frac{3z^2x^2te^{(-2yt)}}{y^3}+\frac{x^2y^2t^2e^{(-2zt)}}{z^2}+\frac{4y^2}{zx}\\
&+ \frac{z^2x^2t^2e^{(-2yt)}}{y^2}+\frac{4z^2e^{-(x+y)t}}{yx}+\frac{4z^2}{yx}+\frac{4x^2}{zy}+\frac{3y^2x^2te^{(-2zt)}}{z^3}\\
&+ \frac{y^2z^2}{2x^4}+\frac{y^2x^2}{2z^4}+\frac{4y^2e^{-(x+z)t}}{zx}+\frac{4x^2e^{-(y+z)t}}{zy}+\frac{z^2x^2}{2y^4}\\
&+ \frac{3y^2z^2te^{(-2xt)}}{x^3},\\
&\vdots
\end{aligned}
$$

which is in full agreement with [1].

6. Conclusions

In this paper, we develop the modified variational iteration method for solving linear problems. We used the modified variational iteration method for solving the heat transfer problems with variable coefficient. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. The proposed method is successfully implemented by using the initial conditions only.

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