Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 3 Issue 2(2011), Pages 61-69.

ON QUASI-POWER INCREASING SEQUENCES GENERAL CONTRACTIVE CONDITION OF INTEGRAL TYPE

(COMMUNICATED BY MOHAMMAD SAL MOSLEHIAN)

W.T. SULAIMAN

Abstract. A general result concerning absolute summability of infinite series by quasi-power increasing sequence is proved. Our result gives three improvements to the result of Sevli and Leindler [4].

1. Introduction

Let $\sum a_n$ be an infinite series with partial sım (s_n), A denote a lower triangular matrix. The series $\sum a_n$ is said to be absolutely A-summable of order $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left| T_n - T_{n-1} \right|^k < \infty, \tag{1.1}$$

where

$$T_n = \sum_{v=0}^{n} a_{nv} s_v. \tag{1.2}$$

The series $\sum a_n$ is summable $|A|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left| T_n - T_{n-1} \right|^k < \infty. \tag{1.3}$$

Let t_n denote the nth $(C, 1)$ mean of the sequence (na_n), i,e.,

$$t_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v.$$

A positive sequence $\gamma = (\gamma_n)$ is said to be a quasi$-\beta-$power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$K n^{\beta} \gamma_n \geq m^{\beta} \gamma_m \tag{1.4}$$

2000 *Mathematics Subject Classification.* 40F35, 40D25.
Key words and phrases. Absolute summability, Quasi-power increasing sequence.

Submitted January 15, 2011. Published May 4, 2011.

holds for all $n \geq m \geq 1$. It may be mentioned that every almost increasing sequence is a quasi-$\beta-$power increasing sequence for any nonnegative β, but the converse need not be true.

A positive sequence $\gamma = (\gamma_n)$ is said to be a quasi$-f-$power increasing sequence if (see[5]) there exists a constant $K = K(\gamma, f) \geq 1$ such that

$$K f_n \gamma_n \geq f_m \gamma_m \tag{1.5}$$

holds for all $n \geq m \geq 1$.

Two lower triangular matrices $\overline{A}$ and $\hat{A}$ are associated with A as follows

$$\overline{a}_{nv} = \sum_{r=v}^{n} a_{nr}, \quad n, v = 0, 1, \ldots, \tag{1.6}$$

$$\hat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}, \quad n = 1, 2, \ldots, \qquad \hat{a}_{00} = \overline{a}_{00} = a_{00}. \tag{1.7}$$

Sevli and Leindler [4] proved the following result

Theorem 1.1. *Let A be lower triangular matrix with nonnegative entries satisfying*

$$a_{n-1,v} \geq a_{n,v} \quad \textit{for} \ \ n \geq v + 1, \tag{1.8}$$

$$\overline{a}_{n0} = 1, \quad n = 0, 1, \ldots, \tag{1.9}$$

$$n a_{nn} = O(1), \quad \textit{as } n \to \infty, \tag{1.10}$$

$$\sum_{n=1}^{m} \lambda_n = o(m), \quad m \to \infty, \tag{1.11}$$

$$\sum_{n=1}^{m} |\Delta \lambda_n| = o(m), \quad m \to \infty. \tag{1.12}$$

If (X_n) is a quasi-$f-$increasing sequence satisfying

$$\sum_{n=1}^{m} n^{-1} |t_n|^k = O(X_m), \quad m \to \infty, \tag{1.13}$$

$$\sum_{n=1}^{\infty} n X_n (\beta, \mu) |\Delta |\Delta \lambda_n|| < \infty, \tag{1.14}$$

then the series $\sum a_n \lambda_n$ is summable $|A|_k$, $k \geq 1$, where $(f_n) = \left(n^\beta (\log n)^\mu\right)$, $\mu \geq 0$, $0 \leq \beta < 1$, and $X_n(\beta, \mu) = \max\left(n^\beta (\log n)^\mu X_n, \log n\right)$. We name the conditions

$$\sum_{n=1}^{m} \frac{1}{n \left(n^\beta \log^\gamma n X_n\right)^{k-1}} |t_n|^k = O\left(m^\beta \log^\gamma m X_m\right), \quad m \to \infty, \tag{1.15}$$

$$\lambda_n \to 0, \ \textit{as } n \to \infty, \tag{1.16}$$

$$\sum_{n=1}^{\infty} n^{\beta+1} \log^\gamma n X_n |\Delta |\Delta \lambda_n|| < \infty, \tag{1.17}$$

$$\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v} = O(a_{nn}), \quad 1/n a_{nn} = O(1). \tag{1.18}$$

2. Lemmas

Lemma 2.1. [1]. *Let A be as defined in theorem 1.1, then*

$$\hat{a}_{n,v+1} \leq a_{nn} \text{ for } n \geq v+1, \tag{2.1}$$

and

$$\sum_{n=v+1}^{m+1} \hat{a}_{n,v+1} \leq 1, \quad v=0,1,\ldots\ . \tag{2.2}$$

Lemma 2.2. *Condition (1.15) is weaker than (1.13).*

Proof. Suppose that (1.13) is satisfied. Since $n^{\beta}\log^{\gamma} nX_n$ is non-decreasing, then

$$\sum_{n=1}^{m} \frac{1}{n\left(n^{\beta}\log^{\gamma} nX_n\right)^{k-1}} |t_n|^k = O(1) \sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m),$$

while if (1.15) is satisfied, we have

$$\begin{aligned}
\sum_{n=1}^{m} \frac{1}{n} |t_n|^k &= \sum_{n=1}^{m} \frac{1}{n\left(n^{\beta}\log^{\mu} nX_n\right)^{k-1}} |t_n|^k \left(n^{\beta}\log^{\mu} nX_n\right)^{k-1} \\
&= \sum_{n=1}^{m-1} \left(\sum_{v=1}^{n} \frac{1}{v\left(v^{\beta}\log^{\mu} vX_v\right)^{k-1}} |t_v|^k\right) \Delta\left(n^{\beta}\log^{\mu} nX_n\right)^{k-1} \\
&\quad + \sum_{n=1}^{m} \frac{1}{n\left(n^{\beta}\log^{\mu} nX_n\right)^{k-1}} |t_n|^k \left(m^{\beta}\log^{\mu} mX_m\right)^{k-1} \\
&= O(1) \sum_{n=1}^{m-1} n^{\beta}\log^{\gamma} nX_n \left|\Delta\left(n^{\beta}\log^{\mu} nX_n\right)^{k-1}\right| \\
&\quad + O\left(m^{\beta}\log^{\gamma} mX_m\right)\left(m^{\beta}\log^{\mu} mX_m\right)^{k-1} \\
&= O\left((m-1)^{\beta}\log^{\gamma}(m-1)X_{m-1}\right) \sum_{n=1}^{m-1} \left(\left((n+1)^{\beta}\log^{\mu}(n+1)X_{n+1}\right)^{k-1}\right. \\
&\quad \left. - \left(n^{\beta}\log^{\mu} nX_n\right)^{k-1}\right) + O\left(m^{\beta}\log^{\mu} mX_m\right)^{k} \\
&= O\left(m^{\beta}\log^{\mu} mX_m\right)\left(m^{\beta}\log^{\mu} mX_m\right)^{k-1} + O\left(m^{\beta}\log^{\mu} mX_m\right)^{k} \\
&= O\left(m^{\beta}\log^{\mu} mX_m\right)^{k} \neq O(X_m).
\end{aligned}$$

Therefore (1.15) implies (1.13) but not conversely. □

Lemma 2.3. *Condition (1.16) and (1.17) imply*

$$m^{\beta+1}\log^{\mu} mX_m\left|\Delta\lambda_m\right| = O(1), \quad m\to\infty \tag{2.3}$$

$$\sum_{n=1}^{\infty} n^{\beta}\log^{\mu} nX_n\left|\Delta\lambda_n\right| = O(1), \tag{2.4}$$

and

$$n^{\beta}\log^{\mu} nX_n\left|\lambda_n\right| = O(1), \quad n\to\infty. \tag{2.5}$$

Proof. As $\Delta\lambda_n \to 0$, and $n^{\beta}\log^{\mu} nX_n$ is non-decreasing, we have

$$\begin{aligned}
n^{\beta+1}\log^{\mu} nX_n\left|\Delta\lambda_n\right| &= n^{\beta+1}\log^{\mu} nX_n\sum_{v=n}^{\infty}\Delta\left|\Delta\lambda_v\right| \\
&= O(1)\sum_{v=n}^{\infty} v^{\beta+1}\log^{\mu} vX_v\left|\Delta\left|\Delta\lambda_v\right|\right| \\
&= O(1).
\end{aligned}$$

This proves (2.3). To prove (2.4), we observe that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\beta}\log^{\mu} nX_n\left|\Delta\lambda_n\right| &= \sum_{n=1}^{\infty} n^{\beta}\log^{\mu} nX_n\sum_{v=n}^{\infty}\Delta\left|\Delta\lambda_v\right| \\
&\leq \sum_{v=1}^{\infty}\left|\Delta\left|\Delta\lambda_v\right|\right|\sum_{n=1}^{v} n^{\beta}\log^{\mu} nX_n \\
&= O(1)\sum_{v=1}^{\infty} v^{\beta+1}\log^{\mu} vX_v\left|\Delta\left|\Delta\lambda_v\right|\right| \\
&= O(1), \quad \text{by (1.17)}.
\end{aligned}$$

Finally

$$\begin{aligned} n^\beta \log^\mu nX_n |\lambda_n| &= n^\beta \log^\mu nX_n \sum_{v=n}^{\infty} \Delta |\lambda_v| \\ &\leq \sum_{v=n}^{\infty} v^\beta \log^\mu vX_v |\Delta\lambda_v| \\ &= O(1), \quad \text{by } (2.4). \end{aligned}$$

□

3. Main Result

Theorem 3.1. *Let A satisfies conditions (1.8)-(1.10) and (1.18), let (λ_n) be a sequence satisfies (1.16). If (X_n) is a quasi-f-power increasing sequence satisfying (1.15) and (1.17), then the series $\sum a_n\lambda_n$ is summable $|A|_k$, $k \geq 1$, where $(f_n) = \left(n^\beta (\log n)^\mu\right)$, $\mu \geq 0$, $0 \leq \beta < 1$.*

Proof. Let x_n be the nth term of the A-transform of the series $\sum a_n\lambda_n$. By definition, we have

$$x_n = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \overline{a}_{nv}\lambda_v a_v,$$

and hence

$$T_n := x_n - x_{n-1} = \sum_{v=0}^{n} v^{-1}\hat{a}_{nv}\lambda_v v a_v. \tag{3.1}$$

Applying Abel's transformation,

$$\begin{aligned} T_n &= \frac{n+1}{n} a_{nn}\lambda_n t_n + \sum_{v=1}^{n-1} \Delta_v \hat{a}_{nv}\lambda_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v t_v + \sum_{v=1}^{n-1} v^{-1}\hat{a}_{n,v}\lambda_v t_v \\ &= T_{n1} + T_{n2} + T_{n3} + T_{n4}. \end{aligned} \tag{3.2}$$

To complete the proof, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} |T_{nj}|^k \quad < \quad \infty, \quad j = 1, 2, 3, 4.$$

Applying Holder's inequality, we have, in view of (2.5),

$$\begin{aligned}
\sum_{n=1}^{m} n^{k-1}\left|T_{n1}\right|^{k} &= \sum_{n=1}^{m} n^{k-1}\left|\frac{n+1}{n}a_{nn}\lambda_n t_n\right|^{k} \\
&= O(1)\sum_{n=1}^{m}\left(na_{nn}\right)^{k}\frac{1}{n}\left|t_n\right|^{k}\left|\lambda_n\right|^{k} \\
&= O(1)\sum_{n=1}^{m}\frac{1}{n}\left|t_n\right|^{k}\left|\lambda_n\right|^{k} \\
&= O(1)\sum_{n=1}^{m}\frac{1}{n\left(n^{\beta}\log^{\mu} nX_n\right)^{k-1}}\left|t_n\right|^{k}\left|\lambda_n\right|\left(\left|\lambda_n\right| n^{\beta}\log^{\mu} nX_n\right)^{k-1} \\
&= O(1)\sum_{n=1}^{m}\frac{1}{n\left(n^{\beta}\log^{\mu} nX_n\right)^{k-1}}\left|t_n\right|^{k}\left|\lambda_n\right| \\
&= O(1)\sum_{n=1}^{m-1}\left|\Delta\lambda_n\right|\sum_{v=1}^{n}\frac{1}{v\left(v^{\beta}\log^{\mu} vX_v\right)^{k-1}}\left|t_v\right|^{k} \\
&\quad +O(1)\left|\lambda_m\right|\sum_{n=1}^{m}\frac{1}{n\left(n^{\beta}\log^{\mu} nX_n\right)^{k-1}}\left|t_n\right|^{k} \\
&= O(1)\sum_{n=1}^{m-1}\left|\Delta\lambda_n\right| n^{\beta}\log^{\mu} nX_n + O(1)\left|\lambda_m\right| m^{\beta}\log^{\mu} mX_m \\
&= O(1).
\end{aligned}$$

As, (see[2]),

$$\sum_{v=0}^{n-1} |\Delta_v \hat{a}_{nv}| = \sum_{v=0}^{n-1} (a_{n-1,v} - a_{n,v}) = 1 - 1 + a_{nn} = a_{nn},$$

therefore, in view of (2.5),

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{k-1} |T_{n2}|^k &= \sum_{n=2}^{m+1} n^{k-1} \left| \sum_{v=1}^{n-1} \Delta_v \hat{a}_{nv} \lambda_v t_v \right|^k \\
&\leq \sum_{n=2}^{m+1} n^{k-1} \sum_{v=1}^{n-1} |\Delta_v \hat{a}_{nv}| |\lambda_v|^k |t_v|^k \left(\sum_{v=1}^{n-1} |\Delta_v \hat{a}_{nv}| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} (n a_{nn})^{k-1} \sum_{v=1}^{n-1} |\Delta_v \hat{a}_{nv}| |\lambda_v|^k |t_v|^k \\
&= O(1) \sum_{v=1}^{m} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m} |\Delta_v \hat{a}_{nv}| \\
&= O(1) \sum_{v=1}^{m} a_{vv} |\lambda_v|^k |t_v|^k \\
&= O(1) \sum_{v=1}^{m} \frac{1}{v} |\lambda_v|^k |t_v|^k \\
&= O(1), \text{ as in the case of } T_{n1}.
\end{aligned}$$

In view of (2.4), (1.10), and (2.2),

$$
\begin{aligned}
\sum_{n=2}^{m+1} n^{k-1}\left|T_{n3}\right|^{k} &= \sum_{n=2}^{m+1} n^{k-1}\left|\sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v t_v\right|^{k} \\
&\le \sum_{n=2}^{m+1} n^{k-1}\sum_{v=1}^{n-1}\left(\hat{a}_{n,v+1}\right)^{k}\left|\Delta\lambda_v\right|\frac{\left|t_v\right|^{k}}{\left(v^{\beta}\log^{\mu} vX_v\right)^{k-1}}\left(\sum_{v=1}^{n-1}\left|\Delta\lambda_v\right| v^{\beta}\log^{\mu} vX_v\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} n^{k-1}\sum_{v=1}^{n-1}\left(\hat{a}_{n,v+1}\right)^{k}\left|\Delta\lambda_v\right|\frac{\left|t_v\right|^{k}}{\left(v^{\beta}\log^{\mu} vX_v\right)^{k-1}} \\
&= O(1)\sum_{v=1}^{m}\left|\Delta\lambda_v\right|\frac{\left|t_v\right|^{k}}{\left(v^{\beta}\log^{\mu} vX_v\right)^{k-1}}\sum_{n=v+1}^{m+1} n^{k-1}\hat{a}_{n,v+1}\left(\hat{a}_{n,v+1}\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left|\Delta\lambda_v\right|\frac{\left|t_v\right|^{k}}{\left(v^{\beta}\log^{\mu} vX_v\right)^{k-1}}\sum_{n=v+1}^{m+1} n^{k-1}\hat{a}_{n,v+1}\left(a_{nn}\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left|\Delta\lambda_v\right|\frac{\left|t_v\right|^{k}}{\left(v^{\beta}\log^{\mu} vX_v\right)^{k-1}}\sum_{n=v+1}^{m+1} \hat{a}_{n,v+1}\left(na_{nn}\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left|\Delta\lambda_v\right|\frac{\left|t_v\right|^{k}}{\left(v^{\beta}\log^{\mu} vX_v\right)^{k-1}}\sum_{n=v+1}^{m+1} \hat{a}_{n,v+1} \\
&= O(1)\sum_{v=1}^{m} v\left|\Delta\lambda_v\right|\frac{\left|t_v\right|^{k}}{v\left(v^{\beta}\log^{\mu} vX_v\right)^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}\Delta\left(v\left|\Delta\lambda_v\right|\right)\sum_{r=1}^{v}\frac{1}{r\left(r^{\beta}\log^{\mu} rX_r\right)^{k-1}}\left|t_r\right|^{k} \\
&\quad +O(1)m\left|\Delta\lambda_m\right|\sum_{v=1}^{m}\frac{1}{v\left(v^{\beta}\log^{\mu} vX_v\right)^{k-1}}\left|t_r\right|^{k} \\
&= O(1)\sum_{v=1}^{m}\left|\Delta\lambda_v\right| v^{\beta}\log^{\mu} vX_v + O(1)\sum_{v=1}^{m}\left|\Delta\left|\Delta\lambda_v\right|\right| v^{\beta+1}\log^{\mu} vX_v \\
&\quad +O(1)\left|\Delta\lambda_m\right| m^{\beta+1}\log^{\mu} mX_m \\
&= O(1).
\end{aligned}
$$

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{k-1}\left|T_{n4}\right|^k &= \sum_{n=2}^{m+1} n^{k-1}\left|\sum_{v=1}^{n-1} v^{-1}\hat{a}_{n,v}\lambda_v t_v\right|^k \\
&= \sum_{n=2}^{m+1} n^{k-1}\sum_{v=1}^{n-1}\left(va_{vv}\right)^{-k} a_{vv}\hat{a}_{n,v}\left|\lambda_v\right|^k\left|t_v\right|^k\left(\sum_{v=1}^{n-1} a_{vv}\hat{a}_{n,v}\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(na_{nn}\right)^{k-1}\sum_{v=1}^{n-1}\left(va_{vv}\right)^{-1} a_{vv}\hat{a}_{n,v}\left|\lambda_v\right|^k\left|t_v\right|^k \\
&= O(1)\sum_{v=1}^{m} a_{vv}\left|\lambda_v\right|^k\left|t_v\right|^k\sum_{n=v+1}^{m+1}\hat{a}_{n,v} \\
&= O(1)\sum_{v=1}^{m} a_{vv}\left|\lambda_v\right|^k\left|t_v\right|^k \\
&= O(1),\ \text{as in the case of } T_{n1}.
\end{aligned}$$

□

Remark 3.2. As an applications to theorem 3.1, improvements to all corollaries mentioned in [4] can also obtained via weaker conditions

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] B.E. Rhodes, *Inclusion theorems for absolute matrix summability methods*,J. Math. Anal. Appl. 238 , (1999), 87-90.

[2] E. Savas, *Quasi-power increasing sequence for generalized absolute summability*, Nonlinear Analysis, 68 (2008), 170-176.

[3] E. Savas and H. Sevli, *A recent note on quasi-power increasing sequence for generalized absolute summability*, J. Ineq. Appl., (2009)Art. Id:675403.

[4] H. Sevli and L. Leindler, *On the absolute summability factors of infinite series involving quasi-power increasing sequences*, Computers and mathematics with applications **57** (2009) 702–709.

[5] W.T. Sulaiman, *Extension on absolute summability factors of infinite series*, J. Math. Anal. Appl. 322 **(2)**, (2006), 1224-1230.

Waad Sulaiman, Department of Computer Engineering, College of Engineering, University of Mosul, Iraq.

E-mail address: waadsulaiman@hotmail.com