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ONE-DIMENSIONAL ASSOCIATED HOMOGENEOUS DISTRIBUTIONS

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Abstract. Let $\mathcal{H}'(R)$ denote the set of Associated Homogeneous Distributions (AHDs) with support in R. The set $\mathcal{H}'(R)$ consists of the distributional analogues of one-dimensional power-log functions. $\mathcal{H}'(R)$ is an important subset of the tempered distributions $\mathcal{S}'(R)$, because (i) it contains the majority of the (one-dimensional) distributions typically encountered in physics applications and (ii) recent work done by the author shows that $\mathcal{H}'(R)$, as a linear space, can be extended to a convolution algebra and an isomorphic multiplication algebra.

This paper (i) reviews the general properties enjoyed by AHDs, (ii) completes the list of properties of the various important basis AHDs by deriving many new and general expressions for their derivatives, Fourier transforms, Taylor and Laurent series with respect to the degree of homogeneity, etc., and (iii) introduces some useful distributional concepts, such as extensions of partial distributions, that play a natural role in the construction of AHD algebras.

1. Introduction

Homogeneous distributions are the distributional analogue of homogeneous functions, such as $|x|^z : R \to \mathbb{C}$, which is homogeneous with complex degree z. Associated to homogeneous functions are power-log functions, which arise when taking the derivative with respect to the degree of homogeneity z. The set of Associated Homogeneous Distributions (AHDs) with support in the line R, and which we denote by $\mathcal{H}'(R)$, is the distributional analogue of the set of one-dimensional power-log functions. One-dimensional associated homogeneous distributions were first introduced in [15]. In the current paper, a comprehensive survey of the set $\mathcal{H}'(R)$ is given, which (i) applies a new approach to regularization and (ii) gives many new properties of AHDs of a greater generality then found in e.g., [15].

The set $\mathcal{H}'(R)$ is an interesting and important subset of the distributions of slow growth (or tempered distributions), $\mathcal{S}'(R)$, [23], [28], for the following reasons.

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(i) $\mathcal{H}'(R)$ contains the majority of the distributions one encounters in physics applications, such as the delta distribution δ, the step distributions $1_{\pm}$, several so called pseudo-functions generated by taking Hadamard's finite part of certain divergent integrals, among which is Cauchy's principal value distribution x^{-1}, the even and odd Riesz kernels, the Heisenberg distributions and many familiar others.

(ii) $\mathcal{H}'(R)$ is, just as $\mathcal{S}'(R)$, closed under Fourier transformation.

(iii) Recent results obtained by the author, [9]–[12], show that it is possible to extend the linear space $\mathcal{H}'(R)$ to a closed (non-commutative and non-associative) convolution algebra over $\mathbb{C}$.

(iv) By combining the two former properties and inspired by the generalized convolution theorem, [28, p. 191], [14, p. 101], we can also define a closed multiplication product for AHDs on R, [13]. Hence, $\mathcal{H}'(R)$ also extends to a closed (non-commutative and non-associative) multiplication algebra over $\mathbb{C}$, which is isomorphic to the former convolution algebra under Fourier transformation. The importance of this distributional multiplication algebra stems from the fact that it now becomes possible to give a rigorous meaning to distributional products such as $\delta^2 \triangleq \delta.\delta$ and many interesting others as a *distribution* (for instance, it is found in [13] that $\delta^2 = c\delta^{(1)}$, $c \in \mathbb{C}$ arbitrary). Attempts to define a multiplication product for the whole set of distributions $\mathcal{D}'$, such as in [3] (using model delta nets and passage to the limit) or in [2] (using Fourier transformation and a functional analytic way, a method which generalizes [16, p. 267, Theorem 8.2.10]), fail to produce meaningful products of certain AHDs as a distribution, e.g., for δ^2 see [5, Chapter 2]. Other approaches allow to define a multiplication product for a different class of generalized functions $\mathcal{G}$, as in Colombeau's work (which belongs to non-standard analysis), [5], and where $\delta^2 \in \mathcal{G}$ but δ^2 has no association in $\mathcal{D}'$. The by the author adopted definition for multiplication of AHDs makes that $(\mathcal{H}'(R), .)$ is internal to Schwartz' distribution theory, in the sense that all products are in $\mathcal{H}'(R) \subset \mathcal{S}'(R) \subset \mathcal{D}'(R)$.

(v) A certain subset of $\mathcal{H}'(R)$ is fundamental to give a distributional justification of "integration of complex degree" over R, for certain subsets of functions and distributions. Integration of complex degree is the generalization to complex degrees of the well-known classical "fractional integration/derivation" calculus, [22]. Certain AHDs appear as kernel in convolution operators that define integration of complex degree over the whole of R. The classical fractional calculus was developed for functions defined on the half-lines $R_{\pm}$ and can be justified in distributional terms, using subalgebras of the well-known convolution algebras $(\mathcal{D}'_L(R), +, *)$ and $(\mathcal{D}'_R(R), +, *)$, [23]. A distributional treatment of integration of complex degree over the whole line however, has not been given yet.

These facts show that the subset $\mathcal{H}'(R)$ occupies a remarkable position within Schwartz' distribution theory and this justifies a more up-to-date review of these important distributions. This work discusses the general properties enjoyed by one-dimensional AHDs and brings together known and many new properties of the various important basis AHDs. All the basis AHDs considered here, play one or more key roles in the construction of the mentioned algebras. The following is an overview of new results presented in this paper.

(i) We extend and generalize the classical exposition, given in [15], for seven types of basis AHDs. This includes expressions for generalized derivatives (of two types), Taylor and Laurent series expansions with respect to the degree of homogeneity, Fourier transforms, etc. The structure theorems for AHDs on R, derived in [9], make use of these basis AHDs and each type has a distinct advantage, allowing common operations to be performed on AHDs with ease and elegance.

(ii) The classical process of the regularization of divergent integrals, as given in [15] and which goes back to Hadamard, has been placed here in the more general context of a functional extension process and is justified by the Hahn-Banach theorem. We will refer to this method as "extension of partial distributions".

(iii) Attention is given to the appropriateness of the suggestive notation commonly used for AHDs. For instance, with $k, m \in \mathbb{Z}_+$, the distributions $x_{\pm}^{-k} \ln^m |x|$ are often tacitly interpreted as the distributional multiplication products $x_{\pm}^{-k} . \ln^m |x|$, which is correct in this case, while the notation $(x \pm i0)^{-k} \ln^m (x \pm i0)$, used in e.g., [15, pp. 96-98] for the distributions $D_z^m (x \pm i0)^z$ at $z = -k$, is prone to be read as the distributional multiplication products $(x \pm i0)^{-k} . \ln^m (x \pm i0)$, but which is incorrect, see eq. (5.156). The justification for using or avoiding such notation is subject to our definition of multiplication product, given by (4.19), and follows from results obtained in [13].

(iv) We emphasize the fact that, besides the well-known generalized derivative D for distributions, one can also define a second natural generalized derivative X. The operator D is a derivation with respect to the multiplication product, and is expressible as the convolution operator $D = \delta^{(1)} *$, while the operator X is a derivation with respect to the convolution product, and is expressible as the multiplication operator $X = x.$. Both generalized derivatives, D and X, appear on an equal footing in the theory of AHDs.

(v) We introduce the distributional concepts, "partial distribution" and "extension of a partial distribution", that proved valuable in the context of AHDs.

In this work we only consider one-dimensional distributions, i.e., with support contained in the line R. This restriction is deliberate for two reasons.

(i) AHDs based on R have a very simple structure, which allows to derive useful structure theorems, see [9]. These theorems give representations for a general AHD in terms of the basis AHDs considered here. The availability of structure theorems leads to more explicit results, such as [12] and [13]. Higher-dimensional analogues of AHDs, with support in R^n with $n > 1$, have far more degrees of freedom, and give rise to a generalization, called almost quasihomogeneous distributions in [26].

(ii) AHDs based on $R^{p,q}$ (pseudo-Euclidean coordinate space with signature (p, q)) can be obtained from AHDs based on R as the pullback along a scalar function. If such a function possesses a special symmetry, e.g., $O(p, q)$-invariance, then the resulting AHDs on $R^{p,q}$ inherit an isomorphic algebraic structure from the AHDs on R. It is thus relevant (and much simpler) to study the algebraic structure of one-dimensional AHDs, as a precursor for constructing algebras of certain n-dimensional AHDs. The latter set of distributions contains members which are of paramount importance in physics, e.g., for solving the ultrahyperbolic wave equation in $R^{p,q}$ and consequently for developing Clifford Analysis over pseudo-Euclidean spaces.

We end this introduction with the organization of the paper.

(i) We first establish some general definitions and notation.

(ii) We state some useful distributional preliminaries.

(iii) We continue with the formal definition of AHDs and state their general properties.

(iv) Thereafter, we introduce a number of un-normalized AHDs, and present a fairly complete collection of properties.

(v) Finally, we discuss in detail and give new properties for a number of important normalized basis AHDs, which, together with the un-normalized ones, play an essential role in the construction of AHD algebras.

2. General definitions

(1) *Number sets.* Define respectively the sets of odd and even positive integers $\mathbb{Z}_{o,+}$ and $\mathbb{Z}_{e,+}$, of odd and even negative integers $\mathbb{Z}_{o,-}$ and $\mathbb{Z}_{e,-}$, of non-negative and non-positive even integers $\mathbb{Z}_{e,[+} \triangleq \{0\} \cup \mathbb{Z}_{e,+}$ and $\mathbb{Z}_{e,-]} \triangleq \mathbb{Z}_{e,-} \cup \{0\}$, of odd and even integers $\mathbb{Z}_o \triangleq \mathbb{Z}_{o,-} \cup \mathbb{Z}_{o,+}$ and $\mathbb{Z}_e \triangleq \mathbb{Z}_{e,-} \cup \{0\} \cup \mathbb{Z}_{e,+}$, of positive and negative integers $\mathbb{Z}_+ \triangleq \mathbb{Z}_{o,+} \cup \mathbb{Z}_{e,+}$ and $\mathbb{Z}_- \triangleq \mathbb{Z}_{o,-} \cup \mathbb{Z}_{e,-}$, of non-negative and non-positive natural numbers $\mathbb{N} \triangleq \mathbb{Z}_{[+} \triangleq \{0\} \cup \mathbb{Z}_+$ and $-\mathbb{N} \triangleq \mathbb{Z}_{-]} \triangleq \mathbb{Z}_- \cup \{0\}$ and the set of integers $\mathbb{Z} \triangleq \mathbb{Z}_- \cup \{0\} \cup \mathbb{Z}_+$. Define finite sets of consecutive integers $\mathbb{Z}_{[i_1,i_2]}$, $\forall i_1, i_2 \in \mathbb{Z}$, as $\mathbb{Z}_{[i_1,i_2]} \triangleq \varnothing$ if $i_1 > i_2$, $\mathbb{Z}_{[i_1,i_1]} \triangleq \{i_1\}$ and $\mathbb{Z}_{[i_1,i_2]} \triangleq \{i_1, i_1+1, ..., i_2\}$ if $i_1 < i_2$. Further, define the open half-lines R_+ (the set of positive real numbers) and R_- (the set of negative real numbers), their half closures $R_{[+} \triangleq \{0\} \cup R_+$ (the set of non-negative real numbers) and $R_{-]} \triangleq R_- \cup \{0\}$ (the set of non-positive real numbers), and the open line $R \triangleq R_- \cup \{0\} \cup R_+$. Open n-dimensional real coordinate space is denoted by R^n and $\mathbb{C}$ stands for the field of complex numbers.

(2) *Indicator functions.* Let $\mathcal{L}_P$ denote the set of logical predicates. Define the indicator (or characteristic) function $1 : \mathcal{L}_P \to \{0, 1\}$, such that

$$p \mapsto 1_p \triangleq \begin{cases} 1 & \text{if} \quad p \text{ is true,} \\ 0 & \text{if} \quad p \text{ is false.} \end{cases} \tag{2.1}$$

We immediately introduce the following convenient even and odd symbols, $\forall k \in \mathbb{Z}$,

$$e_k \triangleq 1_{k \in \mathbb{Z}_e}, \tag{2.2}$$

$$o_k \triangleq 1_{k \in \mathbb{Z}_o}. \tag{2.3}$$

(3) *Factorial polynomials.* Denote by

$$z^{(k)} \triangleq 1_{k=0} + 1_{0<k}\, z(z+1)(z+2)...(z+(k-1)), \tag{2.4}$$

$$= \frac{\Gamma(z+k)}{\Gamma(z)}, \tag{2.5}$$

$$= \sum_{p=0}^{k} (-1)^{k-p} s\,(k,p)\, z^p, \tag{2.6}$$

the rising factorial polynomial (Pochhammer's symbol), $\forall k \in \mathbb{N}$. In particular, $0^{(k)} = 1_{k=0}$ and $m^{(k)} = (m-1+k)!/(m-1)!$ for $m \in \mathbb{Z}_+$. Also, denote by

$$
\begin{aligned}
z_{(k)} &\triangleq 1_{k=0} + 1_{k>0}\, z(z-1)(z-2)...(z-(k-1)), && (2.7)\\
&= \frac{\Gamma(z+1)}{\Gamma(z+1-k)}, && (2.8)\\
&= \sum_{p=0}^{k} s\,(k,p)\, z^{p}, && (2.9)
\end{aligned}
$$

the falling factorial polynomial. In particular, $0_{(k)} = 1_{k=0}$ and $m_{(k)} = 1_{k\leq m}\,(m!/(m-k)!)$ for $m \in \mathbb{Z}_{+}$. In (2.6) and (2.9), $s\,(k,p)$ are Stirling numbers of the first kind, [1, p. 824, 24.1.3]. Both polynomials are related as $z_{(k)} = (-1)^{k}\,(-z)^{(k)}$.

(4) *Modified Euler and Bernoulli polynomials.* We will need the polynomials $\overline{E}_n(x)$ and $\overline{B}_{n+1}(x)$, $\forall n \in \mathbb{N}$, with degrees indicated by their subscripts, defined by the generating functions,

$$
\begin{aligned}
\frac{1}{\cosh t}e^{xt} &= \sum_{n=0}^{+\infty} \overline{E}_n(x)\frac{t^n}{n!}, && (2.10)\\
\frac{t}{\sinh t}e^{xt} &= 1+\sum_{n=1}^{+\infty} n\overline{B}_n(x)\frac{t^n}{n!}. && (2.11)
\end{aligned}
$$

Obviously, $\overline{E}_n(-x) = (-1)^{n}\,\overline{E}_n(x)$ and $\overline{B}_{n+1}(-x) = (-1)^{n+1}\,\overline{B}_{n+1}(x)$. Our polynomials $\overline{E}_n(x)$ and $\overline{B}_{n+1}(x)$ are related to the standard Euler polynomials $E_n(x)$ and standard Bernoulli polynomials $B_{n+1}(x)$, as defined in [1, p. 804, 23.1.1], in the following way,

$$
\begin{aligned}
\overline{E}_n(x) &= 2^{n}E_n\left(\frac{x+1}{2}\right), && (2.12)\\
(n+1)\overline{B}_{n+1}(x) &= 2^{n+1}B_{n+1}\left(\frac{x+1}{2}\right). && (2.13)
\end{aligned}
$$

Define modified Euler and modified Bernoulli numbers by, $\forall n \in \mathbb{N}$,

$$
\begin{aligned}
\overline{E}_n &\triangleq \overline{E}_n(0), && (2.14)\\
\overline{B}_{n+1} &\triangleq \overline{B}_{n+1}(0). && (2.15)
\end{aligned}
$$

The numbers $\overline{E}_n$ and $\overline{B}_{n+1}$ are related to the standard Euler numbers $E_n \triangleq 2^{n}E_n\,(1/2)$ and standard Bernoulli numbers $B_n \triangleq B_n(0)$ as

$$
\begin{aligned}
\overline{E}_n &= E_n, && (2.16)\\
\overline{B}_{n+1} &= -\left(2^{n+1}-2\right)\frac{B_{n+1}}{n+1}. && (2.17)
\end{aligned}
$$

The following orthogonality relations hold for our modified numbers,

$$
\begin{aligned}
\sum_{k=0}^{n} \tbinom{n}{k} e_{n-k}\overline{E}_k &= 1_{n=0}, && (2.18)\\
\sum_{k=0}^{n} \tbinom{n}{k} o_{n-k}\overline{B}_{k+1} &= 1_{n=0} - \frac{e_n}{n+1}, && (2.19)
\end{aligned}
$$

Let d_z denote the ordinary derivative with respect to z. We will also need the following operator relations, holding $\forall n \in \mathbb{N}$,

$$\overline{E}_n(d_z)z = z\overline{E}_n(d_z) + 1_{0<n} n\overline{E}_{n-1}(d_z), \tag{2.20}$$

$$\overline{B}_{n+1}(d_z)z = 1_{n=0} + z\overline{B}_{n+1}(d_z) + 1_{0<n} n\overline{B}_n(d_z). \tag{2.21}$$

Equations (2.18)–(2.21) are easily proved from the generating functions (2.10)–(2.11).

(5) *Function spaces.* Let $n \in \mathbb{Z}_+$ and consider a non-empty open subset $U \subseteq R^n$, called base space. By $\mathcal{C}^k(U,\mathbb{C})$ we denote the space of functions from $U \to \mathbb{C}$ having continuous derivatives of order k, by $\mathcal{E}(U) \triangleq \mathcal{C}^\infty(U,\mathbb{C})$ the space of infinitely differentiable (or smooth) functions from $U \to \mathbb{C}$ and by $\mathcal{S}(U)$ the (Schwartz) space of smooth functions of rapid descent (towards infinity) from $U \to \mathbb{C}$, [23, vol. II, p. 89], [28, p. 99]. We will also need the space $\mathcal{D}_L(R)$ of smooth functions with support bounded on the right and the space $\mathcal{D}_R(R)$ of smooth functions with support bounded on the left, from $R \to \mathbb{C}$. Of central importance is the space $\mathcal{D}(U) \triangleq \mathcal{C}_c^\infty(U,\mathbb{C})$ of smooth, complex valued "test" functions with compact support in the base space U. We will also need the space of smooth functions of slow growth denoted $\mathcal{O}_M(U)$ (kernels of multiplication operators), [23, vol. II, pp. 99–101], the space $\mathcal{P}(U)$ of polynomial functions from $U \to \mathbb{C}$, the space of functions $\mathcal{Z}(V)$ whose Fourier transform is in $\mathcal{D}(U)$, [28, p. 192], and $\mathcal{Z}_M(V)$, the subspace of multipliers in $\mathcal{Z}(V)$, consisting of functions that are the Fourier transform of a distribution of compact support contained in U. In addition, $\mathcal{A}(\Omega,\mathbb{C})$ stands for the space of complex analytic functions from $\Omega \subseteq \mathbb{C} \to \mathbb{C}$ and the space of complex functions, being locally Lebesgue integrable on U, is denoted $\mathcal{L}^1_{loc}(U,\mathbb{C})$.

Let $\mathbb{Z}_1 \subseteq \mathbb{Z}$ and denote by $\mathcal{D}_{\mathbb{Z}_1}(U)$ ($\mathcal{S}_{\mathbb{Z}_1}(U)$) the subspace of test functions $\varphi \in \mathcal{D}(U)$ ($\varphi \in \mathcal{S}(U)$) such that, $\forall l \in \mathbb{Z}_1$, (i) if $l < 0$, all $|l+1|$-th order partial derivatives of φ at the origin are zero and (ii) if $0 \leq l$, all l-th order partial moments of φ are zero. In particular, $\mathcal{D}_{\mathbb{Z}_-}(U)$ is the space of smooth functions with compact support such that the value of the function and the value of all its derivatives at the origin is zero and $\mathcal{D}_{\mathbb{N}}(U)$ is the space of smooth functions with compact support with zero non-negative moments. Similarly, $\mathcal{S}_{\mathbb{Z}_-}(U)$ is the space of smooth functions of rapid descent such that the value of the function and the value of all its derivatives at the origin is zero and $\mathcal{S}_{\mathbb{N}}(U)$ is the space of smooth functions of rapid descent with zero non-negative moments. The space $\mathcal{S}_{\mathbb{N}}(U)$ is also known as Lizorkin's space, [22, p. 148].

(6) *Distribution spaces.* Some well-known generalized function spaces are: (i) the space of distributions $\mathcal{D}'$, (ii) the space of distributions of slow growth (also called tempered distributions) $\mathcal{S}'$, (iii) $\mathcal{Z}'$ (dual to $\mathcal{Z}$), called the space of ultradistributions (i.e., distributions which are the Fourier transform of a distribution in $\mathcal{D}'$). Denote by $\mathcal{E}'(U) \subset \mathcal{D}'(U)$ the space of distributions with compact support, which is dual to the space $\mathcal{E}(U)$ of smooth functions. It will be convenient to also introduce the space of distributions $\mathcal{E}'_0(U) \subset \mathcal{E}'(U)$, having as support $\{0\}$. Denote by $\mathcal{N}'$ the set of regular distributions, which are generated by the set $\mathcal{P}$ of polynomials with complex coefficients. The Fourier transformation $\mathcal{F}_{\mathcal{S}}$ and its inverse

$\mathcal{F}_{\mathcal{S}}^{-1}$ (see below) are homeomorphisms between $\mathcal{N}'(U)$ and $\mathcal{E}_0'(V)$, with V called the spectral base space (or spectral domain) of the base space U. Further, we denote by $\mathcal{D}_R'(R) \subset \mathcal{D}'(R)$ the space of distributions with support bounded on the left (also called right-sided distributions) ($\mathcal{D}_R'$ is dual to $\mathcal{D}_L$) and by $\mathcal{D}_L'(R) \subset \mathcal{D}'(R)$ the space of distributions with support bounded on the right (also called left-sided distributions) ($\mathcal{D}_L'$ is dual to $\mathcal{D}_R$). Clearly, $\mathcal{E}'(R) \subset \mathcal{D}_{L,R}'(R)$. The spaces $\mathcal{D}_{L,R}'(R)$ together with the sum $+$ and convolution product $*$, denoted $\left(\mathcal{D}_{L,R}', +, *\right)$, are commutative integral domains and can be given the additional structure of commutative and associative convolution algebras over $\mathbb{C}$, [23, vol II, pp. 28–30]. We will also need the space of distributions of rapid descent, denoted $\mathcal{O}_C'(U)$ (kernels of convolution operators), [23, vol. II, pp. 99–101]. The following inclusions hold $\mathcal{D}(U) \subset \mathcal{E}'(U) \subset \mathcal{O}_C'(U) \subset \mathcal{S}'(U)$ and also $\mathcal{D}(U) \subset \mathcal{S}(U) \subset \mathcal{O}_M(U) \subset \mathcal{S}'(U)$, [23, vol. II, p. 170].

(7) *Notation.* We almost exclusively consider generalized functions based on R, so we will write $\mathcal{D}$ for $\mathcal{D}(R)$, $\mathcal{D}'$ for $\mathcal{D}'(R)$, etc.. We also adopt the convention that generic symbols like $f(x)$ denote a function value, while f denotes either a function or a distribution, with the distinction being clear from the context. Exceptions to this rule are the customary symbols for particular functions, such as e^x, $\ln|x|$, etc.. Occasionally, the need will arise to explicitly indicate a dummy variable x and write $f_{(x)}$ for a distribution, to show that it is an element of $\mathcal{D}'(\{x \in R\})$, as in the tensor product of two distributions $f_{(x)} \otimes g_{(y)} \in \mathcal{D}'(\{(x,y) \in R^2\})$.

3. Preliminaries

Any function $f \in \mathcal{L}_{loc}^1(U, \mathbb{C})$ generates a distribution $f \in \mathcal{D}'(U)$ by defining, $\forall \varphi \in \mathcal{D}(U)$,

$$\langle f, \varphi \rangle \triangleq \int_U f(x)\varphi(x)dx. \tag{3.1}$$

The resulting distribution f is called regular. Distributions which are not regular are called singular.

3.1. **Products of distributions.**

Products of distributions can in general not be defined. Under certain restrictions however, a multiplication product or a convolution product can be defined, which is commutative but generally not associative, as is illustrated by the special cases below. In these cases, the defined product is assumed to satisfy the usual distributive and algebra axioms over $\mathbb{C}$.

3.1.1. *Multiplication product.* The multiplication product of two distributions can easily be defined in the following cases.

(1) If f and g are both regular distributions, generated from locally integrable functions f and g from $U \to \mathbb{C}$, and if the (codomain pointwise) product function fg is also locally integrable on U, then the multiplication product of distributions $f.g = g.f$ can be defined as the regular distribution given by

$$\langle f.g, \varphi \rangle \triangleq \int_U f(x)g(x)\varphi(x)dx \triangleq \langle g.f, \varphi \rangle. \tag{3.2}$$

In this case, the multiplication product inherits associativity from the associative product of locally integrable functions. A typical example is the set of regular distributions generated from the set of continuous functions, which together with addition and the multiplication (3.2) forms an algebra over $\mathbb{C}$.

(2) More general, if h is a regular distribution, generated from any smooth function $h \in \mathcal{E}(U)$, and f is any distribution, the multiplication product of distributions $h.f = f.h$ can be defined by

$$\langle h.f, \varphi \rangle \triangleq \langle f, h\varphi \rangle \triangleq \langle f.h, \varphi \rangle \tag{3.3}$$

and it is easily shown that $h.f \in \mathcal{D}'(U)$, [28, p. 28]. In particular, if $h \in \mathcal{D}(U)$, $h.f \in \mathcal{E}'(U)$.

(3) If h is a regular distribution, generated from a smooth function of slow growth $h \in \mathcal{O}_M(U)$ and $f \in \mathcal{S}'(U)$ is a tempered distribution, the multiplication product $h.f$ can also be defined by (3.3) and $h.f \in \mathcal{S}'(U)$, [23, vol. II, pp. 101–102].

(4) One can also define the multiplication product of a smooth function with a distribution as $\mathcal{E}(U) \times \mathcal{D}'(U) \to \mathcal{D}'(U)$ such that $(h, f) \mapsto hf = fh$, with hf defined by $\langle hf, \varphi \rangle \triangleq \langle f, h\varphi \rangle \triangleq \langle fh, \varphi \rangle$. This is an example of an external operation, which turns $\mathcal{D}'(U)$ into a module over $\mathcal{E}(U)$. In the particular case that $h = \varphi \in \mathcal{D}(U)$, this operation is called *localization* of a distribution $f \in \mathcal{D}'(U)$, with respect to φ, and the resulting distribution of compact support is regarded as a localized approximation to f. In this sense, test functions can be thought of as "localized" approximations of the one distribution 1.

(5) Let $\mathcal{Z}'_+$ and $\mathcal{Z}'_-$ denote those subspaces of the ultradistributions $\mathcal{Z}'$ whose elements are the Fourier transform of a distribution in $\mathcal{D}'_L$ and $\mathcal{D}'_R$, respectively. If $f, g \in \mathcal{Z}'_+$ ($f, g \in \mathcal{Z}'_-$), then a multiplication product $f.g \in \mathcal{Z}'_+$ ($f.g \in \mathcal{Z}'_-$) can be defined by

$$f.g \triangleq \mathcal{F}_{\mathcal{D}'} \left(\left(\mathcal{F}^{-1}_{\mathcal{Z}'} f \right) * \left(\mathcal{F}^{-1}_{\mathcal{Z}'} g \right) \right). \tag{3.4}$$

Then, the spaces $\mathcal{Z}'_\pm$ together with the sum $+$ and multiplication product ., and denoted $\left(\mathcal{Z}'_\pm, +, . \right)$, are commutative integral domains and can be given the additional structure of commutative and associative convolution algebras over $\mathbb{C}$. The structures $\left(\mathcal{Z}'_\pm, +, . \right)$ are isomorphic under the Fourier transformation to the structures $\left(\mathcal{D}'_{L,R}, +, * \right)$, considered below, and are two other examples of generalized function spaces that can be extended to a multiplication algebra over $\mathbb{C}$.

(6) Fix any $w \in \mathcal{D}'(U)$ and let $\mathcal{D}'_w(U) \subset \mathcal{D}'(U)$ be that subset of distributions for which the multiplication with w exists. The operator $T_w : \mathcal{D}'_w(U) \to \mathcal{D}'(U)$ such that $f \mapsto w.f$ is called a *multiplication operator* with kernel w. A necessary and sufficient condition that an operator T, from a subset of $\mathcal{D}'(U)$ to $\mathcal{D}'(U)$, is a multiplication operator is that T is (i) single-valued, (ii) linear, (iii) sequentially continuous and (iv) commutes with any phase operator $E_a \triangleq e^{i(x \cdot a)}$. (by isomorphism with convolution operator, see below).

In engineering, a multiplication operator $w.$ is called a (phase-invariant) filter and its kernel w is called the transfer function of the filter.

3.1.2. *Convolution product.* The convolution product of two distributions is always defined in the following cases.

(1) In particular, if f and g are regular distributions, generated from locally integrable functions f and g from $U \to \mathbb{C}$, respectively, and if the function

$$h(x) \triangleq \int_U f(x-y) g(y)\, dy, \tag{3.5}$$

exists almost everywhere and is also locally integrable, then $f * g = g * f$ is defined as the regular distribution generated by h, [28, p. 126].

(2) Let $\varphi \in \mathcal{D}(U)$ and $\varphi_{x+y} \in \mathcal{D}(U \times U) : (x, y) \mapsto \varphi(x+y)$. The convolution product $f * g = g * f$ of two distributions $f, g \in \mathcal{D}'(U)$, can be defined in terms of the tensor product $f_{(x)} \otimes g_{(y)} \in \mathcal{D}'(U \times U)$ as

$$\begin{aligned} \langle f * g, \varphi \rangle &\triangleq \left\langle f_{(x)} \otimes g_{(y)}, \varphi(x+y) \right\rangle, && (3.6) \\ &\triangleq \left\langle f_{(x)}, \left\langle g_{(y)}, \varphi(x+y) \right\rangle \right\rangle, && (3.7) \end{aligned}$$

provided $\mathrm{supp}(f_{(x)} \otimes g_{(y)}) \cap \mathrm{supp}(\varphi(x+y))$ in $U \times U$ is compact, and then $f * g \in \mathcal{D}'(U)$, [28, p. 122]. If $f \in \mathcal{E}'(U)$ or $g \in \mathcal{E}'(U)$, or $f \in \mathcal{D}'_L$ and $g \in \mathcal{D}'_L$, or $f \in \mathcal{D}'_R$ and $g \in \mathcal{D}'_R$, then $f * g$ is always defined by (3.6)–(3.7). The convolution product of m distributions is associative if at least $m-1$ distributions in this product belong to $\mathcal{E}'(U)$ or they all belong to either $\mathcal{D}'_L$ or $\mathcal{D}'_R$. It is well-known that under these same conditions the convolution is a sequentially continuous operation, [28, p. 136].

(3) If $h \in \mathcal{O}'_C(U)$ and $f \in \mathcal{S}'(U)$, the convolution product $h * f = f * h$ is defined by (3.7) and $h * f \in \mathcal{S}'(U)$, [23, vol. II, pp. 102–103].

(4) One can also define the convolution product of a test function with a distribution as $\mathcal{D}(U) \times \mathcal{D}'(U) \to \mathcal{E}(U)$ such that $(\varphi, f) \mapsto \varphi * f = f * \varphi$, with $\varphi * f$ defined by $(\varphi * f)(x) \triangleq \left\langle f_{(y)}, \varphi(x-y) \right\rangle \triangleq (f * \varphi)(x)$. This definition agrees with (3.6)–(3.7) if the test function φ is replaced by the regular distribution φ it generates, see [28, p. 132, Theorem 5.5-1]. This turns $\mathcal{D}'(U)$ into a module over $\mathcal{D}(U)$. This operation is called *regularization* of a distribution $f \in \mathcal{D}'(U)$, with respect to a mollifier $\varphi \in \mathcal{D}(U)$, and the resulting smooth function is regarded as an approximation to f. In this sense, test functions in $\mathcal{D}(U)$ can be thought of as "mollified" approximations of the delta distribution δ.

(5) Fix any $w \in \mathcal{D}'(U)$ and let $\mathcal{D}'_w(U) \subset \mathcal{D}'(U)$ be that subset of distributions for which the convolution with w exists. The operator $T_w : \mathcal{D}'_w(U) \to \mathcal{D}'(U)$ such that $f \mapsto w * f$ is called a *convolution operator* with kernel w. A necessary and sufficient condition that an operator T, from a subset of $\mathcal{D}'(U)$ to $\mathcal{D}'(U)$, is a convolution operator is that T is (i) single-valued, (ii) linear, (iii) sequentially continuous and (iv) commutes with any translation operator T_a (eq. (3.39)), [28, p. 147].

In engineering, a convolution operator $w*$ is called a (time-invariant) linear system and its kernel w is called the impulse response of the system.

3.2. **Operations on distributions.**

3.2.1. *Generalized derivations.* Let $j \in \mathbb{Z}_{[1,n]}$. Let $h, f, g \in \mathcal{D}'(U)$, with h a regular distribution generated by a smooth function and g a distribution of compact support.

(1) *Generalized multiplication partial derivation.* The generalized partial derivation $D_j : \mathcal{D}'(U) \to \mathcal{D}'(U)$ such that $f \mapsto D_j f$, is the continuous linear

operator defined by

$$\langle D_j f, \varphi \rangle \triangleq - \langle f, d_j \varphi \rangle . \tag{3.8}$$

The symbol d_j in the right-hand side of (3.8) stands for the ordinary partial derivation in $\mathcal{D}(U)$ with respect to the j-th coordinate. The generalized partial derivation is a derivation with respect to multiplication of distributions, as defined by (3.3), but not with respect to the convolution product, as defined by (3.7),

$$\begin{aligned} D_j(h.f) &= (D_j h).f + h.\left(D_j f\right), & (3.9)\\ D_j(f * g) &= (D_j f) * g = f * (D_j g). & (3.10) \end{aligned}$$

It is important to notice that in general, D_j is not a derivation with respect to multiplication of distributions as defined by (3.2). If $n = 1$ we write D for D_1. We will call D_j the generalized multiplication partial derivation, in order to distinguish it from the one defined in the next item.

(2) *Generalized convolution partial derivation.* The continuous operator X^j : $\mathcal{D}'(U) \to \mathcal{D}'(U)$ such that $f \mapsto X^j f$, defined by (x^j is the j-th coordinate function)

$$\left\langle X^j f, \varphi \right\rangle \triangleq \left\langle f, x^j \varphi \right\rangle , \tag{3.11}$$

is also a derivation, now with respect to the convolution product, but not with respect to multiplication. Indeed, one easily deduces from (3.3) and (3.7) that,

$$\begin{aligned} X^j(h.f) &= (X^j h).f = h.\left(X^j f\right), & (3.12)\\ X^j(f * g) &= (X^j f) * g + f * (X^j g). & (3.13) \end{aligned}$$

It is again important to notice that in general, X^j is not a derivation with respect to convolution of distributions as defined by (3.5). If $n = 1$ we write X for X^1. We will call X_j the generalized convolution partial derivation.

(3) We have, $\forall i, j \in \mathbb{Z}_{[1,n]}$,

$$D_i \circ X^j - X^j \circ D_i = \delta_i^j \, \mathrm{Id}, \tag{3.14}$$

wherein δ_i^j are the components of the $(1,1)$-unit tensor and $\mathrm{Id} : \mathcal{D}'(U) \to \mathcal{D}'(U)$ denotes the common identity operator (either realized as the multiplication operator $\mathrm{Id} = 1.$ or as the convolution operator $\mathrm{Id} = \delta *$). For further convenience we also define the following operators,

$$\begin{aligned} \mathbf{X} \cdot \mathbf{D} &\triangleq \sum_{i=1}^{n} X^i \circ D_i, & (3.15)\\ \mathbf{D} \cdot \mathbf{X} &\triangleq \sum_{i=1}^{n} D_i \circ X^i, & (3.16) \end{aligned}$$

satisfying

$$\mathbf{D} \cdot \mathbf{X} - \mathbf{X} \cdot \mathbf{D} = n \, \mathrm{Id}, \tag{3.17}$$

with n the dimension of the base space U. In (3.15), $\left(X^i \circ D_i\right) f \triangleq X^i \left(D_i f\right) = x^i . \left(\delta_i^{(1)} * f\right)$ and in (3.16), $\left(D_i \circ X^i\right) f \triangleq D_i \left(X^i f\right) = \delta_i^{(1)} * \left(x^i . f\right)$. The operator defined in (3.15) is the generalized Euler operator.

3.2.2. *Fourier transformation.*

A. Distributions of slow growth.

(1) Define the Fourier transformation $\mathcal{F}_{\mathcal{S}} : \mathcal{S}(U) \to \mathcal{S}(V)$ such that $\varphi \mapsto \psi \triangleq \mathcal{F}_{\mathcal{S}}\varphi$ with

$$(\mathcal{F}_{\mathcal{S}}\varphi)(\chi) \triangleq \int_U e^{-i2\pi(\chi\cdot x)}\varphi(x)\, dx. \tag{3.18}$$

The inverse transformation $\mathcal{F}_{\mathcal{S}}^{-1} : \mathcal{S}(V) \to \mathcal{S}(U)$ such that $\psi \mapsto \varphi = \mathcal{F}_{\mathcal{S}}^{-1}\psi$ is given by, [28, p. 178],

$$\left(\mathcal{F}_{\mathcal{S}}^{-1}\psi\right)(x) = \int_V e^{+i2\pi(\chi\cdot x)}\psi(\chi)\, d\chi. \tag{3.19}$$

The Fourier transformation $\mathcal{F}_{\mathcal{S}}$ and its inverse $\mathcal{F}_{\mathcal{S}}^{-1}$ are linear homeomorphisms between $\mathcal{S}(U)$ and $\mathcal{S}(V)$, [28, pp. 182–183].

Let $e^{\pm i2\pi(\chi\cdot x)}$ denote the regular distributions based on $V \times U$, generated by the smooth functions $e^{\pm i2\pi(\chi\cdot x)} : V \times U \to \mathbb{C}$. The distributions $e^{\pm i2\pi(\chi\cdot x)}$ are regular (Schwartz) kernels, [4, p. 471], [14, p. 68]. The transformation $\mathcal{F}_{\mathcal{S}}$ ($\mathcal{F}_{\mathcal{S}}^{-1}$) is thus also given by right (left) contraction as follows,

$$\begin{aligned} (\mathcal{F}_{\mathcal{S}}\varphi)(\chi) &= \left\langle e^{-i2\pi(\chi\cdot x)}, \varphi_x \right\rangle, & (3.20)\\ \left(\mathcal{F}_{\mathcal{S}}^{-1}\psi\right)(x) &= \left\langle e^{+i2\pi(\chi\cdot x)}, \psi_\chi \right\rangle. & (3.21)\end{aligned}$$

(2) The Fourier transformation on the dual space $\mathcal{F}_{\mathcal{S}'} : \mathcal{S}'(V) \to \mathcal{S}'(U)$ such that $f \mapsto g = \mathcal{F}_{\mathcal{S}'}f$ is defined by, $\forall\varphi \in \mathcal{S}(U)$,

$$\langle \mathcal{F}_{\mathcal{S}'}f, \varphi\rangle \triangleq \langle f, \mathcal{F}_{\mathcal{S}}\varphi\rangle, \tag{3.22}$$

The legitimacy of this definition stems from the fact that $\mathcal{F}_{\mathcal{S}'}f$ coincides with the ordinary Fourier transformation of a function $f \in \mathcal{L}^1_{loc}$, due to Parseval-Plancherel's theorem, [28, p. 184]. The inverse transformation $\mathcal{F}_{\mathcal{S}'}^{-1} : \mathcal{S}'(U) \to \mathcal{S}'(V)$ such that $g \mapsto f = \mathcal{F}_{\mathcal{S}'}^{-1}g$ is readily given by, $\forall\psi \in \mathcal{S}(V)$,

$$\left\langle \mathcal{F}_{\mathcal{S}'}^{-1}g, \psi\right\rangle = \left\langle g, \mathcal{F}_{\mathcal{S}}^{-1}\psi\right\rangle. \tag{3.23}$$

The Fourier transformation $\mathcal{F}_{\mathcal{S}'}$ and its inverse $\mathcal{F}_{\mathcal{S}'}^{-1}$ are linear homeomorphisms between $\mathcal{S}'(V)$ and $\mathcal{S}'(U)$, [23, vol II, pp. 105–107].

(3) The transformations $\mathcal{F}_{\mathcal{S}'}$ and $\mathcal{F}_{\mathcal{S}'}^{-1}$ are also linear homeomorphisms between $\mathcal{O}_M(V) \subset \mathcal{S}'(V)$ and $\mathcal{O}'_C(U) \subset \mathcal{S}'(U)$, that exchange the multiplication product and the convolution product, [23, vol II, p. 124], i.e., the generalized convolution theorem. In particular, $\mathcal{F}_{\mathcal{S}'}$ and $\mathcal{F}_{\mathcal{S}'}^{-1}$ are linear homeomorphisms between $\mathcal{P}(V)$ and $\mathcal{E}'_0(U)$.

B. Ultradistributions.

(1) Define the Fourier transformation $\mathcal{F}_{\mathcal{Z}} : \mathcal{Z}(U) \to \mathcal{D}(V)$ such that $\psi \mapsto \varphi = \mathcal{F}_{\mathcal{Z}}\psi$ by

$$\varphi(\chi) \triangleq \int_U e^{-i2\pi(\chi\cdot x)}\psi(x)\, dx. \tag{3.24}$$

The inverse transformation $\mathcal{F}_{\mathcal{D}}^{-1} : \mathcal{D}(V) \to \mathcal{Z}(U)$ such that $\varphi \mapsto \psi = \mathcal{F}_{\mathcal{D}}^{-1}\varphi$ is given by

$$\psi(x) = \int_V e^{+i2\pi(\chi\cdot x)}\varphi(\chi)\, d\chi. \tag{3.25}$$

The Fourier transformation $\mathcal{F}_{\mathcal{Z}}$ and its inverse $\mathcal{F}_{\mathcal{D}}^{-1}$ are homeomorphisms between $\mathcal{Z}(U)$ and $\mathcal{D}(V)$.

(2) The Fourier transformation on the dual space $\mathcal{F}_{\mathcal{D}'} : \mathcal{D}'(V) \to \mathcal{Z}'(U)$ such that $f \mapsto g = \mathcal{F}_{\mathcal{D}'} f$ is defined by, $\forall \psi \in \mathcal{Z}(U)$,

$$\langle \mathcal{F}_{\mathcal{D}'} f, \psi \rangle \triangleq \langle f, \mathcal{F}_{\mathcal{Z}} \psi \rangle . \tag{3.26}$$

The inverse transformation $\mathcal{F}_{\mathcal{Z}'}^{-1} : \mathcal{Z}'(U) \to \mathcal{D}'(V)$ such that $g \mapsto f = \mathcal{F}_{\mathcal{Z}'}^{-1} g$ is readily given by, $\forall \varphi \in \mathcal{D}(V)$,

$$\left\langle \mathcal{F}_{\mathcal{Z}'}^{-1} g, \varphi \right\rangle = \left\langle g, \mathcal{F}_{\mathcal{D}}^{-1} \varphi \right\rangle . \tag{3.27}$$

The Fourier transformation $\mathcal{F}_{\mathcal{D}'}$ and its inverse $\mathcal{F}_{\mathcal{Z}'}^{-1}$ are homeomorphisms between $\mathcal{D}'(V)$ and $\mathcal{Z}'(U)$.

(3) We have

$$\mathcal{F}_{\mathcal{D}'} D_j = +(2\pi i) X^j \mathcal{F}_{\mathcal{D}'}, \tag{3.28}$$

$$\mathcal{F}_{\mathcal{D}'} X_j = -(2\pi i)^{-1} D^j \mathcal{F}_{\mathcal{D}'}. \tag{3.29}$$

Herein are D_j, X^j generalized partial derivations with respect to the base space U and D^j, X_j generalized partial derivations with respect to the base space V.

The following inclusions of subspaces hold: $\mathcal{Z}(U) \subset \mathcal{S}(U) \subset \mathcal{S}'(U) \subset \mathcal{Z}'(U)$, [28, p. 201].

3.3. Extension of a partial distribution.

(1) *Partial distribution.* A linear and sequentially continuous functional f, which is defined $\forall \psi \in \mathcal{D}_r \subset \mathcal{D}$ and is undefined $\forall \varphi \in \mathcal{D} \backslash \mathcal{D}_r$, will be called a partial distribution.

(2) *Extension of a partial distribution.* A distribution $f_\varepsilon \in \mathcal{D}'$, defined $\forall \varphi \in \mathcal{D}$, such that $\langle f_\varepsilon, \psi \rangle = \langle f, \psi \rangle$, $\forall \psi \in \mathcal{D}_r$, will be called an extension of the partial distribution f from $\mathcal{D}_r$ to $\mathcal{D}$. The existence of a functional f_ε, that is also a distribution, is guaranteed by the Hahn-Banach theorem, but f_ε is in general not unique, [21, p. 56], [4, p. 424]. Let $\mathcal{D}'_r$ denote the continuous dual of $\mathcal{D}_r$. The subset of $\mathcal{D}'_r$ which maps $\mathcal{D}_r$ to zero is called the annihilator of $\mathcal{D}_r$ and is denoted by $\mathcal{D}'^{\perp}_r$. Any two extensions $f_{\varepsilon,1}$ and $f_{\varepsilon,2}$ differ by a generalized function $g \in \mathcal{D}'^{\perp}_r$. However, if the subspace $\mathcal{D}_r$ is dense in $\mathcal{D}$, then the extension f_ε is unique [4, p. 425].

Some authors call a distribution such as f_ε a regularization. This stems from the fact that f_ε is usually obtained after applying a procedure to give a meaning to the defining divergent integral for f and the *integral* is then regarded as being regularized. We will reserve the term regularization of a generalized function for its usual meaning, as defined in the previous subsection, and use the term "extension of a partial distribution" for the procedure described in this subsection.

(3) If a function $f : U \subset R \to \mathbb{C}$, is integrable on some but not all finite intervals, then there exists a subspace $\mathcal{D}_s \subset \mathcal{D}$ such that the integral in (3.1) does not exist iff $\varphi \in \mathcal{D}_s$. The functional (3.1) will then only be defined on the subspace $\mathcal{D}_r \triangleq \mathcal{D} \backslash \mathcal{D}_s$. As an example, consider a function f with a countable number of algebraic singularities at isolated points $x_0 \in \Lambda \subset R$. In this case, $\mathcal{D}_s$ is the subspace of test functions which do not vanish at

a sufficient rate (depending on the degree of the singularities) at Λ. The (closed) set Λ is called the *singular support* of f.

(4) More explicitly, let $\Lambda = \{0\}$ be the singular support of a linear and sequentially continuous functional $f \in \mathcal{D}'$, defined in terms of a function f with an algebraic singularity of degree $k \in \mathbb{Z}_+$, such that $x^k f \in \mathcal{L}^1_{loc}(R, \mathbb{C})$. Let (i) $\lambda \in \mathcal{D}(R)$ with support a neighborhood of Λ and such that all its derivatives are zero at $x = 0$ or (ii) $\lambda = 1$. Then, an extension $f_\varepsilon \in \mathcal{D}'$ of f can be obtained as

$$\langle f_\varepsilon, \varphi \rangle = \int_{-\infty}^{+\infty} f(x) \left(T_{p,q}\varphi\right)(x) dx, \tag{3.30}$$

wherein $T_{p,q} : \mathcal{D} \to \mathcal{D}_{\mathbb{Z}_{[-k,-(p+1)]}}$ such that $\varphi \mapsto T_{p,q}\varphi$ with

$$\left(T_{p,q}\varphi\right)(x) \triangleq \varphi(x) - \sum_{l=0}^{p+q} \varphi^{(l)}(0) \frac{x^l}{l!} \lambda(x) \left(1_{l<p} + 1_{p \le l} 1_{[+}(1-x^2)\right), \tag{3.31}$$

$\forall p, q \in \mathbb{N} : p + q = k - 1$ and the step function $1_{[+}(x) = 1$ iff $x \geq 0$. Indeed, for each allowed combination of p, q and λ the integral in (3.30) is now a regular integral and it defines a linear and sequentially continuous functional on $\mathcal{D}$. Further, $\forall \psi \in \mathcal{D}_{\mathbb{Z}_{[-k,-(p+1)]}}$, $T_{p,q}\psi = \psi$ and (3.30) reduces to $\langle f_\varepsilon, \psi \rangle = \langle f, \psi \rangle$, the value of which is independent of the choice we make for p, q and λ. Hence, any f_ε given by (3.30), is an extension of f from $\mathcal{D}_{\mathbb{Z}_{[-k,-(p+1)]}}$ to $\mathcal{D}$. For any $\varphi \in \mathcal{D} \backslash \mathcal{D}_{\mathbb{Z}_{[-k,-(p+1)]}}$, the value $\langle f_\varepsilon, \varphi \rangle$ depends on the chosen extension. Any pair of extensions f_ε, constructed from (3.30), differ by a linear combination of $\delta^{(l)}$ distributions, with $l \in \mathbb{Z}_{[p,k-1]}$, and with coefficients depending on λ. Note that all f_ε obtained from (3.30) are singular distributions. A remark on the notation: we will use the subscript ε to denote any extension obtained through (3.30) and the subscript e will be reserved for an extension that is an AHD.

(5) Let $\varphi \in \mathcal{D}$. For the linear operator $T_{p,q}$, defined by (3.31), hold the commutator relations,

$$\begin{aligned} T_{p,q}(d\varphi) &= d\left(T_{p+1,q}\varphi\right) & (3.32) \\ &+ \left(\sum_{l=0}^{(p+1)+q} \varphi^{(l)}(0) \frac{x^l}{l!} \begin{pmatrix} 1_{l<(p+1)} \\ +1_{(p+1)\le l} 1_{[+}(1-x^2) \end{pmatrix}\right)(d\lambda), & (3.33) \\ \left(T_{p,q}(x\varphi)\right) &= x\left(T_{p-1,q}\varphi\right), & (3.34) \end{aligned}$$

and the scaling property, $\forall r > 0$,

$$\left(T_{p,q}\varphi_{x/r}\right)(x) = \left(T_{p,q}\varphi_x\right)(x/r) + \sum_{l=p}^{p+q} \varphi^{(l)}(0) \frac{(x/r)^l}{l!} \lambda_1(x/r) \begin{pmatrix} 1_{[+}(1-(x/r)^2) \\ -1_{[+}(r^{-2}-(x/r)^2) \end{pmatrix}, \tag{3.35}$$

wherein $\lambda_1(x/r) \triangleq \lambda(x)$, $\forall x \in R$.

3.4. Distributions depending on a parameter.

(1) *Completeness.* Due to the topology of the space $\mathcal{D}'$ is the limit of any sequence of distributions $\{f^z \in \mathcal{D}' : z \to a\}$, parametrized by a complex number z, is again a unique distribution, denoted $f^a \in \mathcal{D}'$, and we can write $\lim_{z\to a} f^z = f^a$, [28, p. 36], [4, pp. 453-454]. This means that

$\lim_{z\to a}\langle f^z,\varphi\rangle=\langle\lim_{z\to a}f^z,\varphi\rangle=\langle f^a,\varphi\rangle$ or more explicitly, that $\forall\epsilon>0$, $\exists\delta>0$ such that, whenever $|z-a|<\delta$, $|\langle f^z-f^a,\varphi\rangle|<\epsilon$, $\forall\varphi\in\mathcal{D}$.

(2) *Continuity.* A distribution f^z is said to be C^k, $k\in\mathbb{N}$, in a parameter $z\in\Omega\subseteq\mathbb{C}$, iff $d_z^k\langle f^z,\varphi\rangle$, $\forall\varphi\in\mathcal{D}$, is a continuous function of z, $\forall z\in\Omega$. A distribution $f^{a,b}$ is said to be jointly C^k, $k\in\mathbb{N}$, in two parameters $(a,b)\in\Omega\subseteq\mathbb{C}^2$, iff $d_a^m d_b^n\langle f^{a,b},\varphi\rangle$, $\forall m,n\in\mathbb{N}:m+n=k$ and $\forall\varphi\in\mathcal{D}$, is jointly continuous in a and b, $\forall(a,b)\in\Omega$.

(3) *Monogenicy.* A distribution f^z, depending on a complex parameter z, is called monogenic (also: complex analytic or holomorphic) in $z\in\Omega\subseteq\mathbb{C}$, iff $d_z\langle f^z,\varphi\rangle$ exists everywhere in Ω, $\forall\varphi\in\mathcal{D}$. It is necessary and sufficient that the complex functions $d_z\langle f^z,\varphi\rangle$ exist, for the distribution D_zf^z to exist and to be given by $\langle D_zf^z,\varphi\rangle=d_z\langle f^z,\varphi\rangle$, $\forall z\in\Omega$, [15, pp. 147-151]. Then, also $\langle f^z,\varphi\rangle\in\mathcal{A}(\Omega,\mathbb{C})$. For any distribution f^z monogenic in $z\in\Omega$, $D_z^mf^z$ is C^∞, $\forall z\in\Omega$ and $\forall m\in\mathbb{N}$.

(4) Let f^z be a monogenic distribution in $z\in\Omega\subseteq\mathbb{C}$. Combining the previous item with the definitions of the generalized derivations D and X yields the important properties,

$$D^kD_z^l\,f^z = D_z^lD^k\,f^z, \tag{3.36}$$
$$X^kD_z^l\,f^z = D_z^lX^k\,f^z, \tag{3.37}$$

$\forall z\in\Omega$ and $\forall k,l\in\mathbb{N}$. Equations (3.36)–(3.37) greatly simplify the calculation of generalized derivatives of AHDs.

3.5. **Pullbacks of distributions.** We here only consider pullbacks along diffeomorphisms.

(1) *General.* Let $X,Y\subseteq R^n$ and $T:X\to Y$ such that $\mathbf{x}\mapsto\mathbf{y}=T(\mathbf{x})$ denotes a C^∞-diffeomorphism, and let $\varphi\in\mathcal{D}(X)$. The pullback T^*f of any $f\in\mathcal{D}'(Y)$ is defined, compatible with (3.1), by

$$\langle T^*f,\varphi\rangle\triangleq\left\langle f,\left|\det\left(T^{-1}\right)'\right|\left(T^{-1}\right)^*\varphi\right\rangle, \tag{3.38}$$

with $\left|\det\left(T^{-1}\right)'\right|$ the modulus of the Jacobian determinant of the inverse of T, which exists $\forall\mathbf{y}\in Y$ by assumption. It can be shown that $\left|\det\left(T^{-1}\right)'\right|\left(T^{-1}\right)^*\varphi\in\mathcal{D}(Y)$ and $T^*f\in\mathcal{D}'(X)$, [14, Theorem 7.1.1]. Hereafter, we consider three particular diffeomorphisms from the base space $U=R^n$ to itself and write them in terms of the somewhat more explicit symbols $f_{(T(\mathbf{x}))}\triangleq T^*f$ and $\varphi\left(T^{-1}(\mathbf{y})\right)\triangleq\left(T^{-1}\right)^*\varphi$.

(2) *Translation.* For any $\mathbf{x}_0\in R^n$ denote by $T_{\mathbf{x}_0}:R^n\to R^n$ translation over $\mathbf{x}_0$ such that $\mathbf{x}\mapsto T_{\mathbf{x}_0}\mathbf{x}\triangleq\mathbf{x}-\mathbf{x}_0$. The translated distribution $f_{(\mathbf{x}-\mathbf{x}_0)}\triangleq T_{\mathbf{x}_0}^*f\in\mathcal{D}'(R^n)$ of any $f\in\mathcal{D}'(R^n)$ is given by

$$\left\langle f_{(\mathbf{x}-\mathbf{x}_0)},\varphi(\mathbf{x})\right\rangle=\left\langle f_{(\mathbf{x})},\varphi(\mathbf{x}+\mathbf{x}_0)\right\rangle. \tag{3.39}$$

A distribution is called periodic with period $\mathbf{x}_0\neq0$ iff $f_{(\mathbf{x}-\mathbf{x}_0)}=f_{(\mathbf{x})}$.

(3) *Reflection.* Let $R:R^n\to R^n$ such that $\mathbf{x}\mapsto R\mathbf{x}\triangleq-\mathbf{x}$ denote reflection with respect to the origin. The reflected distribution $f_{(-\mathbf{x})}\triangleq R^*f\in\mathcal{D}'(R^n)$ of any $f\in\mathcal{D}'(R^n)$ is given by

$$\left\langle f_{(-\mathbf{x})},\varphi(\mathbf{x})\right\rangle=\left\langle f_{(\mathbf{x})},\varphi(-\mathbf{x})\right\rangle. \tag{3.40}$$

A distribution has even parity (is even) if $f_{(-\mathbf{x})} = f_{(\mathbf{x})}$ and odd parity (is odd) if $f_{(-\mathbf{x})} = -f_{(\mathbf{x})}$. A useful identity is

$$((R^* f) * \varphi)(\mathbf{x}) = \left\langle f_{(\mathbf{y})}, \varphi(\mathbf{x}+\mathbf{y}) \right\rangle, \tag{3.41}$$

wherein R^* is the pullback of the reflection operator. From combining (3.41) with (3.6)–(3.7), it readily follows that the adjoint operator of a distributional convolution operator $F' \triangleq f*$, is the test function convolution operator $F \triangleq (R^* f)*$, which are such that

$$\langle F' g, \varphi \rangle = \langle g, F\varphi \rangle. \tag{3.42}$$

(4) *Dilatation.* For any $r \in R_+$ denote by $U_r : R^n \to R^n$ dilatation by r such that $\mathbf{x} \mapsto U_r \mathbf{x} \triangleq r\mathbf{x}$. The dilated distribution $f_{(r\mathbf{x})} \triangleq U_r^* f \in \mathcal{D}'(R^n)$ of any $f \in \mathcal{D}'(R^n)$ is given by

$$\left\langle f_{(r\mathbf{x})}, \varphi(\mathbf{x}) \right\rangle = r^{-n} \left\langle f_{(\mathbf{x})}, \varphi(\mathbf{x}/r) \right\rangle. \tag{3.43}$$

A distribution is called self-similar with scaling factor $r \neq 1$ iff $f_{(r\mathbf{x})} = f_{(\mathbf{x})}$.

4. Associated homogeneous distributions

This section formally defines AHDs on R and reviews their general properties.

4.1. **Definition.**

(1) A distribution $f_0^z \in \mathcal{D}'$ is called a (positively) homogeneous distribution of degree of homogeneity $z \in \mathbb{C}$ iff it satisfies for any $r > 0$,

$$\left\langle (f_0^z)_{(x)}, \varphi(x/r) \right\rangle = r^{z+1} \left\langle (f_0^z)_{(x)}, \varphi(x) \right\rangle, \forall \varphi \in \mathcal{D}, \tag{4.1}$$

or, using (3.43), $(f_0^z)_{(rx)} = r^z (f_0^z)_{(x)}$. A homogeneous distribution is also called an associated homogeneous distribution of order $m = 0$. A homogeneous distribution of degree $z = 0$ is a self-similar distribution.

A distribution $f_m^z \in \mathcal{D}'$ is called an associated (positively) homogeneous distribution of degree of homogeneity $z \in \mathbb{C}$ and order of association $m \in \mathbb{Z}_+$ iff there exists a sequence of associated homogeneous distributions f_{m-l}^z of degree of homogeneity z and order of association $m - l$, $\forall l \in \mathbb{Z}_{[1,m]}$, not depending on r and with $f_0^z \neq 0$, satisfying for any $r > 0$,

$$\left\langle (f_m^z)_{(x)}, \varphi(x/r) \right\rangle = r^{z+1} \left\langle \left(f_m^z + \sum_{l=1}^{m} \frac{(\ln r)^l}{l!} f_{m-l}^z \right)_{(x)}, \varphi(x) \right\rangle, \forall \varphi \in \mathcal{D}. \tag{4.2}$$

The notion of order used here, is not to be confused with what is commonly called the order of a distribution, see e.g., [28, p. 94].

(2) Comment. The following two main definitions of one-dimensional AHDs can be found in the literature.

(i) The one originally given by Gel'fand and Shilov in [15, p. 84, eq. (3)], and a close variant of it used in the books by Estrada and Kanwal, [7], [8], both of which are eventually stated as

$$\langle f_m^z, \varphi(x/r) \rangle = r^{z+1} \left\langle \left(f_m^z + (\ln r) f_{m-1}^z \right), \varphi(x) \right\rangle, \tag{4.3}$$

wherein f_{m-1}^z in the right-hand side is an AHD of degree of homogeneity z and order of association $m - 1$.

(ii) The definition given by Shelkovich in [25, eq. (3.7)] and which is equivalent to (4.1)–(4.2). Shelkovich calls his distributions Quasi Associated Homogeneous Distributions (QAHDs), to distinguish them from those defined by (4.3), which he calls Associated Homogeneous Distributions (AHDs). The point stressed in [25] is that definition (4.3) is self-contradictory for $m \geq 2$ and only produces HDs and AHDs of order 1. E.g., for $m = 2$, (4.3) gives on the one hand

$$\langle f_2^z, \varphi(x/r)\rangle = r^{z+1} \langle (f_2^z + (\ln r) f_1^z), \varphi(x)\rangle,$$

while on the other hand, with $r = ab$,

$$\begin{aligned} & \langle f_2^z, \varphi(x/r)\rangle \\ = \quad & a^{z+1} \langle (f_2^z + (\ln a) f_1^z), \varphi(x/b)\rangle, \\ = \quad & r^{z+1} \langle (f_2^z + (\ln b) f_1^z), \varphi(x)\rangle + r^{z+1} (\ln a) \langle (f_1^z + (\ln b) f_0^z), \varphi(x)\rangle, \\ = \quad & r^{z+1} \langle (f_2^z + (\ln r) f_1^z + (\ln a)(\ln b) f_0^z), \varphi(x)\rangle. \end{aligned}$$

The vacuous situation of the Gel'fand–Shilov definition is avoided by using definition (4.2). This definition not only imposes a specific dependence on r to f_{m-1}^z in the right-hand side of (4.3), but also implies that the sum of an AHD of order m and an AHD of order n, with $0 \leq n \leq m$, both of the same degree of homogeneity z, is again an AHD of order m and of degree z. This property is essential to accommodate the action of the dilatation operator on AHDs for orders $m \geq 2$.

In this paper we use definition (4.2), but still call the resulting distributions AHDs, because the resulting distributions are already widely known under this name and are essentially the same set as studied in [15, Chapter I, Section 4]. Furthermore, the adjective quasi may be easily confused with the same prefix in the name quasihomogeneous distributions, introduced in [26], where this prefix refers to a more general form of homogeneity which arises for higher-dimensional distributions. In [26], the equivalent of associated homogeneous distributions in n dimensions are defined and called almost quasihomogeneous distributions.

A more detailed discussion of the Gel'fand–Shilov and Shelkovich definitions for one-dimensional distributions and the von Grudzinski definitions for higher-dimensional distributions are given in Appendix A.1.

(3) Another comment is in order on the use of the words *order* (of association) and *degree* (of homogeneity) of an AHD. We are forced to make a strict distinction between these two terms in the theory of AHDs. What is usually called in the literature the "order of differentiation or integration", is actually a number related to the degree of the kernel of the integration convolution operator. It would therefore be more appropriate to speak of the degree of differentiation or integration and of the degree of a differential or integral equation. This matter is important to avoid a clash of terminology, when considering generalizations of integral equations involving associated homogeneous (i.e., power-log) kernels, which are characterized by a particular order of association (power of the logarithm) and a degree of homogeneity (power of the independent variable).

(4) We will denote the set of AHDs based on R, with degree of homogeneity $z \in \mathbb{C}$ and order of association $m \in \mathbb{N}$, by $\mathcal{H}'^z_m(R)$. The set of all AHDs

based on R with common degree of homogeneity z will be denoted $\mathcal{H}'^{z}(R) \triangleq \cup_{\forall m\in\mathbb{N}}\mathcal{H}'^{z}_{m}(R)$. The set of all AHDs based on R with common order of association m will be denoted $\mathcal{H}'_{m}(R) \triangleq \cup_{\forall z\in\mathbb{C}}\mathcal{H}'^{z}_{m}(R)$. The set of all AHDs based on R will be denoted by $\mathcal{H}'(R) \triangleq \cup_{\forall m\in\mathbb{N}}\mathcal{H}'_{m}(R)$. Since we will only consider AHDs based on R, we will further drop (R) in this notation.

(5) *The operator* X_z. Differentiate (4.2) l times with respect to r, put $r = 1$, use the definitions (3.8) and (3.11) of the generalized derivatives, and (3.17). This yields the system, $\forall l \in \mathbb{Z}_{[1,m+1]}$,

$$X_z f^z_{m-(l-1)} = f^z_{m-l}, \tag{4.4}$$

wherein we set $f^z_{-1} \triangleq 0$ and, with Id the identity operator,

$$X_z \triangleq \mathbf{X}\cdot\mathbf{D} - z\,\mathrm{Id}\,. \tag{4.5}$$

The system (4.4) can be used as an equivalent for definition (4.2), see [25]. System (4.4) generalizes Euler's theorem on homogeneous functions, see e.g., [17, p. 10], to AHDs. In particular, for any homogeneous distribution $f^z_0 \in \mathcal{D}'$ holds,

$$(\mathbf{X}\cdot\mathbf{D})\, f^z_0 = z f^z_0, \tag{4.6}$$

so homogeneous distributions are eigendistributions of the generalized Euler operator $\mathbf{X}\cdot\mathbf{D}$, as expected.

(6) *The operator* x_z. We have, $\forall f^z_m \in \mathcal{H}'_m$ and $\forall\varphi \in \mathcal{D}$,

$$\begin{aligned} \langle X_z f^z_m, \varphi\rangle &= -\left\langle f^z_m, x_{-(z+1)}\varphi\right\rangle, && (4.7)\\ x_z &\triangleq \mathbf{x}\cdot\mathbf{d} - z\,\mathrm{Id}\,. && (4.8)\end{aligned}$$

Hence, the adjoint of the operator X_z is the operator $-x_{-(z+1)}$. Applying (4.7) to any homogeneous distribution f^z_0, we see that the operator $x_{-(z+1)}$ maps $\mathcal{D}$ to $\mathcal{D}^z \subset \mathcal{D}$, being that subspace which is the kernel (i.e., the pre-image of 0) of any HD f^z_0 of degree z. Hence, $\mathcal{H}'_0$ is the annihilator of $\mathcal{D}^z$.

4.2. **General properties.** The following properties of AHDs easily follow from (4.2) or (4.4).

(1) AHDs of the same order m, but of different degrees $\{z_1, ..., z_k\}$, are linearly independent.
(2) AHDs of the same degree z, but of different orders $\{m_1, ..., m_k\}$, are linearly independent. Any such linear combination is again an AHD of degree z and of order $m \le \max\{m_1, ..., m_k\}$.
(3) Let f^z_m be an AHD of order m and which is monogenic in its degree $z \in \Omega \subset \mathbb{C}$. If f^z_m has an analytic extension $(f^z_m)_{a.e.}$ to a region $\Omega_1 \supset \Omega$, then $(f^z_m)_{a.e.}$ is an AHD of degree z and order m due to the uniqueness of the process of analytic continuation, [15, p. 150].
(4) AHDs based on R are distributions of slow growth: $\mathcal{H}' \subset \mathcal{S}'$. A proof for homogeneous distributions can be found in [6, pp. 154–155]. By using in addition [9, Theorem 1], property 5(ii) in the next subsection, linearity and induction, it follows that any AHD is a distribution of slow growth.

4.3. **Derivations.**

(1) Let f_p^a and g_q^b be AHDs on R, both monogenic in their degree in $\Omega \subseteq \mathbb{C}$, and such that $f_p^a.g_q^b \in \mathcal{D}'$ and is monogenic in $a+b \in \Omega$. We find that D_z and, by (3.9), (3.12) and (4.5), that X_z are derivations with respect to multiplication,

$$D_z\left(f_p^a.g_q^b\right)_{z=a+b} = \left(D_a f_p^a\right).g_q^b + f_p^a.\left(D_b g_q^b\right), \quad (4.9)$$

$$X_z\left(f_p^a.g_q^b\right)_{z=a+b} = \left(X_a f_p^a\right).g_q^b + f_p^a.\left(X_b g_q^b\right). \quad (4.10)$$

(2) Let f_p^{a-1} and g_q^{b-1} be AHDs on R, both monogenic in their degree in $\Omega \subseteq \mathbb{C}$, and such that $f_p^{a-1} * g_q^{b-1} \in \mathcal{D}'$ and is monogenic in $a+b-1 \in \Omega$. We see that D_z and, by (3.13), (3.10), (3.17) and (4.5), that X_z are both also derivations with respect to convolution,

$$D_z\left(f_p^{a-1} * g_q^{b-1}\right)_{z=a+b-1} = \left(D_z f_p^z\right)_{z=a-1} * g_q^{b-1} + f_p^{a-1} * \left(D_z g_q^z\right)_{z=b-1}, \quad (4.11)$$

$$X_z\left(f_p^{a-1} * g_q^{b-1}\right)_{z=a+b-1} = \left(X_z f_p^z\right)_{z=a-1} * g_q^{b-1} + f_p^{a-1} * \left(X_z g_q^z\right)_{z=b-1}. \quad (4.12)$$

(3) The following commutation relation hold:

$$X_z D_z - D_z X_z = \mathrm{Id}. \quad (4.13)$$

(4) Let $f^z \in \mathcal{D}'$ be a monogenic distribution in $z \in \Omega \subseteq \mathbb{C}$. The properties (3.36)–(3.37), holding for D_z, can now be supplemented with the following properties, holding for X_z, $\forall k, l \in \mathbb{N}$,

$$D^k X_z^l f^z = X_{z-k}^l D^k f^z, \quad (4.14)$$

$$X^k X_z^l f^z = X_{z+k}^l X^k f^z. \quad (4.15)$$

(5) It is readily found from (4.4) and (4.13)–(4.15) that, if f_m^z is an AHD on R of order m and monogenic in its degree z in Ω, then:
(i) $X_z f_m^z$ is associated of order $m-1$ and of the same degree z,
(ii) $D_z f_m^z$ is associated of order $m+1$ and of the same degree z,
(iii) $D f_m^z$ is associated of the same order m and of degree $z-1$, and
(iv) $X f_m^z$ is associated of the same order m and of degree $z+1$.
Moreover, X_z and D_z are parity preserving operators and X and D are parity exchanging operators.

(6) The results stated in 1–5 imply (by induction) the following.
(i) If $f_p^a.g_q^b$ exists as a distribution and neither f_p^a nor g_q^b is a zero divisor, then this multiplication product is associated of order $m = p+q$ and of degree $a+b$.

Under these conditions, any injective multiplication operator with a homogeneous kernel of degree 0 is a map from $\mathcal{H}'_m \to \mathcal{H}'_m$. In particular, the parity reversal transformation (i.e., the multiplication operator $S \triangleq -i\,\mathrm{sgn}\,.$, see (5.4)) then preserves the degree of homogeneity and order of association.

(ii) If $f_p^{a-1} * g_q^{b-1}$ exists as a distribution and neither f_p^{a-1} nor g_q^{b-1} is a zero divisor, then this convolution product is associated of order $m = p+q$ and of degree $a+b-1$.

Under these conditions, any injective convolution operator with a homogeneous kernel of degree -1 is a map from $\mathcal{H}'_m \to \mathcal{H}'_m$. In particular, the

Hilbert transformation (i.e., the convolution operator $H \triangleq \eta *$, see (5.99)) then preserves the degree of homogeneity and order of association.

4.4. **Fourier transformation.** The Fourier transform of any distribution in $\mathcal{D}'$ always exists as an element of the ultradistributions $\mathcal{Z}'$.

(1) By (3.28)–(3.29), we obtain

$$\mathcal{F}_{\mathcal{D}'} X_z + X_{-(z+1)} \mathcal{F}_{\mathcal{D}'} = 0. \tag{4.16}$$

For any f_m^z satisfying $X_z f_m^z = f_{m-1}^z$, with f_{m-1}^z some AHD of degree z and order $m-1$, (4.16) immediately gives

$$X_{-(z+1)} \left(\mathcal{F}_{\mathcal{D}'} f_m^z\right) = -\mathcal{F}_{\mathcal{D}'} f_{m-1}^z, \tag{4.17}$$

which implies (by induction) that $\mathcal{F}_{\mathcal{D}'}$ maps any AHD based on R to an AHD based on R, such that:
(i) the order of association m is preserved,
(ii) the degree of homogeneity z is mapped to $-(z+1)$,
(iii) the parity of the distribution is preserved.

Hence, $\mathcal{F}_{\mathcal{H}'_m} : \mathcal{H}'_m \to \mathcal{H}'_m$ such that $f_m^z \mapsto g_m^{-(z+1)} \triangleq \mathcal{F}_{\mathcal{S}'} f_m^z$ is an automorphism of $\mathcal{H}'_m$. Also, $\mathcal{F}_{\mathcal{H}'} : \mathcal{H}' \to \mathcal{H}'$ is an automorphism of $\mathcal{H}'$.

(2) By (3.26) and continuity with respect to the degree of homogeneity z, we easily deduce that, when operating on any $f_m^z \in \mathcal{H}'$, $\forall k \in \mathbb{Z}_+$,

$$\mathcal{F}_{\mathcal{H}'} D_z^k = D_z^k \mathcal{F}_{\mathcal{H}'}. \tag{4.18}$$

(3) Let f_p^a and g_q^b be AHDs on R. By item 1, $\left(\mathcal{F}_{\mathcal{H}'}^{-1} f_p^a\right)$ and $\left(\mathcal{F}_{\mathcal{H}'}^{-1} g_q^b\right)$ are also AHDs on R. If $\left(\mathcal{F}_{\mathcal{H}'}^{-1} f_p^a\right) * \left(\mathcal{F}_{\mathcal{H}'}^{-1} g_q^b\right)$ exists, define

$$f_p^a . g_q^b \triangleq \mathcal{F}_{\mathcal{H}'} \left(\left(\mathcal{F}_{\mathcal{H}'}^{-1} f_p^a\right) * \left(\mathcal{F}_{\mathcal{H}'}^{-1} g_q^b\right)\right). \tag{4.19}$$

This definition is natural, since it coincides with the convolution theorem in case one of the distributions $\mathcal{F}_{\mathcal{H}'}^{-1} f_p^a$ or $\mathcal{F}_{\mathcal{H}'}^{-1} g_q^b$ is in $\mathcal{E}'$, [28, p. 206], or more generally in $\mathcal{O}'_C$, [23, vol. II, p. 124]. Then, it also coincides with the multiplication defined in (3.3), since one of the distributions f_p^a or g_q^b will be in $\mathcal{Z}_M$ or in $\mathcal{O}_M$. We see from [9, e.g., Theorem 6] that if $-1 < \mathrm{Re}\,(a)$ and $-1 < \mathrm{Re}\,(b)$, then f_p^a and g_q^b are regular AHDs. If $\left(\mathcal{F}_{\mathcal{H}'}^{-1} f_p^a\right) * \left(\mathcal{F}_{\mathcal{H}'}^{-1} g_q^b\right)$ exists, $\mathcal{F}_{\mathcal{H}'} \left(\left(\mathcal{F}_{\mathcal{H}'}^{-1} f_p^a\right) * \left(\mathcal{F}_{\mathcal{H}'}^{-1} g_q^b\right)\right)$ will be an AHD of degree $a+b$. If in addition $-1 < \mathrm{Re}\,(a+b)$, $\mathcal{F}_{\mathcal{H}'} \left(\left(\mathcal{F}_{\mathcal{H}'}^{-1} f_p^a\right) * \left(\mathcal{F}_{\mathcal{H}'}^{-1} g_q^b\right)\right)$ is also a regular AHD. Then it follows from the generalized convolution theorem, that definition (4.19) coincides with definition (3.2).

An example of a multiplication algebra of distributions, based on definition (4.19), is the subset of ultradistributions, which are the Fourier transform of a regular distribution, generated from a continuous function, and which form, together with addition and convolution, an algebra over $\mathbb{C}$.

In [10]–[11], it was shown that for AHDs based on R, f_p^a and g_q^b, the convolution product $f_p^a * g_q^b$ always exists (either directly or as an extension of a partial distribution). This result together with definition (4.19) then paves the way to give a meaning to the multiplication of AHDs based on R.

5. Basic associated homogeneous distributions

5.1. Simple distributions.

(1) The zero distribution 0 is defined by $\langle 0, \varphi \rangle \triangleq 0$. It is a regular homogeneous distribution of undefined degree, which is both even and odd.

(2) We introduce the step functions, $1_\pm : R \rightarrow R$ such that

$$x \quad \mapsto \quad 1_+(x) \triangleq \begin{cases} 1 & \text{if} \quad x > 0 \\ c_+ & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x < 0 \end{cases}, \tag{5.1}$$

$$x \quad \mapsto \quad 1_-(x) \triangleq \begin{cases} 0 & \text{if} \quad x > 0 \\ c_- & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x < 0 \end{cases}, \tag{5.2}$$

wherein the parameters $c_\pm \in R$ are left unspecified. These two parametrized sets of functions $1_\pm$ give rise to two unique regular distributions $1_\pm \in \mathcal{D}'$, respectively, both of which are homogeneous distributions of degree 0.

The regular distributions

$$1 \quad \triangleq \quad 1_+ + 1_+, \tag{5.3}$$

$$\text{sgn} \quad \triangleq \quad 1_+ - 1_+, \tag{5.4}$$

are even and odd homogeneous distributions of degree 0, respectively. The distribution 1 is called the *one* distribution. A constant distribution is a distribution proportional to the distribution 1.

Applying (4.19) and (5.50) with $z = 0$ yields the following multiplication products for the regular distributions $1_\pm$,

$$1_+.1_+ \quad = \quad 1_+, \tag{5.5}$$

$$1_-.1_- \quad = \quad 1_-, \tag{5.6}$$

$$1_-.1_+ \quad = \quad 0 = 1_+.1_-. \tag{5.7}$$

The same multiplication products are also obtained from definition (3.2). One should not confuse the multiplication product of distributions $1_+.1_+ = 1_+$ with the pointwise multiplication product of functions $1_+.1_+ = 1_+$. The former is always true, while the latter depends on the value c_+ and only holds iff $c_+ \in \{0, 1\}$.

(3) With $\varphi \in \mathcal{D}$ and $k \in \mathbb{N}$, define distributions $x^k \in \mathcal{D}'$ by

$$\left\langle x^k, \varphi \right\rangle \triangleq \left(m^{k+1} \varphi \right)(0), \tag{5.8}$$

with $m^{k+1} : \mathcal{D} \rightarrow \mathcal{P}$ the operator of degree k such that

$$\varphi \mapsto \left(m^{k+1} \varphi \right) \triangleq (-1)^k \int_{-\infty}^{+\infty} (x - \tau)^k \varphi(\tau)\, d\tau = (-1)^k \left(x^k * \varphi \right). \tag{5.9}$$

Hence, $\left(m^{k+1} \varphi \right)(0)$ is the k-th moment of φ. Clearly, $\forall k \in \mathbb{N}$: (i) $x^k \in \mathcal{P}$ and $\operatorname{supp}\left(x^k \right) = R$, (ii) x^k is a homogeneous distribution of degree k.

From the definitions (3.11), (5.8) and (3.3) it follows that the generalized convolution derivative of degree k, X^k, is the multiplication operator with kernel x^k, i.e., $X^k = x^k.$. Hence, $x^k = X^k 1$. In particular for $k = 0$, $x^0 = 1$. The one distribution is an identity element in any distributional multiplication algebra.

Further, since x is a regular distribution and because the pointwise multiplication of functions x is associative, we have that, $\forall f \in \mathcal{D}'$ and $\forall k, l \in \mathbb{N}$,

$$X^k \left(X^l f\right) = \left(X^k X^l\right) f = X^{k+l} f. \tag{5.10}$$

The distributions x^k, $\forall k \in \mathbb{N}$, are regular distributions. Obviously, each element of $\mathcal{P}$ is a finite linear combination of x^k distributions.

(4) With $\varphi \in \mathcal{D}$ and $k \in \mathbb{N}$, define distributions $\delta^{(k)} \in \mathcal{D}'$ by

$$\left\langle \delta^{(k)}, \varphi \right\rangle \triangleq (-1)^k \left(d^k \varphi\right)(0), \tag{5.11}$$

with $d^k : \mathcal{D} \to \mathcal{D}$ the ordinary derivative of degree k in $\mathcal{D}$. Note that, $\forall k \in \mathbb{N}$: (i) $\delta^{(k)} \in \mathcal{E}'_0$ with $\operatorname{supp}\left(\delta^{(k)}\right) = \{0\}$, (ii) $\delta^{(k)}$ is a homogeneous distribution of degree $-(k+1)$.

From the definitions (3.8), (5.11) and (3.7) follows that the generalized ordinary derivative of degree k, D^k, is the convolution operator with kernel $\delta^{(k)}$, i.e., $D^k = \delta^{(k)} *$. Hence, $\delta^{(k)} = D^k \delta$. In particular for $k = 0$, $\delta^{(0)} \triangleq \delta$ is called the *delta* distribution and it is an identity element in any distributional convolution algebra.

Further, since convolution of any number of compact support distributions with one of arbitrary support is always associative, we have that, $\forall f \in \mathcal{D}'$ and $\forall k, l \in \mathbb{N}$,

$$D^k \left(D^l f\right) = \left(D^k D^l\right) f = D^{k+l} f. \tag{5.12}$$

The $\delta^{(k)}$, $\forall k \in \mathbb{N}$, are singular distributions. Further, it can be shown that each element of $\mathcal{E}'_0$ is a finite linear combination of $\delta^{(k)}$ distributions, [28, p. 96].

(5) For completeness we also define the regular distributions, $\forall k, m \in \mathbb{N}$,

$$\left\langle x^k \ln^m |x|, \varphi \right\rangle \triangleq \int_{-\infty}^{+\infty} \left(x^k \ln^m |x|\right) \varphi(x) dx, \tag{5.13}$$

$$\left\langle x^k \operatorname{sgn} \ln^m |x|, \varphi \right\rangle \triangleq \int_{-\infty}^{+\infty} \left(x^k \operatorname{sgn}(x) \ln^m |x|\right) \varphi(x) dx, \tag{5.14}$$

both of which are AHD of order m and degree k.

(6) Notice that $-i \operatorname{sgn} . : \mathcal{S} \to \mathcal{O}'_C$ such that $\varphi \mapsto -i \operatorname{sgn} . \varphi$, is the classical parity reversal transformation on $\mathcal{S}$ (the transformation in the spectral (frequency) domain that is equivalent to the Hilbert transformation in the original (time) domain). We will call the multiplication operator $S^{k+1} \triangleq -ix^k \operatorname{sgn} . : \mathcal{D}'_{S^{k+1}} \subset \mathcal{D}' \to \mathcal{D}'$ the *generalized parity reversal transformation* of degree k, wherein $\mathcal{D}'_{S^{k+1}}$ is that subset of $\mathcal{D}'$ for which S^{k+1} is defined.

By definition (4.19), we have the distributional relation (for a simple proof see [10, Appendix]),

$$\left(-ix^k \operatorname{sgn}\right) . \left(-ix^l \operatorname{sgn}\right) = -x^{k+l}, \tag{5.15}$$

which for $k = l = 0$ states the well-known anti-involution property of the distributional Hilbert transformation, here expressed in the spectral domain. This makes the kernel $-i \operatorname{sgn}$ a generator of complex structure in those associative distributional multiplication algebras where S is defined.

5.2. The distributions $x_{\pm}^z$.

5.2.1. *Definition.* Let $x \in R$ and $z \in \mathbb{C}$. The distributions $x_{\pm}^{z} \in \mathcal{D}'$ are defined in the half plane $-1 < \operatorname{Re}(z)$ as regular distributions in terms of the integrals, [15, p. 48],

$$\left\langle x_{\pm}^{z}, \varphi \right\rangle \triangleq \int_{-\infty}^{+\infty} \left(|x|^{z} 1_{\pm}(x) \right) \varphi(x) dx, \tag{5.16}$$

and $\operatorname{supp}(x_{\pm}^{z}) = \{0\} \cup R_{\pm}$. Equations such as (5.16), which contain double signs in either side, are to be understood as double equations: one equation holding for the upper signs and the other holding for the lower signs.

Notice that $x_{\pm}^{z}$, given by (5.16), can be regarded as a distributional multiplication $|x|^{z} .1_{\pm}$ for the regular distributions $|x|^{z}$ (i.e., for $-1 < \operatorname{Re}(z)$) and $1_{\pm}$, as the notation suggests.

5.2.2. *Analytic continuation and extension.* The process of analytic continuation of distributions goes back to M. Riesz. The functional (5.16) can be rewritten in the following equivalent form, $\forall k \in \mathbb{Z}_{+}$,

$$\begin{aligned} \left\langle x_{\pm}^{z}, \varphi \right\rangle &= \int_{-\infty}^{+\infty} \left(|x|^{z} 1_{\pm}(x) \right) \left(\varphi(x) - \sum_{l=0}^{k-1} \varphi^{(l)}(0) \frac{x^{l}}{l!} \right) 1_{[+}(1 - x^{2}) dx \\ &\quad + \int_{-\infty}^{+\infty} \left(|x|^{z} 1_{\pm}(x) \right) \varphi(x) 1_{-}(1 - x^{2}) dx \qquad (5.17) \\ &\quad + \sum_{l=0}^{k-1} \frac{\frac{(\pm 1)^{l}}{l!} \varphi^{(l)}(0)}{z + l + 1}, \qquad (5.18) \end{aligned}$$

the right-hand side of which is now valid for $-(k+1) < \operatorname{Re}(z)$ and $z \notin \mathbb{Z}_{[-k,-1]}$. The expression (5.18), holding $\forall k \in \mathbb{Z}_{+}$, thus uniquely extends the definition of $x_{\pm}^{z}$ to all $z \in \mathbb{C} \backslash \mathbb{Z}_{-}$. The so obtained distributions $x_{\pm}^{z}$, for $\operatorname{Re}(z) \leq -1$ and $z \notin \mathbb{Z}_{-}$, are no longer regular distributions.

The analytic continuation process leading to (5.18) also extends the meaning of the distributional multiplication $|x|^{z} .1_{\pm}$, defined above for $-1 < \operatorname{Re}(z)$, now to all $z \in \mathbb{C} \backslash \mathbb{Z}_{-}$. In this sense, it is still legitimate to regard $x_{\pm}^{z}$ as a product, notwithstanding the fact that after continuation $x_{\pm}^{z}$ is no longer a regular distribution.

Expression (5.18) can be further rearranged in the form,

$$\begin{aligned} \left\langle x_{\pm}^{z}, \varphi \right\rangle &= \frac{\left\langle \frac{(\mp 1)^{k-1}}{(k-1)!} \delta^{(k-1)}, \varphi \right\rangle}{z + k} + 1_{p \leq k-2} \sum_{l=p}^{k-2} \frac{\left\langle \frac{(\mp 1)^{l}}{l!} \delta^{(l)}, \varphi \right\rangle}{z + l + 1} \\ &\quad + \int_{-\infty}^{+\infty} \left(|x|^{z} 1_{\pm}(x) \right) \left(T_{p,q} \varphi \right)(x) dx, \end{aligned} \tag{5.19}$$

now holding in the strip $-(k+1) < \operatorname{Re}(z) < -p$ and $z \notin \{-k, -(k-1), ..., -(p+1)\}$. Herein we used (3.31) with $\lambda = 1$ and we can choose $p, q \in \mathbb{N}$ subject to the condition $p + q = k - 1$. Expression (5.19) shows that the distributions $x_{\pm}^{z}$ are complex analytic for all z, except at the points $z = -k \in \mathbb{Z}_{-}$, which are simple poles with residues $x_{\pm,-1}^{-k}$, read off as

$$x_{\pm,-1}^{-k} = \frac{(\mp 1)^{k-1} \delta^{(k-1)}}{(k-1)!}. \tag{5.20}$$

Denote by $x^z_{\pm,a(z=-k)}$ the analytic part of the Laurent series of $x^z_\pm$ about the point $z=-k$. From (5.19) follows

$$\left\langle x^z_{\pm,a(z=-k)},\varphi\right\rangle = 1_{p\leq k-2}\sum_{l=p}^{k-2}\frac{\left\langle \frac{(\mp1)^l}{l!}\delta^{(l)},\varphi\right\rangle}{z+l+1} \tag{5.21}$$

$$+\int_{-\infty}^{+\infty}\left(|x|^z\,1_\pm(x)\right)\left(T_{p,q}\varphi\right)(x)dx, \tag{5.22}$$

$$= \int_{-\infty}^{+\infty}\left(|x|^z\,1_\pm(x)\right)\left(T_{k-1,0}\varphi\right)(x)dx. \tag{5.23}$$

Define new distributions $x^z_{\pm,\varepsilon}$, complex analytic in $-(k+1)<\mathrm{Re}(z)<-p$ and $\forall\varphi\in\mathcal{D}$, by

$$\left\langle x^z_{\pm,\varepsilon},\varphi\right\rangle \triangleq \int_{-\infty}^{+\infty}\left(|x|^z\,1_\pm(x)\right)\left(T_{p,q}\varphi\right)(x)dx, \tag{5.24}$$

$\forall p,q\in\mathbb{N}: p+q=k-1$ and $T_{p,q}$ given by (3.31), but now with general λ. We have $\left\langle x^z_{\pm,\varepsilon},\psi\right\rangle = \left\langle x^z_\pm,\psi\right\rangle$, $\forall\psi\in\mathcal{D}_{\mathbb{Z}_{[-k,-(p+1)]}}$. Thus, $x^z_{\pm,\varepsilon}$ are extensions of $x^z_\pm$ from $\mathcal{D}_{\mathbb{Z}_{[-k,-(p+1)]}}$ to $\mathcal{D}$. We see from (5.23) and (5.24) that $x^z_{\pm,a(z=-k)}$ is just the extension (or regularization of the integral in (5.16) at $z=-k$ in the sense of [15, p. 10]), corresponding to $p=k-1$ and $q=0$ and $\lambda=1$. In particular at the pole $z=-k$, $x^{-k}_{\pm,a(z=-k)}\triangleq x^{-k}_{\pm,0}$ is the extension of the partial distribution $x^{-k}_\pm$, obtained with $p=k-1$, $q=0$ and $\lambda=1$, and given by

$$\left\langle x^{-k}_{\pm,0},\varphi\right\rangle = \int_{-\infty}^{+\infty}\left(|x|^{-k}\,1_\pm(x)\right)\left(T_{k-1,0}\varphi\right)(x)dx. \tag{5.25}$$

The distribution $x^{-k}_{\pm,0}$ (called a "pseudo-function" in [23, vol. I, p. 41]) will be called the analytic finite part of $x^z_\pm$ at $z=-k$. It is easily shown that $x^{-k}_{\pm,0}$ is identical to Hadamard's finite part $\mathrm{Fp}_H\,x^{-k}_\pm$, provided this is defined for x^z_+ by

$$\left\langle \underset{H}{\mathrm{Fp}}\,x^{-k}_+,\varphi\right\rangle \triangleq \lim_{\varepsilon\to0}\left(\begin{array}{c}\int_\varepsilon^{+\infty}x^{-k}\varphi(x)dx\\ +\sum_{l=0}^{k-2}\frac{\varphi^{(l)}(0)}{l!}\frac{\varepsilon^{-k+l+1}}{-k+l+1}+\frac{\varphi^{(k-1)}(0)}{(k-1)!}\ln\varepsilon\end{array}\right),$$

and similarly for x^z_-. The above development neatly explains how Hadamard's approach, of only retaining a finite part of a divergent integral at an isolated singularity, is actually equivalent to the introduction of a particular extension of the partial distribution $x^{-k}_\pm$.

Choosing different values for p and q in (5.24) yield extensions that differ by a linear combination of $\delta^{(l)}$ distributions, $\forall l\in\mathbb{Z}_{[p,k-1]}$. Demanding that an extension $x^{-k}_{\pm,\varepsilon}$ of the partial homogeneous distribution $x^{-k}_\pm$ is again a homogeneous distribution is not sufficient to make this extension unique. For we can always add a term $c\delta^{(k-1)}$ to $x^{-k}_{\pm,0}$, with arbitrary $c\in\mathbb{C}$, without changing its degree of homogeneity and the distribution $x^{-k}_{\pm,e}\triangleq x^{-k}_{\pm,0}+c\delta^{(k-1)}$ is an equally acceptable extension from $\mathcal{D}_{\{-k\}}$ to $\mathcal{D}$, besides $x^{-k}_{\pm,0}$. This can be made more explicit by checking the homogeneity of the distributions $x^z_{\pm,\varepsilon}$, defined in (5.24), using (4.1). Invoking (3.35), we get

$$\left\langle x^z_{\pm,\varepsilon},\varphi_{x/r}\right\rangle = r^{z+1}\left\langle x^z_{\pm,\varepsilon},\varphi_x\right\rangle + r^{z+1}\sum_{l=p}^{p+q}(\pm1)^l\,\frac{\varphi^{(l)}(0)}{l!}\int_{1/r}^{1}y^{z+l}\lambda\left(ry\right)dy. \tag{5.26}$$

If $0 \leq p < k-1$, $x_{\pm,\varepsilon}^{z}$ are not homogeneous extensions of $x_{\pm}^{z}$. If $p = k-1$, (5.26) takes the form

$$\left\langle x_{\pm,e}^{z}, \varphi_{x/r} \right\rangle = r^{z+1} \left\langle x_{\pm,e}^{z} + c\delta^{(k-1)}, \varphi_{x} \right\rangle, \tag{5.27}$$

for some $c \in \mathbb{C}$ (depending on λ). Eq. (5.27) shows that $x_{\pm,e}^{z}$ are homogeneous of degree z and of first order of association. This also holds at $z = -k$ for $x_{\pm,0}^{-k}$. Putting $c \triangleq b - a$, we can state the rescaling property of $x_{\pm,0}^{-k}$ equivalently as

$$\left\langle x_{\pm,0}^{-k} + a\delta^{(k-1)}, \varphi_{x/r} \right\rangle = r^{-k+1} \left\langle x_{\pm,0}^{-k} + b\delta^{(k-1)}, \varphi_{x} \right\rangle, \tag{5.28}$$

with arbitrary $a, b \in \mathbb{C}$. This shows that under a rescaling any extension $x_{\pm,e}^{-k}$ is mapped into another extension $x_{\pm,e'}^{-k}$. We thus see that the concept of homogeneity of a homogeneous extension $x_{\pm,e}^{-k}$ requires that we regard $x_{\pm,e}^{-k}$ as an equivalence set of distributions, which is a subtlety that is rarely mentioned in the literature. A hint at it was given by Hörmander in [16, p. 71], when he wrote that for $P\left(x_{\pm}^{-k}\right)$ "the homogeneity is partly lost".

For instance if $k = 1$, we get for the analytic finite part $x_{\pm,0}^{-1}$ the following particular extension of the partial distribution $x_{\pm}^{-1}$,

$$\left\langle x_{\pm,0}^{-1}, \varphi \right\rangle = \int_{0}^{1} \frac{\varphi(\pm x) - \varphi(0)}{x} dx + \int_{1}^{+\infty} \frac{\varphi(\pm x)}{x} dx, \tag{5.29}$$

while $x_{\pm,e}^{-1}$ is its most general extension of homogeneous degree, given by

$$\left\langle x_{\pm,e}^{-1}, \varphi \right\rangle = \int_{0}^{1} \frac{\varphi(\pm x) - \varphi(0)}{x} dx + \int_{1}^{+\infty} \frac{\varphi(\pm x)}{x} dx + c\varphi(0), \tag{5.30}$$

with arbitrary $c \in \mathbb{C}$.

5.2.3. *Associated distributions.* For $-1 < \mathrm{Re}(z)$, the integral in (5.16) generates a complex analytic function of z, so we have $D_{z}^{m} x_{\pm}^{z} = x_{\pm}^{z} \ln^{m}|x|$, $\forall m \in \mathbb{N}$, where

$$\left\langle x_{\pm}^{z} \ln^{m}|x|, \varphi \right\rangle = \int_{-\infty}^{+\infty} \left(|x|^{z} 1_{\pm}(x) \ln^{m}|x|\right) \varphi(x) dx. \tag{5.31}$$

The distribution $x_{\pm}^{z} \ln^{m}|x|$, given by (5.31), defines the distributional multiplication $x_{\pm}^{z}.\ln^{m}|x|$ for the regular distributions $x_{\pm}^{z}$ (i.e., for $-1 < \mathrm{Re}(z)$) and $\ln^{m}|x|$, as the notation suggests.

In particular, letting $z = k$ in (5.31) gives us the coefficients, $\forall m \in \mathbb{N}$, of the Taylor series of $x_{\pm}^{z}$ about the ordinary points $z = k \in \mathbb{N}$. For $-(k+1) < \mathrm{Re}(z)$ and $z \notin \mathbb{Z}_{[-k,-1]}$ we obtain from (5.18), $\forall k \in \mathbb{Z}_{+}$ and $\forall m \in \mathbb{N}$,

$$\begin{aligned}
&\left\langle x_{\pm}^{z} \ln^{m}|x|, \varphi \right\rangle \\
= &\int_{-\infty}^{+\infty} \left(|x|^{z} 1_{\pm}(x) \ln^{m}|x|\right) \left(\varphi(x) - \sum_{l=0}^{k-1} \varphi^{(l)}(0) \frac{x^{l}}{l!}\right) 1_{[+}(1-x^{2}) dx \\
&+ \int_{-\infty}^{+\infty} \left(|x|^{z} 1_{\pm}(x) \ln^{m}|x|\right) \varphi(x) 1_{-}(1-x^{2}) dx && (5.32) \\
&+ (-1)^{m} \sum_{l=0}^{k-1} \frac{\frac{(\pm 1)^{l}}{l!} \varphi^{(l)}(0)}{(z+l+1)^{m+1}}. && (5.33)
\end{aligned}$$

The analytic continuation process leading to (5.33) can be used to extend the distributional multiplication $x_{\pm}^{z}.\ln^{m}|x|$, defined above for $-1<\mathrm{Re}(z)$, now to all $z\in\mathbb{C}\backslash\mathbb{Z}_{-}$. In this sense, it is still legitimate to regard $x_{\pm}^{z}\ln^{m}|x|$ as a multiplication product.

Further, from (5.19) we get the expression

$$\begin{aligned}\left\langle x_{\pm}^{z}\ln^{m}|x|,\varphi\right\rangle &= (-1)^{m}\frac{\left\langle\frac{(\mp1)^{k-1}}{(k-1)!}\delta^{(k-1)},\varphi\right\rangle}{(z+k)^{m+1}} \qquad (5.34)\\ &\quad+1_{0\leq p\leq k-2}(-1)^{m}\sum_{l=p}^{k-2}\frac{\left\langle\frac{(\mp1)^{l}}{l!}\delta^{(l)},\varphi\right\rangle}{(z+l+1)^{m+1}}\\ &\quad+\int_{-\infty}^{+\infty}\left(|x|^{z}1_{\pm}(x)\ln^{m}|x|\right)\left(T_{p,q}\varphi\right)(x)dx, \qquad (5.35)\end{aligned}$$

with $T_{p,q}$ given by (3.31) and $\lambda=1$, holding in $0<|z+k|<1$. Denote by $x_{\pm,0}^{-k}\ln^{m}|x|$ the constant term of the analytic part of the Laurent series of $x_{\pm}^{z}\ln^{m}|x|$ about the point $z=-k$. We get from (5.35), $\forall m\in\mathbb{N}$,

$$\begin{aligned}&\left\langle x_{\pm,0}^{-k}\ln^{m}|x|,\varphi\right\rangle\\ &= 1_{0\leq p\leq k-2}(-1)^{m}\sum_{l=p}^{k-2}\frac{\left\langle\frac{(\mp1)^{l}}{l!}\delta^{(l)},\varphi\right\rangle}{(-k+l+1)^{m+1}} \qquad (5.36)\\ &\quad+\int_{-\infty}^{+\infty}\left(|x|^{-k}1_{\pm}(x)\ln^{m}|x|\right)\left(T_{p,q}\varphi\right)(x)dx,\\ &= \int_{-\infty}^{+\infty}\left(|x|^{-k}1_{\pm}(x)\ln^{m}|x|\right)\left(T_{k-1,0}\varphi\right)(x)dx. \qquad (5.37)\end{aligned}$$

The distributions $x_{\pm,0}^{-k}\ln^{m}|x|$, as given by (5.37), are extensions of $x_{\pm}^{z}\ln^{m}|x|$ at the pole $z=-k$, in the sense of (3.30), for the particular values $p=k-1$ and $q=0$. Combining (5.19) and (5.37) gives us the Laurent series of $x_{\pm}^{z}$ about $z=-k$ in the form

$$x_{\pm}^{z}=\frac{\frac{(\mp1)^{k-1}}{(k-1)!}\delta^{(k-1)}}{z+k}+\sum_{m=0}^{+\infty}x_{\pm,0}^{-k}\ln^{m}|x|\frac{(z+k)^{m}}{m!}, \qquad (5.38)$$

holding in $0<|z+k|<1$.

Regarded as distributions, the $\lim_{z\to-k}D_{z}^{m}x_{\pm}^{z}$ does not exist, but we do have $\lim_{z\to-k}D_{z}^{m}x_{\pm,\varepsilon}^{z}=\left(D_{z}^{m}x_{\pm,\varepsilon}^{z}\right)_{z=-k}=x_{\pm,\varepsilon}^{-k}\ln^{m}|x|$, due to (5.24). From

$$\begin{aligned}\left\langle x_{\pm,\varepsilon}^{z}\ln^{m}|x|,\varphi\right\rangle &\triangleq \left\langle D_{z}^{m}x_{\pm,\varepsilon}^{z},\varphi\right\rangle,\\ &= D_{z}^{m}\left\langle x_{\pm,\varepsilon}^{z},\varphi\right\rangle \qquad (5.39)\\ &= \int_{-\infty}^{+\infty}\left(|x|^{z}1_{\pm}(x)\ln^{m}|x|\right)\left(T_{p,q}\varphi\right)(x)dx, \qquad (5.40)\end{aligned}$$

and $T_{p,q}$ given by (3.31) but now with general λ, and the continuity of $x_{\pm,\varepsilon}^{z}$ at $z=-k$, we see that we can regard $x_{\pm,\varepsilon}^{-k}\ln^{m}|x|$ as the distributional multiplication of $x_{\pm,\varepsilon}^{-k}$ and $\ln^{m}|x|$, with this product being defined by (5.40).

The distributions $D_{z}^{m}x_{\pm}^{z}=x_{\pm}^{z}\ln^{m}|x|$, with $z\in\mathbb{C}\backslash\mathbb{Z}_{-}$, are AHDs of order m and degree z, while $x_{\pm,e}^{-k}\ln^{m}|x|\triangleq x_{\pm,e}^{-k}.\ln^{m}|x|$, $\forall k\in\mathbb{Z}_{+}$, are AHDs of order $m+1$ and degree $-k$. The extensions $x_{\pm,e}^{-k}\ln^m|x|$ are again to be understood as associated homogeneous equivalence sets, in the sense of the previous subsection. This is made more explicit, now by starting from (4.2) and invoking result [13, eq. (20)], as

$$\left\langle x_{\pm,e}^{-k}.\ln^m|x|\,,\varphi_{x/r}\right\rangle=r^{-k+1}\left\langle x_{\pm,e}^{-k}.\ln^m|x|+(\ln r)\,f_m^{-k},\varphi_x\right\rangle, \tag{5.41}$$

wherein f_m^{-k} is associated of order m. Again the extensions in the left-hand and right-hand members of (5.41) are different elements of the equivalence set $x_{\pm,e}^{-k}\ln^m|x|$.

5.2.4. *Generalized multiplication derivatives.* For $-(1-n)<\mathrm{Re}(z)$, we apply definition (3.8), the associativity property (5.12) and induction to (5.16). This results in, $\forall n\in\mathbb{Z}_+$,

$$D^n x_\pm^z=(\pm1)^n\,z_{(n)}\,x_\pm^{z-n}, \tag{5.42}$$

with $z_{(n)}$ given by (2.7). The generalized derivatives $D^n\left(x_\pm^z\ln^m|x|\right)$, $\forall n\in\mathbb{Z}_+$ and for $-(1-n)<\mathrm{Re}(z)$, now easily follow from the commutation of D^n and D_z^m. This yields

$$\begin{aligned}D^n\left(x_\pm^z\ln^m|x|\right)&=D_z^m\left((\pm1)^n\,z_{(n)}\,x_\pm^{z-n}\right),\\&=(\pm1)^n\,x_\pm^{z-n}\sum_{p=0}^m\binom{m}{p}\left(D_z^{m-p}z_{(n)}\right)\ln^p|x|\,.\end{aligned} \tag{5.43}$$

By analytic continuation, this holds $\forall z\in\mathbb{C}\backslash\left(\mathbb{Z}_-\cup\mathbb{Z}_{[0,n-1]}\right)$.

The generalized derivatives $D\left(1_\pm\ln^m|x|\right)$, $\forall m\in\mathbb{N}$, are found by first applying definition (3.8) to (5.31) with $z=0$. We get

$$D\left(1_\pm\ln^m|x|\right)=\pm1_{m=0}\delta\pm1_{m>0}\,m\,x_{\pm,0}^{-1}\ln^{m-1}|x|\,. \tag{5.44}$$

For $z=-k\in\mathbb{Z}_-$ and $\forall m\in\mathbb{N}$, we start from the extension (5.37) and use (3.33) with $\lambda=1$. This gives

$$\begin{aligned}\left\langle D\left(x_{\pm,0}^{-k}\ln^m|x|\right),\varphi\right\rangle&=-\left\langle x_{\pm,0}^{-k}\ln^m|x|\,,d\varphi\right\rangle,\\&=-\int_{-\infty}^{+\infty}\left(|x|^{-k}\,1_\pm(x)\ln^m|x|\right)d\left(T_{k,0}\varphi\right)(x)dx.\end{aligned} \tag{5.45}$$

By partial integration of (5.45) we deduce the following expression, holding $\forall k\in\mathbb{Z}_+$ and $\forall m\in\mathbb{N}$,

$$\begin{aligned}D\left(x_{\pm,0}^{-k}\ln^m|x|\right)&=\pm1_{m=0}\frac{(\mp1)^k}{k!}\delta^{(k)}\\&\quad\pm(-k)\,x_{\pm,0}^{-(k+1)}\ln^m|x|\pm1_{m>0}\,m\,x_{\pm,0}^{-(k+1)}\ln^{m-1}|x|.\end{aligned} \tag{5.46}$$

From this result with $m=0$ follows, using (5.12) and induction, that, $\forall k,n\in\mathbb{Z}_+$,

$$\begin{aligned}D^n\left((k-1)!x_{\pm,0}^{-k}\right)&=-(\mp1)^{k-1}\left(H_{k-1+n}-H_{k-1}\right)\delta^{(k-1+n)}\\&\quad+(\mp1)^n\,(k-1+n)!\,x_{\pm,0}^{-(k+n)},\end{aligned} \tag{5.47}$$

wherein $H_k\triangleq1_{k>0}\sum_{p=1}^k\frac{1}{p}$ are the harmonic numbers. Eq. (5.47) generalizes the result [15, p. 87, eq. (5)] and (5.46) appears to be new.

5.2.5. *Generalized convolution derivatives.* For $-1 < \mathrm{Re}(z)$, we apply definition (3.11), the associativity property (5.10) and induction to (5.31). We get, $\forall n \in \mathbb{Z}_+$ and $\forall m \in \mathbb{N}$,

$$X^n \left(x_\pm^z \ln^m |x|\right) = (\pm 1)^n x_\pm^{z+n} \ln^m |x| . \tag{5.48}$$

By analytic continuation, this holds $\forall z \in \mathbb{C}\backslash\mathbb{Z}_-$.

For $z = -k \in \mathbb{Z}_-$ and $\forall m, n \in \mathbb{N}$, we start from the extension (5.37) and use (3.34) together with $T_{0,0}(x\varphi) = x\varphi$. By induction we get,

$$X^n \left(x_{\pm,0}^{-k} \ln^m |x|\right) = 1_{k\leq n} (\pm 1)^n \; x_\pm^{n-k} \ln^m |x| + 1_{n<k} (\pm 1)^n \; x_{\pm,0}^{-(k-n)} \ln^m |x| . \tag{5.49}$$

5.2.6. *Fourier transforms.* The Fourier transform of the distributions $x_\pm^z$, $\forall z \in \mathbb{C}\backslash\mathbb{Z}_-$, is easily obtained from (6.36) as ($\mathcal{F}$ is from here on shorthand for $\mathcal{F}_{\mathcal{H}'}$)

$$\mathcal{F}\left[(2\pi x)_\pm^z\right] = \Phi_{\chi\mp i0}^{-z}. \tag{5.50}$$

More generally, $\forall m \in \mathbb{N}$ and $\forall z \in \mathbb{C}\backslash\mathbb{Z}_-$, we have by the commutativity of the $\mathcal{F}$ and D_z operators,

$$\mathcal{F}\left[D_z^m x_\pm^z\right] = (-1)^m (2\pi)^{-z} \sum_{p=0}^{m} \binom{m}{p} \ln^{m-p}(2\pi) \left(D_w^p \Phi_{\chi\mp i0}^w\right)_{w=-z} . \tag{5.51}$$

The distributions $D_z^m \Phi_{\chi\pm i0}^z$ are given by (6.111).

The Fourier transform of the distributions $x_{\pm,0}^{-k} \ln^m |x|$, $\forall k \in \mathbb{Z}_+$ and $\forall m \in \mathbb{N}$, is obtained as follows. Use (5.38), the Laurent series of the distributions $\Phi_{\chi\mp i0}^{-z}$ about their simple pole $-z = k$, the Taylor series of $(2\pi)^{-z}$ about the ordinary point $-z = k$, and the Fourier transform pair $\mathcal{F}\left[\delta^{(k-1)}\right] = (2\pi i\chi)^{k-1}$. This results in

$$\begin{aligned} \mathcal{F}\left[x_{\pm,0}^{-k} \ln^m |x|\right] \;\; &= \;\; (-1)^{m+1} \frac{\ln^{m+1}(2\pi)}{m+1} \frac{(\mp 2\pi i\chi)^{k-1}}{(k-1)!} \\ &\quad + (2\pi)^k (-1)^m \sum_{p=0}^{m} \binom{m}{p} \ln^{m-p}(2\pi) \left(D_k^p \Phi_{\chi\mp i0}^k\right)_0 . \end{aligned} \tag{5.52}$$

In particular for $m = 0$, using (6.109), (5.52) reduces to

$$\mathcal{F}\left[x_{\pm,0}^{-k}\right] = (\mp\pi i) \frac{1}{2} \frac{(\mp 2\pi i\chi)^{k-1}}{(k-1)!} \operatorname{sgn} - \frac{(\mp 2\pi i\chi)^{k-1}}{(k-1)!} \left(\ln |2\pi\chi| - \psi(k)\right) . \tag{5.53}$$

The distributions $\left(D_k^p \Phi_{\chi\pm i0}^k\right)_0$ are given by (6.112)–(6.113). Expressions (5.52)–(5.53) are new.

We will call the $x_\pm^z$ *homogeneous half-line kernels* and the $D_z^m x_\pm^z$, $\forall m \in \mathbb{Z}_+$, *associated homogeneous half-line kernels*. The distributions $D_z^m x_\pm^z$ will turn out to be convenient basis distributions for the set of AHDs, in order to easily compute multiplication products.

5.3. **The even distribution $|x|^z$.**

5.3.1. *Definition.* The even distribution $|x|^z \in \mathcal{D}'$ is defined $\forall z \in \mathbb{C}\backslash\mathbb{Z}_-$ by the sum, [15, p. 50],

$$|x|^z \triangleq x_+^z + x_-^z, \tag{5.54}$$

and $\operatorname{supp}(|x|^z) = R$. From (5.54) and (5.19) follows, for $-k \in \mathbb{Z}_{o,-}$,

$$\langle |x|^z, \varphi\rangle = \frac{\left\langle \frac{2o_k}{(k-1)!}\delta^{(k-1)}, \varphi\right\rangle}{z+k} + \int_{-\infty}^{+\infty} |x|^z \left(T_{k-1,0}\varphi\right)(x)dx. \tag{5.55}$$

The distribution $|x|^z$ can be analytically extended to all $z \in \mathbb{C}$, except for the points $z = -(2p+1) \in \mathbb{Z}_{o,-}$, which are simple poles with residue

$$|x|_{-1}^{-(2p+1)} = 2\frac{\delta^{(2p)}}{(2p)!}. \tag{5.56}$$

Each distribution denoted by

$$\left(x^{-(2p+1)} \operatorname{sgn}\right)_0 \triangleq |x|_0^{-(2p+1)} \triangleq x_{+,0}^{-(2p+1)} + x_{-,0}^{-(2p+1)}, \tag{5.57}$$

is an extension of the distribution $|x|^z$ at its poles. At the ordinary points $z = -2p \in \mathbb{Z}_{e,-]}$,

$$x^{-2p} \triangleq |x|^{-2p} \triangleq x_{+,0}^{-2p} + x_{-,0}^{-2p}. \tag{5.58}$$

From (5.54) and (5.19) follows, for $-k \in \mathbb{Z}_{o,-}$, that

$$\langle |x|^z \operatorname{sgn}, \varphi\rangle = \frac{\left\langle \frac{2o_k}{(k-1)!}\delta^{(k-1)}, \varphi\right\rangle}{z+k} + \int_{-\infty}^{+\infty} \left(|x|^z \operatorname{sgn}(x)\right) \left(T_{k-1,0}\varphi\right)(x)dx, \tag{5.59}$$

holding in the strip $-(k+2) < \operatorname{Re}(z) < -(k-2)$ and $z \neq -k$.

5.3.2. *Associated distributions.* We have $D_z^m |x|^z = |x|^z \ln^m |x|$, $\forall z \in \mathbb{C}\backslash\mathbb{Z}_{o,-}$ and $\forall m \in \mathbb{N}$. For $-(k+1) < \operatorname{Re}(z)$ and $z \notin \mathbb{Z}_{[-k,-1]} \cap \mathbb{Z}_{o,-}$ we obtain from (5.33), $\forall k \in \mathbb{Z}_+$ and $\forall m \in \mathbb{N}$,

$$\begin{aligned}
& \langle |x|^z \ln^m |x|, \varphi\rangle \\
= \; & \int_{-\infty}^{+\infty} \left(|x|^z \ln^m |x|\right) \left(\varphi(x) - \sum_{l=0}^{k-1} \varphi^{(l)}(0)\frac{x^l}{l!}\right) 1_{[+}(1-x^2)dx \\
& + \int_{-\infty}^{+\infty} \left(|x|^z \ln^m |x|\right) \varphi(x) 1_-(1-x^2)dx \qquad (5.60) \\
& + (-1)^m \sum_{l=0}^{k-1} \frac{\frac{2e_l}{l!}\varphi^{(l)}(0)}{(z+l+1)^{m+1}}. \qquad (5.61)
\end{aligned}$$

From (5.59) follows, for $-k \in \mathbb{Z}_{o,-}$,

$$\left\langle \left(x^{-k} \operatorname{sgn}\right)_0 \ln^m |x|, \varphi\right\rangle = \int_{-\infty}^{+\infty} \left(x^{-k} \operatorname{sgn}(x) \ln^m |x|\right) \left(T_{k-1,0}\varphi\right)(x)dx, \tag{5.62}$$

and for the ordinary points $-k \in \mathbb{Z}_{e,-]}$,

$$\left\langle x^{-k} \ln^m |x|, \varphi\right\rangle = \int_{-\infty}^{+\infty} \left(x^{-k} \ln^m |x|\right) \left(T_{k-1,0}\varphi\right)(x)dx. \tag{5.63}$$

5.3.3. *Generalized multiplication derivatives.* From (5.54), (5.44) and (5.43) follows

$$D\left(\ln^m |x|\right) = 1_{m>0} m\, x^{-1} \ln^{m-1} |x| , \tag{5.64}$$

and, $\forall z \in \mathbb{C}\backslash(-\mathbb{N})$,

$$D\left(|x|^z \ln^m |x|\right) = z\, |x|^{z-1} \operatorname{sgn} \ln^m |x| + 1_{m>0} m\ |x|^{z-1} \operatorname{sgn} \ln^{m-1} |x| . \tag{5.65}$$

Combining (5.46) with (5.57) gives the generalized multiplication derivatives, for $-k = -(2p+1) \in \mathbb{Z}_{o,-}$,

$$\begin{aligned} D\left(|x|_0^{-(2p+1)} \ln^m |x|\right) &= -1_{m=0} 2 \frac{\delta^{(2p+1)}}{(2p+1)!} - (2p+1)\left(|x|^{-(2p+2)} \operatorname{sgn}\right)_0 \ln^m |x| \\ &\quad + 1_{m>0} m \left(|x|^{-(2p+2)} \operatorname{sgn}\right)_0 \ln^{m-1} |x| , \end{aligned} \tag{5.66}$$

and combining (5.46) with (5.58) gives, for $-k = -2p \in \mathbb{Z}_{e,-]}$,

$$D\left(|x|^{-2p} \ln^m |x|\right) = -(2p)\, |x|^{-(2p+1)} \operatorname{sgn} \ln^m |x| + 1_{m>0} m\ |x|^{-(2p+1)} \operatorname{sgn} \ln^{m-1} |x| . \tag{5.67}$$

Expressions (5.66)–(5.67) are new results.

5.3.4. *Generalized convolution derivatives.* From (5.54) and (5.48) follows, $\forall z \in \mathbb{C}\backslash\mathbb{Z}_-$ and $\forall m, n \in \mathbb{N}$,

$$X^n\left(|x|^z \ln^m |x|\right) = |x|^{z+n} \left(e_n 1 + o_n \operatorname{sgn}\right) \ln^m |x| . \tag{5.68}$$

Combining (5.49) with (5.76) gives the generalized multiplication derivatives, for $-k = -(2p+1) \in \mathbb{Z}_{o,-}$ and $\forall m, n \in \mathbb{N}$,

$$X^n\left(|x|_0^{-(2p+1)} \ln^m |x|\right) = \begin{pmatrix} 1_{2p+1 \le n} x^{n-(2p+1)} \operatorname{sgn} \\ +1_{n<2p+1}\left(x^{-(2p+1-n)} \operatorname{sgn}\right)_0 \end{pmatrix} \ln^m |x| , \tag{5.69}$$

and combining (5.49) with (5.77) gives, for $-k = -2p \in \mathbb{Z}_{e,-]}$ and $\forall m, n \in \mathbb{N}$,

$$X^n\left(|x|^{-2p} \ln^m |x|\right) = x^{n-2p} \ln^m |x| . \tag{5.70}$$

The $|x|^z \ln^m |x|$, with $z \in \mathbb{C}\backslash\mathbb{Z}_{o,-}$, are AHDs of order m and degree z and $|x|_0^{-(2p+1)} \ln^m |x|$, $\forall p \in \mathbb{N}$, are AHDs of order $m+1$ and degree $-(2p+1)$.

5.3.5. *Fourier transforms.* The Fourier transform of the distribution $|x|^z$, $\forall z \in \mathbb{C}\backslash\mathbb{Z}_{o,-}$, is easily obtained from (5.54), (5.50) and (6.106) as

$$\mathcal{F}\left[|2\pi x|^z\right] = \Phi_e^{-z}. \tag{5.71}$$

More generally, $\forall m \in \mathbb{N}$ and $\forall z \in \mathbb{C}\backslash\mathbb{Z}_{o,-}$, we have by the commutativity of the $\mathcal{F}$ and D_z operators,

$$\mathcal{F}\left[|x|^z \ln^m |x|\right] = (2\pi)^{-z} (-1)^m \sum_{p=0}^{m} \tbinom{m}{p} \ln^{m-p}(2\pi) \left(D_w^p \Phi_e^w\right)_{w=-z} . \tag{5.72}$$

The distributions $D_z^m \Phi_e^z$ are given by (6.56).

Adding both Fourier transform in (5.52) gives, $\forall p \in \mathbb{N}$,

$$\begin{aligned} & \mathcal{F}\left[|x|_0^{-(2p+1)} \ln^m |x|\right] \\ = & -(-1)^m \, 2 \frac{\ln^{m+1}(2\pi)}{m+1} \frac{(2\pi i\chi)^{2p}}{(2p)!} \\ & + (-1)^m (2\pi)^{2p+1} \sum_{p=0}^{m} \binom{m}{p} \ln^{m-p}(2\pi) \left(\left(D_w^p \Phi_e^w\right)_{w=2p+1} \right)_0 . \end{aligned} \tag{5.73}$$

The distributions $\left(\left(D_w^p \Phi_e^w\right)_{w=2p+1} \right)_0$ are given by (6.58). Eq. (5.73) is new.

We will call the distribution $|x|^z$ a *homogeneous even kernel* and the distributions $D_z^m |x|^z \operatorname{sgn}$, $\forall m \in \mathbb{Z}_+$, *associated homogeneous even kernels.*

5.4. **The odd distribution $|x|^z \operatorname{sgn}$.**

5.4.1. *Definition.* The odd distribution $|x|^z \operatorname{sgn}$ is defined $\forall z \in \mathbb{C} \backslash \mathbb{Z}_-$ by the difference, [15, p. 50],

$$|x|^z \operatorname{sgn} \triangleq x_+^z - x_-^z, \tag{5.74}$$

and $\operatorname{supp}(|x|^z \operatorname{sgn}) = R$. The distribution $|x|^z \operatorname{sgn}$ can be analytically extended to all $z \in \mathbb{C}$, except for the points $z = -(2p+2) \in \mathbb{Z}_{e,-}$, which are simple poles with residue

$$\left(|x|^{-(2p+2)} \operatorname{sgn} \right)_{-1} = \frac{-2\delta^{(2p+1)}}{(2p+1)!}. \tag{5.75}$$

Each distribution denoted by

$$\left(x^{-(2p+2)} \operatorname{sgn} \right)_0 \triangleq \left(|x|^{-(2p+2)} \operatorname{sgn} \right)_0 \triangleq x_{+,0}^{-(2p+2)} - x_{-,0}^{-(2p+2)} \tag{5.76}$$

is an extension of the distribution $|x|^z \operatorname{sgn}$ at its poles. At the ordinary points $z = -(2p+1) \in \mathbb{Z}_{o,-}$,

$$x^{-(2p+1)} \triangleq |x|^{-(2p+1)} \operatorname{sgn} \triangleq x_{+,0}^{-(2p+1)} - x_{-,0}^{-(2p+1)}. \tag{5.77}$$

For similar reasons as given above for the distributions $x_\pm^z \ln^m |x|$, the distribution $|x|^z \operatorname{sgn}$ defines the multiplication of the distributions $|x|^z$ and sgn, $\forall z \in \mathbb{C} \backslash \mathbb{Z}_{e,-}$.

From (5.74) and (5.19) follows, for $-k \in \mathbb{Z}_{e,-}$, that

$$\langle |x|^z \operatorname{sgn}, \varphi \rangle = \frac{\left\langle -\frac{2e_k}{(k-1)!} \delta^{(k-1)}, \varphi \right\rangle}{z+k} + \int_{-\infty}^{+\infty} \left(|x|^z \operatorname{sgn}(x)\right) \left(T_{k-1,0}\varphi\right)(x) dx, \tag{5.78}$$

holding in the strip $-(k+2) < \operatorname{Re}(z) < -(k-2)$ and $z \neq -k$.

5.4.2. *Associated distributions.* We have $D_z^m (|x|^z \operatorname{sgn}) = |x|^z \operatorname{sgn} \ln^m |x|$, $\forall z \in \mathbb{C} \backslash \mathbb{Z}_{e,-}$ and $\forall m \in \mathbb{N}$. For $-(k+1) < \operatorname{Re}(z)$ and $z \notin \mathbb{Z}_{[-k,-1]} \cap \mathbb{Z}_{e,-}$ we obtain from (5.33),

$\forall k \in \mathbb{Z}_+$ and $\forall m \in \mathbb{N}$,

$$\begin{aligned}
&\langle |x|^z \operatorname{sgn} \ln^m |x|, \varphi \rangle \\
= &\int_{-\infty}^{+\infty} \left(|x|^z \operatorname{sgn}(x) \ln^m |x| \right) \left(\varphi(x) - \sum_{l=0}^{k-1} \varphi^{(l)}(0) \frac{x^l}{l!} \right) 1_{[+}(1-x^2) dx \\
&+ \int_{-\infty}^{+\infty} \left(|x|^z \operatorname{sgn}(x) \ln^m |x| \right) \varphi(x) 1_-(1-x^2) dx \qquad (5.79) \\
&+ (-1)^m \sum_{l=0}^{k-1} \frac{\frac{2o_l}{l!} \varphi^{(l)}(0)}{(z+l+1)^{m+1}}. \qquad (5.80)
\end{aligned}$$

From (5.78) follows, for the ordinary points $-k \in \mathbb{Z}_{o,-}$,

$$\left\langle x^{-k} \ln^m |x|, \varphi \right\rangle = \int_{-\infty}^{+\infty} \left(x^{-k} \ln^m |x| \right) \left(T_{k-1,0} \varphi \right)(x) dx, \qquad (5.81)$$

and for $-k \in \mathbb{Z}_{e,-}$,

$$\left\langle \left(x^{-k} \operatorname{sgn} \right)_0 \ln^m |x|, \varphi \right\rangle = \int_{-\infty}^{+\infty} \left(x^{-k} \operatorname{sgn}(x) \ln^m |x| \right) \left(T_{k-1,0} \varphi \right)(x) dx. \qquad (5.82)$$

5.4.3. *Generalized multiplication derivatives.* From (5.74), (5.44) and (5.43) follows

$$D \left(\operatorname{sgn} \ln^m |x| \right) = 1_{m=0} 2\delta + 1_{m>0} m |x|_0^{-1} \ln^{m-1} |x|, \qquad (5.83)$$

and, $\forall z \in \mathbb{C} \backslash (-\mathbb{N})$,

$$D \left(|x|^z \operatorname{sgn} \ln^m |x| \right) = z\, |x|^{z-1} \ln^m |x| + 1_{m>0} m |x|^{z-1} \ln^{m-1} |x|. \qquad (5.84)$$

Combining (5.46) with (5.77) gives the generalized multiplication derivatives, for $-k = -(2p+1) \in \mathbb{Z}_{o,-}$,

$$D \left(|x|^{-(2p+1)} \operatorname{sgn} \ln^m |x| \right) = -(2p+1) |x|^{-(2p+2)} \ln^m |x| + 1_{m>0} m |x|^{-(2p+2)} \ln^{m-1} |x|, \qquad (5.85)$$

and combining (5.46) with (5.76) gives, for $-k = -(2p+2) \in \mathbb{Z}_{e,-}$,

$$\begin{aligned}
&D \left(\left(|x|^{-(2p+2)} \operatorname{sgn} \right)_0 \ln^m |x| \right) \\
= \;& 1_{m=0} 2 \frac{\delta^{(2p+2)}}{(2p+2)!} - (2p+2) |x|_0^{-(2p+3)} \ln^m |x| \qquad (5.86) \\
&+ 1_{m>0} m |x|_0^{-(2p+3)} \ln^{m-1} |x|. \qquad (5.87)
\end{aligned}$$

Expressions (5.85)–(5.87) are new results.

5.4.4. *Generalized convolution derivatives.* From (5.74) and (5.48) follows, $\forall z \in \mathbb{C} \backslash \mathbb{Z}_-$ and $\forall m, n \in \mathbb{N}$,

$$X^n \left(|x|^z \operatorname{sgn} \ln^m |x| \right) = |x|^{z+n} \left(o_n 1 + e_n \operatorname{sgn} \right) \ln^m |x|. \qquad (5.88)$$

Combining (5.49) with (5.58) gives the generalized multiplication derivatives, for $-k = -(2p+1) \in \mathbb{Z}_{o,-}$ and $\forall m, n \in \mathbb{N}$,

$$X^n \left(|x|^{-(2p+1)} \operatorname{sgn} \ln^m |x| \right) = x^{n-(2p+1)} \ln^m |x|, \qquad (5.89)$$

and combining (5.49) with (5.57) gives, for $-k = -(2p+2) \in \mathbb{Z}_{e,-}$ and $\forall m, n \in \mathbb{N}$,

$$\begin{aligned} & X^n \left(\left(|x|^{-(2p+2)} \operatorname{sgn} \right)_0 \ln^m |x| \right) \\ = & \left(1_{2p+2 \leq n} x^{n-(2p+2)} \operatorname{sgn} + 1_{n<2p+2} \left(x^{-(2p+2-n)} \operatorname{sgn} \right)_0 \right) \ln^m |x| , \end{aligned} \tag{5.90}$$

which is a new expression.

The $|x|^z \operatorname{sgn} \ln^m |x|$, with $z \in \mathbb{C} \backslash \mathbb{Z}_{e,-}$, are AHDs of order m and degree z and $\left(x^{-(2p+2)} \operatorname{sgn}\right)_0 \ln^m |x|$, $\forall p \in \mathbb{N}$, are AHDs of order $m+1$ and degree $-(2p+2)$.

5.4.5. *Fourier transforms.* The Fourier transform of the distribution $|x|^z \operatorname{sgn}$, $\forall z \in \mathbb{C} \backslash \mathbb{Z}_{e,-}$, is easily obtained from (5.74), (5.50) and (6.106) as

$$\mathcal{F}\left[|2\pi x|^z \operatorname{sgn} \right] = -i \Phi_o^{-z}. \tag{5.91}$$

More generally, $\forall m \in \mathbb{N}$ and $\forall z \in \mathbb{C} \backslash \mathbb{Z}_{e,-}$, we have by the commutativity of the $\mathcal{F}$ and D_z operators,

$$\mathcal{F}\left[|x|^z \operatorname{sgn} \ln^m |x| \right] = -i (2\pi)^{-z} (-1)^m \sum_{p=0}^{m} \binom{m}{p} \ln^{m-p} (2\pi) \left(D_w^p \Phi_o^w \right)_{w=-z}. \tag{5.92}$$

The distributions $D_z^m \Phi_o^z$ are given by (6.57).

Subtracting both Fourier transform in (5.52) gives, $\forall p \in \mathbb{N}$,

$$\begin{aligned} & \mathcal{F}\left[\left(\left(|x|^{-(2p+2)} \operatorname{sgn} \right)_0 \ln^m |x| \right] \right. \\ = & (-1)^m 2 \frac{\ln^{m+1}(2\pi)}{m+1} \frac{(2\pi i \chi)^{2p+1}}{(2p+1)!} \\ & - (-1)^m i (2\pi)^{2p+2} \sum_{p=0}^{m} \binom{m}{p} \ln^{m-p} (2\pi) \left(\left(D_w^p \Phi_o^w \right)_{w=2p+2} \right)_0 . \end{aligned} \tag{5.93}$$

The distributions $\left(\left(D_w^p \Phi_o^w \right)_{w=2p+2} \right)_0$ are given by (6.61). Eq. (5.93) is new.

We will call the distribution $|x|^z \operatorname{sgn}$ a *homogeneous odd kernel* and the distributions $D_z^m |x|^z \operatorname{sgn}$, $\forall m \in \mathbb{Z}_+$, *associated homogeneous odd kernels.*

5.5. **The distributions $\eta_{\pm,0}^{(k)}$.** Combining the definitions in (5.63) and (5.81), we see that the distributions $x^{-k} \ln^m |x|$, $\forall k \in \mathbb{Z}_+$ and $\forall m \in \mathbb{N}$, are given by

$$\left\langle x^{-k} \ln^m |x| , \varphi \right\rangle = \int_{-\infty}^{+\infty} \left(x^{-k} \ln^m |x| \right) \left(T_{k-1,0} \varphi \right)(x) dx. \tag{5.94}$$

Further, combining the definitions in (5.62) and (5.82), gives for the distributions $\left(x^{-k} \operatorname{sgn}\right)_0 \ln^m |x|$, $\forall k \in \mathbb{Z}_+$ and $\forall m \in \mathbb{N}$,

$$\left\langle \left(x^{-k} \operatorname{sgn} \right)_0 \ln^m |x| , \varphi \right\rangle = \int_{-\infty}^{+\infty} \left(x^{-k} \operatorname{sgn}(x) \ln^m |x| \right) \left(T_{k-1,0} \varphi \right)(x) dx. \tag{5.95}$$

Note that $x^{-k} \ln^m |x|$ is an AHD of order m, which is an analytic continuation, and $\left(x^{-k} \operatorname{sgn}\right)_0 \ln^m |x|$ an AHD of order $m+1$, which is an extension, both of degree $-k$.

The well-known distribution $x^{-1} \in \mathcal{D}'$, sometimes written as the pseudo-function $\mathrm{Pf}\,\frac{1}{x}$ or as Cauchy's principal value $\mathrm{Pv}\,\frac{1}{x}$, is the particular case of (5.94) with $m = 0$ and $k = 1$, and thus given by

$$\langle x^{-1}, \varphi \rangle \;=\int_0^{+\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx. \tag{5.96}$$

Similarly, from (5.95) we get the particular case,

$$\left\langle \left(x^{-1}\,\mathrm{sgn}\right)_0, \varphi \right\rangle = \int_0^{+\infty} \frac{\varphi(x) + \varphi(-x) - 2\varphi(0) 1_{[+}(1-x)}{x} dx. \tag{5.97}$$

5.5.1. *Definition.* We shall find it convenient to define the following distributions, $\forall k \in \mathbb{N}$ and using (5.47),

$$\eta_{\pm,0}^{(k)} \triangleq D^k \eta_{\pm,0} \triangleq D^k \left(\frac{1}{\pi}\, x_{\pm,0}^{-1} \right) = -\frac{1}{\pi} H_k \delta^{(k)} + \frac{1}{\pi} (\mp 1)^k k!\, x_{\pm,0}^{-(k+1)}. \tag{5.98}$$

The $\eta_{\pm,0}^{(k)}$ are AHDs of first order and degree $-(k+1)$ with $\mathrm{supp}(\eta_{\pm,0}^{(k)}) = \{0\} \cup R_{\pm}$.

Further, using (5.98), (5.58), (5.77), (5.57) and (5.76), we define the linear combinations

$$\eta^{(k)} \;\triangleq\; \eta_{+,0}^{(k)} - \eta_{-,0}^{(k)} = \frac{1}{\pi} (-1)^k k!\, x^{-(k+1)}, \tag{5.99}$$

$$\left(\eta^{(k)}\,\mathrm{sgn}\right)_0 \;\triangleq\; \eta_{+,0}^{(k)} + \eta_{-,0}^{(k)} + \frac{2}{\pi} H_k \delta^{(k)} = \frac{1}{\pi} (-1)^k k! \left(x^{-(k+1)}\,\mathrm{sgn} \right)_0 \tag{5.100}$$

Like $\delta^{(k)}$ is also $\eta^{(k)}$ a homogeneous distribution of degree $-(k+1)$, while $\left(\eta^{(k)}\,\mathrm{sgn}\right)_0$ is an AHD of first order and degree $-(k+1)$. We will call η the *eta* distribution and (5.99) shows that $\pi\eta = \mathrm{Pf}\,\frac{1}{x}$.

Notice that $H_{\mathcal{S}} : \mathcal{S} \to \mathcal{O}_M$ such that $\varphi \mapsto \eta * \varphi$ is the classical Hilbert transformation on $\mathcal{S}$. The transformation $\eta *$ is the one that phase shifts the positive frequencies in φ by $-\pi/2$ and the negative frequencies by $+\pi/2$, while $-\eta *$ shifts the positive frequencies by $+\pi/2$ and the negative frequencies by $-\pi/2$ (for the Fourier transformation defined by (3.18). For these reasons, we will call the convolution operator $H^k \triangleq \eta^{(k)} * : \mathcal{D}'_{H^k} \subset \mathcal{D}' \to \mathcal{D}'$ for $k > 0$ the *generalized Hilbert (multiplication) derivative* of degree k (with $\mathcal{D}'_{H^k}$ the subset of $\mathcal{D}'$ where H^k is defined).

We have the distributional relation (for a simple proof see [10, Appendix]),

$$\eta^{(k)} * \eta^{(l)} = -\delta^{(k+l)}, \tag{5.101}$$

which for $k = l = 0$ states the well-known anti-involution property of the distributional Hilbert transformation $H \triangleq H^0$. This makes η a generator of complex structure in those associative distributional convolution algebras where H is defined.

Further, as is shown in [9, eq. (14)], a general homogeneous distribution of degree $-(k+1)$ is a linear combination of $\delta^{(k)}$ and $\eta^{(k)}$.

All these facts place η on the same level of importance as δ and justify the introduction of a proper name and symbol for this important distribution.

5.5.2. *Associated distributions.* From (5.94) and (5.95) follows, $\forall k, m \in \mathbb{N}$,

$$\left\langle \eta^{(k)} \ln^m |x|, \varphi \right\rangle = \frac{1}{\pi}(-1)^k k! \int_{-\infty}^{+\infty} \left(x^{-(k+1)} \ln^m |x| \right) (T_{k,0}\varphi)(x) dx, \tag{5.102}$$

$$\begin{aligned} &\left\langle \left(\eta^{(k)} \operatorname{sgn} \right)_0 \ln^m |x|, \varphi \right\rangle \\ = &\ \frac{1}{\pi}(-1)^k k! \int_{-\infty}^{+\infty} \left(x^{-(k+1)} \operatorname{sgn}(x) \ln^m |x| \right) (T_{k,0}\varphi)(x) dx. \end{aligned} \tag{5.103}$$

5.5.3. *Generalized multiplication derivatives.* From the definition (5.98) follows, $\forall k, n \in \mathbb{N}$,

$$D^n \eta^{(k)}_{\pm,0} = \eta^{(k+n)}_{\pm,0}. \tag{5.104}$$

Combining (5.67) with (5.85) and (5.66) with (5.87), respectively, gives the generalized multiplication derivatives, $\forall k \in \mathbb{N}$,

$$D\left(\eta^{(k)} \ln^m |x| \right) = \eta^{(k+1)} \ln^m |x| - 1_{m>0} \frac{m}{k+1} \eta^{(k+1)} \ln^{m-1} |x|, \tag{5.105}$$

and

$$\begin{aligned} &D\left(\left(\eta^{(k)} \operatorname{sgn} \right)_0 \ln^m |x| \right) \\ = &\ -1_{m=0} \frac{2}{\pi} \frac{\delta^{(k+1)}}{k+1} \\ &+ \left(\eta^{(k+1)} \operatorname{sgn} \right)_0 \left(\ln^m |x| - 1_{m>0} \frac{m}{k+1} \ln^{m-1} |x| \right). \end{aligned} \tag{5.106}$$

These are new expressions.

5.5.4. *Generalized convolution derivatives.* Using (6.34), (5.49) and (5.98) we obtain, $\forall k, n \in \mathbb{N}$,

$$\begin{aligned} (-X)^n \eta^{(k)}_{\pm,0} = &\ 1_{k<n} \frac{1}{\pi} (\mp 1)^{n-k} k!\, x_\pm^{n-(k+1)} \\ &+ 1_{n\le k} \frac{k!}{(k-n)!} \left(-\frac{1}{\pi} \left(H_k - H_{k-n} \right) \delta^{(k-n)} + \eta^{(k-n)}_{\pm,0} \right). \end{aligned} \tag{5.107}$$

Combining (5.70) with (5.89) and (5.69) with (5.90), respectively, gives the generalized multiplication derivatives, $\forall k, m, n \in \mathbb{N}$,

$$X^n \left(\eta^{(k)} \ln^m |x| \right) = \left(1_{k<n} \frac{1}{\pi} (-1)^k k! x^{n-(k+1)} + 1_{n\le k} (-1)^n \frac{k!}{(k-n)!} \eta^{(k-n)} \right) \ln^m |x|, \tag{5.108}$$

and

$$X^n \left(\left(\eta^{(k)} \operatorname{sgn} \right)_0 \ln^m |x| \right) = \left(\begin{array}{c} 1_{k<n} \frac{1}{\pi} (-1)^k k! x^{n-(k+1)} \operatorname{sgn} \\ +1_{n\le k} (-1)^n \frac{k!}{(k-n)!} \left(\eta^{(k-n)} \operatorname{sgn} \right)_0 \end{array} \right) \ln^m |x|. \tag{5.109}$$

These are also new expressions.

5.5.5. *Fourier transforms.* From (5.72) and (5.92) follows, $\forall k \in \mathbb{Z}_+$ and $\forall m \in \mathbb{N}$,

$$\mathcal{F}\left[x^{-k}\ln^m|x|\right] = (2\pi)^k(-1)^m\sum_{p=0}^{m}\tbinom{m}{p}\ln^{m-p}(2\pi)\left(e_kD_k^p\Phi_e^k - io_kD_k^p\Phi_o^k\right). \quad (5.110)$$

From (5.73) and (5.93) follows, $\forall k \in \mathbb{Z}_+$ and $\forall m \in \mathbb{N}$,

$$\begin{aligned} & \mathcal{F}\left[\left(x^{-k}\operatorname{sgn}\right)_0\ln^m|x|\right] \\ = \;& -(-1)^m\,2\frac{\ln^{m+1}(2\pi)}{m+1}\frac{(-2\pi i\chi)^{k-1}}{(k-1)!} \\ & +(-1)^m(2\pi)^k\sum_{p=0}^{m}\tbinom{m}{p}\ln^{m-p}(2\pi)\left(o_k\left(D_k^p\Phi_e^k\right)_0 - ie_k\left(D_k^p\Phi_o^k\right)_0\right) \quad (5.111)\end{aligned}$$

This is a new result.

In particular for $m=0$ and $\forall k \in \mathbb{Z}_+$, (5.110)–(5.111) become

$$\mathcal{F}\left[x^{-k}\right] = -i\pi\frac{(-2\pi i\chi)^{k-1}}{(k-1)!}\operatorname{sgn}, \quad (5.112)$$

$$\mathcal{F}\left[\left(x^{-k}\operatorname{sgn}\right)_0\right] = -2\frac{(-2\pi i\chi)^{k-1}}{(k-1)!}\left(\ln(2\pi) - \psi(k) + \ln|\chi|\right). \quad (5.113)$$

From this immediately follows by (5.99) and (5.100), $\forall k \in \mathbb{N}$, that

$$\mathcal{F}\left[\eta^{(k)}\right] = -i\,(2\pi i\chi)^k\operatorname{sgn}, \quad (5.114)$$

$$\mathcal{F}\left[\left(\eta^{(k)}\operatorname{sgn}\right)_0\right] = -\frac{2}{\pi}(2\pi i\chi)^k\left(\ln(2\pi) - \psi(k+1) + \ln|\chi|\right). \quad (5.115)$$

Herein is $\psi(k+1) = -\gamma + H_k$, with γ the Euler-Mascheroni constant and H_k the harmonic numbers.

Using (6.37), (5.114)–(5.115), (5.99) and (5.100) gives, $\forall k \in \mathbb{N}$,

$$\mathcal{F}\left[\eta_{\pm,0}^{(k)}\right] = -\frac{1}{\pi}(2\pi i\chi)^k\left(\left(\ln(2\pi)+\gamma\right)1 \pm i\frac{\pi}{2}\operatorname{sgn} + \ln|\chi|\right). \quad (5.116)$$

5.6. **The sets $\mathcal{H}_0'^{-1}$ and $\mathcal{H}_0'^{0}$.** Linear combinations of the delta and eta distributions and their Fourier transforms generate the following interesting subsets of homogeneous distributions.

5.6.1. *The set $\mathcal{H}_0'^{-1}$.* Define the subset of distributions

$$\mathcal{H}_0'^{-1} \triangleq \left\{\omega \in \mathcal{D}' : \omega \triangleq p\delta + q\eta, \forall p,q \in \mathbb{C}\right\}. \quad (5.117)$$

The set $\mathcal{H}_0'^{-1}$ is a two-dimensional linear space of homogeneous distributions of degree -1 over $\mathbb{C}$. In addition, by the usual algebra axioms, $\left(\mathcal{H}_0'^{-1},+,*;\mathbb{C}\right)$ becomes an associative convolution algebra over $\mathbb{C}$. Moreover, $\left(\mathcal{H}_0'^{-1},+,*;R\right)$ is also an associative convolution algebra over R, isomorphic to the bicomplex numbers, [19].

The structure $\left(\mathcal{H}_0'^{-1},+,*\right)$ contains the following two proper, prime and principal ideals: $I_{0,+}^{-1} \triangleq \left\{\frac{1}{2}(\delta+i\eta)*\omega : \omega \in \mathcal{H}_0'^{-1}\right\}$ and $I_{0,-}^{-1} \triangleq \left\{\frac{1}{2}(\delta-i\eta)*\omega : \omega \in \mathcal{H}_0'^{-1}\right\}$, whose elements are zero divisors. Since $\left(\mathcal{H}_0'^{-1},+,*\right)$ is an associative and commutative ring with identity δ and $I_{0,\pm}^{-1}$ are prime ideals, $\left(\mathcal{H}_0'^{-1}/I_{0,\pm}^{-1},+,*\right)$ are integral domains.

Define the map $\exp_* : \mathcal{H}_0'^{-1} \to \mathcal{H}_0'^{-1}$ such that $\omega \mapsto \exp_*(\omega)$ with

$$\begin{aligned}\exp_*(\omega) &\triangleq \sum_{k=0}^{+\infty} \frac{\omega^{*k}}{k!}, \\ &= e^p \left(\cos(q)\delta + \sin(q)\eta\right). \end{aligned} \tag{5.118}$$

Herein we used $\omega^{*0} \triangleq \delta$, $\omega^{*1} \triangleq \omega$, $\omega^{*2} \triangleq \omega * \omega$, etc.. The relation (5.118) is the distributional equivalent of Euler's formula. We will call distributions of the form $\exp_*(q\eta) = \cos(q)\delta + \sin(q)\eta$, *distributional convolution phase factors* with phase q. In particular, we retrieve from (5.118) the distributional convolution equivalent of Euler's identity,

$$\exp_*(\pi\eta) + \delta = 0. \tag{5.119}$$

Further, $\forall \omega_a, \omega_b \in \mathcal{H}_0'^{-1}$ holds that

$$\exp_*(\omega_a) * \exp_*(\omega_b) = \exp_*(\omega_a + \omega_b), \tag{5.120}$$

and in particular, $\forall \omega \in \mathcal{H}_0'^{-1}$,

$$\exp_*(-\omega) * \exp_*(+\omega) = \delta. \tag{5.121}$$

5.6.2. *The set* $\mathcal{H}_0'^{0}$. Define the subset of distributions

$$\mathcal{H}_0'^{0} \triangleq \left\{\omega \in \mathcal{D}' : \omega \triangleq p1 + q\,(-i\,\mathrm{sgn})\,, \forall p, q \in \mathbb{C}\right\}. \tag{5.122}$$

The Fourier transformation $\mathcal{F}_{\mathcal{H}'}$ and its inverse $\mathcal{F}_{\mathcal{H}'}^{-1}$ are homeomorphisms between $\mathcal{H}_0'^{0}$ and $\mathcal{H}_0'^{-1}$, so $\left(\mathcal{H}_0'^{0}, +, .\right)$ is isomorphic to $\left(\mathcal{H}_0'^{-1}, +, *\right)$. The structure $\left(\mathcal{H}_0'^{0}, +, .\right)$ contains the following two proper, prime and principal ideals: $I_{0,+}^0 \triangleq \left\{1_+.\omega : \omega \in \mathcal{H}_0'^{0}\right\}$ and $I_{0,-}^0 \triangleq \left\{1_-.\omega : \omega \in \mathcal{H}_0'^{0}\right\}$, whose elements are zero divisors. Since $\left(\mathcal{H}_0'^{0}, +, .\right)$ is an associative and commutative ring with identity 1 and $I_{0,\pm}^0$ are prime ideals, $\left(\mathcal{H}_0'^{0}/I_{0,\pm}^0, +, .\right)$ are integral domains.

Define the map $\exp_. : \mathcal{H}_0'^{0} \to \mathcal{H}_0'^{0}$ such that $\omega \mapsto \exp_.(\omega)$ with

$$\begin{aligned}\exp_.(\omega) &\triangleq \sum_{k=0}^{+\infty} \frac{\omega^{\cdot k}}{k!}, \\ &= e^p \left(\cos(q)1 + \sin(q)\,(-i\,\mathrm{sgn})\right). \end{aligned} \tag{5.123}$$

Herein we used $\omega^{\cdot 0} \triangleq \delta$, $\omega^{\cdot 1} \triangleq \omega$, $\omega^{\cdot 2} \triangleq \omega.\omega$, etc.. The relation (5.123) is another distributional equivalent of Euler's formula. We will call distributions of the form $\exp_.(q\eta) = \cos(q)1 + \sin(q)\,(-i\,\mathrm{sgn})$, *distributional multiplication phase factors* with phase q. In particular, we retrieve from (5.123) the distributional multiplication equivalent of Euler's identity,

$$\exp_.(-i\pi\,\mathrm{sgn}) + 1 = 0. \tag{5.124}$$

Further, $\forall \omega_a, \omega_b \in \mathcal{H}_0'^{0}$ holds that

$$\exp_.(\omega_a).\exp_.(\omega_b) = \exp_.(\omega_a + \omega_b), \tag{5.125}$$

and in particular, $\forall \omega \in \mathcal{H}_0'^{0}$,

$$\exp_.(-\omega)\,.\exp_.(+\omega) = 1. \tag{5.126}$$

5.7. **The distributions** $\ln(x \pm i0)$**.**

5.7.1. *Definition.* Define $\forall x \in R\backslash\{0\}$ the functions,

$$\begin{aligned}\ln(x \pm i0) &\triangleq \lim_{\varepsilon\to 0} \ln(x \pm i\varepsilon), \\ &= \ln|x| \pm i\pi 1_{-}.\end{aligned} \tag{5.127}$$

The function $\ln|x| : R\backslash\{0\} \to R$ and the set of functions 1_{-} (see (5.2)) are locally integrable in R, so the set of functions $\ln(x \pm i0)$ generate a unique regular distribution, of first order of association and degree 0, also denoted by $\ln(x \pm i0)$.

More generally, $\forall m \in \mathbb{N}$ and $\forall x \in R\backslash\{0\}$, the regular distribution $\ln^m(x \pm i0)$ is defined as the m-th power of $\ln(x \pm i0)$ and we can derive the useful distributional identity,

$$\ln^m(x \pm i0) = 1_{+} \ln^m|x| + 1_{-} \sum_{p=0}^{m} \binom{m}{p} (\pm i\pi)^p \ln^{m-p}|x| \,. \tag{5.128}$$

For each m, the set of functions $\ln^m(x \pm i0)$ define a unique regular distribution, of m-th order of association and degree 0, also denoted by $\ln^m(x \pm i0)$.

5.7.2. *Generalized multiplication derivatives.* The generalized multiplication derivatives of the distributions $\ln(x \pm i0)$ are, $\forall k \in \mathbb{Z}_{+}$,

$$D^k \ln(x \pm i0) = (-1)^{k-1}(k-1)!(x \pm i0)^{-k}, \tag{5.129}$$

with the distributions $(x \pm i0)^z$ defined in (5.135) below. Substituting (5.127) and (5.139) in (5.129) yields

$$D^k \ln|x| = +\pi\eta^{(k-1)}, \tag{5.130}$$

$$D^k 1_{-} = -\delta^{(k-1)}. \tag{5.131}$$

5.7.3. *Generalized convolution derivatives.* The generalized convolution derivatives of the distributions $\ln(x \pm i0)$ are, $\forall k \in \mathbb{Z}_{+}$,

$$X^k \ln(x \pm i0) = x^k . \ln(x \pm i0) = x^k \ln|x| \pm i\pi \,(-1)^k\, x_{-}^k = D_k(x \pm i0)^k, \tag{5.132}$$

wherein we used (5.155) below.

5.7.4. *Fourier transforms.* From (5.168) with $z = 0$, (6.12) and (6.18)–(6.19) we find, $\forall m \in \mathbb{N}$,

$$\begin{aligned}&\mathcal{F}\left[\ln^m(x \pm i0)\right] \\ =\;& (-1)^m \left(\sum_{p=0}^{m} \binom{m}{p} \left(\mathrm{Log}\,(\mp 2\pi i)\right)^{m-p} \frac{c_{p+1}(0)}{p+1}\right)\delta + 1_{0<m}\,(-1)^m\, m \\ & \left(\sum_{p=0}^{m-1} \binom{m-1}{p} \sum_{q=0}^{p} \binom{p}{q} \left(\mathrm{Log}\,(\mp 2\pi i)\right)^{m-1-p} \frac{c_{p-q+1}(0)}{p-q+1} \ln^q|x|\right) x_{\pm,0}^{-1},\end{aligned} \tag{5.133}$$

again a new result. In (5.133), $\mathrm{Log}\,(\mp 2\pi i) = \ln(2\pi) \mp i\pi/2$. In particular for $m = 1$,

$$\mathcal{F}\left[\ln(x \pm i0)\right] = -\left(\ln(2\pi) + \gamma \mp i\pi/2\right)\delta - \, x_{\pm,0}^{-1}. \tag{5.134}$$

5.8. **Complex kernels** $(x \pm i0)^z$**.**

5.8.1. *Definition.* The distributions $(x \pm i0)^z \in \mathcal{D}'$ are defined by [15, p. 59],

$$
\begin{aligned}
(x \pm i0)^z &\triangleq x_+^z + e^{\pm i\pi z} x_-^z, && (5.135)\\
&= e^{\pm i(\pi/2)z}\left(\cos(\pi z/2)\,|x|^z \mp i \sin(\pi z/2)\,(|x|^z \operatorname{sgn})\right), && (5.136)
\end{aligned}
$$

and $\operatorname{supp}((x \pm i0)^z) = R$. An equivalent form is

$$(x \pm i0)^z = e^{z \ln(x \pm i0)}. \tag{5.137}$$

Proof. By Maclaurin's series of the exponential and (5.127) holds

$$
\begin{aligned}
e^{z \ln(x \pm i0)} &= \sum_{p=0}^{+\infty} \ln^p(x \pm i0) \frac{z^p}{p!},\\
&= \sum_{p=0}^{+\infty} \left(1_+ \ln|x| + 1_- (\ln|x| \pm i\pi)\right)^p \frac{z^p}{p!}.
\end{aligned}
$$

By (5.5)–(5.6) and the associativity of the multiplication product of regular distributions, we get

$$
\begin{aligned}
e^{z \ln(x \pm i0)} &= \sum_{p=0}^{+\infty} \left(1_+ . \ln^p|x| + 1_- . (\ln|x| \pm i\pi)^p\right) \frac{z^p}{p!},\\
&= \left(\sum_{p=0}^{+\infty} \ln^p|x| \frac{z^p}{p!}\right) .1_+ + \left(\sum_{p=0}^{+\infty} (\ln|x| \pm i\pi)^p \frac{z^p}{p!}\right) .1_-,\\
&= e^{z \ln|x|} .1_+ + e^{z(\ln|x| \pm i\pi)} .1_-.
\end{aligned}
$$

First for $-1 < \operatorname{Re}(z)$ and (5.16) and thereafter by analytic continuation, we obtain

$$e^{z \ln(x \pm i0)} = x_+^z + e^{\pm i\pi z} x_-^z.$$

□

We have from (5.135),

$$(x \pm i0)^k = x^k, \forall k \in \mathbb{N}. \tag{5.138}$$

Hence, the distributions $(x \pm i0)^z$ are linearly independent $\forall z \in \mathbb{C} \backslash \mathbb{N}$ and linearly dependent $\forall z \in \mathbb{N}$.

At $z = -k \in \mathbb{Z}_-$, from (5.147) below follows

$$
\begin{aligned}
(x \pm i0)^{-k} &= \mp i\pi \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)} + x^{-k}, && (5.139)\\
&= \mp i\pi \frac{(-1)^{k-1}}{(k-1)!} \left(\delta^{(k-1)} \pm i\eta^{(k-1)}\right). && (5.140)
\end{aligned}
$$

Herein is x^{-k} given by (5.94) with $m = 0$ and $\eta^{(k-1)}$ by (5.99).

The expressions (5.139) for $k = 1$ are known as Sokhotskii-Plemelj equations, [18, p. 28], and

$$\Phi^0_{x \pm i0} \triangleq \mp \frac{1}{2\pi i} (x \pm i0)^{-1} = \frac{1}{2} (\delta \pm i\eta) \tag{5.141}$$

are called Heisenberg distributions (also denoted by $\delta_\pm$), [18, p. 27]. Obviously, $\Phi^0_{x \pm i0}$ are the two extensions, complex analytic on R, of the delta distribution. A distribution complex analytic on R, is the generalization of what in physics and engineering is called an "analytic signal" (a particular complex function defined on R with prescribed real part, which is the boundary value of a complex analytic function in the upper or lower half plane). Also, $\pm i\Phi^0_{x\pm i0}$ are the two extensions, complex analytic on R, of the eta distribution.

5.8.2. *Associated distributions.* From (5.38) and (5.37) we get the Laurent series of $x^z_\pm$ about the point $z=-k\in\mathbb{Z}_-$. Substitution of these series in (5.135) gives the Taylor series about the point $z=-k\in\mathbb{Z}_-$,

$$\begin{aligned}
&\langle (x\pm i0)^z,\varphi\rangle \\
= &\sum_{m=0}^{+\infty}\left(\frac{(\pm i\pi)^{m+1}}{m+1}\left\langle \frac{-(-1)^{k-1}}{(k-1)!}\delta^{(k-1)},\varphi\right\rangle\right)\frac{(z+k)^m}{m!} \\
&+\sum_{m=0}^{+\infty}\int_{-\infty}^{+\infty}\left(\begin{array}{c} |x|^{-k}\,1_+(x)\ln^m|x| \\ +(-1)^k\sum_{p=0}^m\binom{m}{p}(\pm i\pi)^p\,|x|^{-k}\,1_-(x)\ln^{m-p}|x|\end{array}\right) \\
&(T_{k-1}\varphi)(x)dx\frac{(z+k)^m}{m!}.
\end{aligned}\tag{5.142}$$

The quantity in parentheses in the integrand of the integral in (5.142) becomes, $\forall x\in R\backslash\{0\}$,

$$\begin{aligned}
&|x|^{-k}\,1_+(x)\ln^m|x|+(-1)^k\sum_{p=0}^m\binom{m}{p}(\pm i\pi)^p\,|x|^{-k}\,1_-(x)\ln^{m-p}|x| \\
= &x^{-k}\left(\operatorname{sgn}(x)\,1_+(x)\ln^m|x|-\sum_{p=0}^m\binom{m}{p}(\pm i\pi)^p\operatorname{sgn}(x)\,1_-(x)\ln^{m-p}|x|\right), \\
= &x^{-k}\left(1_+(x)\ln^m|x|+\sum_{p=0}^m\binom{m}{p}(\pm i\pi)^p\,1_-(x)\ln^{m-p}|x|\right).
\end{aligned}\tag{5.143}$$

By (5.128), the expression in parentheses is just $\ln^m(x\pm i0)$ so (5.142) simplifies to

$$\begin{aligned}
&\langle (x\pm i0)^z,\varphi\rangle \\
= &\sum_{m=0}^{+\infty}\frac{(\pm i\pi)^{m+1}}{m+1}\left\langle \frac{-(-1)^{k-1}}{(k-1)!}\delta^{(k-1)},\varphi\right\rangle\frac{(z+k)^m}{m!} \\
&+\sum_{m=0}^{+\infty}\int_{-\infty}^{+\infty}x^{-k}\ln^m(x\pm i0)\,(T_{k-1,0}\varphi)(x)dx\,\frac{(z+k)^m}{m!}.
\end{aligned}\tag{5.144}$$

Define distributions $x^{-k}\ln^m(x\pm i0)$, $\forall k\in\mathbb{Z}_+$ and $\forall m\in\mathbb{N}$, by

$$\left\langle x^{-k}\ln^m(x\pm i0),\varphi\right\rangle\triangleq\int_{-\infty}^{+\infty}x^{-k}\ln^m(x\pm i0)\,(T_{k-1,0}\varphi)(x)dx.\tag{5.145}$$

Substituting (5.143) in the integrand of (5.145) and identifying with (5.37) gives

$$x^{-k}\ln^m(x\pm i0)=x^{-k}_{+,0}\ln^m|x|+(-1)^k\sum_{p=0}^m\binom{m}{p}(\pm i\pi)^p\,x^{-k}_{-,0}\ln^{m-p}|x|\,.\tag{5.146}$$

Now (5.144) yields the Taylor series of $(x\pm i0)^z$ about $z=-k\in\mathbb{Z}_-$,

$$(x\pm i0)^z=\sum_{m=0}^{+\infty}\left(-\frac{(\pm i\pi)^{m+1}}{m+1}\frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}+x^{-k}\ln^m(x\pm i0)\right)\frac{(z+k)^m}{m!}.\tag{5.147}$$

Expression (5.147) shows that $z \in \mathbb{Z}_-$ are ordinary points, so the distributions $(x \pm i0)^z$ are entire in z.

(i) At $z = -k \in \mathbb{Z}_-$, we can read off from (5.147) the associated distributions of order m and degree $-k$,

$$\left(D_w^m (x \pm i0)^w\right)_{w=-k} = -\frac{(\pm i\pi)^{m+1}}{m+1} \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)} + x^{-k} \ln^m (x \pm i0). \quad (5.148)$$

Equation (5.148) is the generalization of the Sokhotskii-Plemelj equations for AHDs of all orders of association and for all negative integer degrees and appears to be new.

(ii) $\forall z \in \mathbb{C} \backslash \mathbb{Z}_-$, (5.135) gives

$$D_z^m (x \pm i0)^z = \left(D_z^m x_+^z\right) + e^{\pm i\pi z} \sum_{p=0}^{m} \binom{m}{p} (\pm i\pi)^p \left(D_z^{m-p} x_-^z\right). \quad (5.149)$$

By (5.31), $D_z^m x_\pm^z = x_\pm^z \ln^m |x|$, $\forall z \in \mathbb{C} \backslash \mathbb{Z}_-$, so that (5.149) can be written as

$$D_z^m (x \pm i0)^z = x_+^z \ln^m |x| + e^{\pm i\pi z} \sum_{p=0}^{m} \binom{m}{p} (\pm i\pi)^p x_-^z \ln^{m-p} |x|. \quad (5.150)$$

By the meaning given to the distributional multiplication product $|x|^z .1_\pm$, $\forall z \in \mathbb{C} \backslash \mathbb{Z}_-$, and by invoking (5.5)–(5.7), (5.150) is seen to be equivalent to

$$D_z^m (x \pm i0)^z = \left(x_+^z + e^{\pm i\pi z} x_-^z\right) . \left(1_+ \ln^m |x| + \sum_{p=0}^{m} \binom{m}{p} (\pm i\pi)^p 1_- \ln^{m-p} |x|\right). \quad (5.151)$$

By (5.135) and (5.128), (5.151) reduces to

$$D_z^m (x \pm i0)^z = (x \pm i0)^z . \ln^m (x \pm i0), \quad (5.152)$$

which also serves as the equation that gives meaning to the distributional multiplication $(x \pm i0)^z . \ln^m (x \pm i0)$ and which will be denoted by $(x \pm i0)^z \ln^m (x \pm i0)$, $\forall z \in \mathbb{C} \backslash \mathbb{Z}_-$. Expression (5.152) now makes it legitimate to write, $\forall z \in \mathbb{C} \backslash \mathbb{Z}_-$,

$$D_z^m (x \pm i0)^z = D_z^m e^{z \ln(x \pm i0)} = (x \pm i0)^z \ln^m (x \pm i0). \quad (5.153)$$

The distributions $D_z^m (x \pm i0)^z$, $\forall z \in \mathbb{C}$ and $\forall m \in \mathbb{N}$, are AHDs of order m, which are entire in their degree z.

In particular at $z = k \in \mathbb{N}$,

$$D_k^m (x \pm i0)^k = x^k \ln^m (x \pm i0). \quad (5.154)$$

with $x^k \ln^m (x \pm i0) \triangleq (x \pm i0)^k . \ln^m (x \pm i0)$. This result can be expressed in terms of the generalized convolution derivative as

$$D_k^m (x \pm i0)^k = X^k \ln^m (x \pm i0). \quad (5.155)$$

The multiplication product $(x \pm i0)^{-k} . \ln^m (x \pm i0)$ is defined using (4.19). At $z = -k \in \mathbb{Z}_-$, and in view of the result [13, eq. (24)], eqs. (5.148) and (5.139) show that, $\forall k, m \in \mathbb{Z}_+$,

$$\left(D_w^m (x \pm i0)^w\right)_{w=-k} \neq (x \pm i0)^{-k} . \ln^m (x \pm i0). \quad (5.156)$$

For this reason, we will not denote $\left(D_w^m (x \pm i0)^w\right)_{w=-k}$ by $(x \pm i0)^{-k} \ln^m (x \pm i0)$ (as is done in [15, p. 98]), since the latter notation is prone to interpretational error.

Interesting linear combinations of these AHDs are,

$$\begin{aligned} & D_z^{m+1}(x+i0)^z - D_z^{m+1}(x-i0)^z \\ = \; & \sum_{p=0}^{m+1} \binom{m+1}{p} \left((+i\pi)^p\, e^{+i\pi z} - (-i\pi)^p\, e^{-i\pi z}\right) x_-^z \ln^{m+1-p}|x|\,, \end{aligned} \tag{5.157}$$

and

$$\begin{aligned} & D_z^{m+1}(x+i0)^z + D_z^{m+1}(x-i0)^z \\ = \; & 2\left(x_+^z \ln^{m+1}|x|\right) \\ & + \sum_{p=0}^{m+1} \binom{m+1}{p} \left((+i\pi)^p\, e^{+i\pi z} + (-i\pi)^p\, e^{-i\pi z}\right) x_-^z \ln^{m+1-p}|x|\,. \end{aligned} \tag{5.158}$$

Notice that for $\forall z \in \mathbb{N}$, (5.157) reduces to an AHDs of order m.

5.8.3. *Generalized multiplication derivatives.* From (5.135) and (5.42) follows, $\forall n \in \mathbb{Z}_+$,

$$D^n(x \pm i0)^z = z_{(n)}(x \pm i0)^{z-n}. \tag{5.159}$$

From the commutativity of D and D_z follows immediately

$$D^n\left(D_z^m(x \pm i0)^z\right) = (x \pm i0)^{z-n} \sum_{p=0}^{m} \binom{m}{p} \left(D_z^{m-p} z_{(n)}\right) \ln^p(x \pm i0). \tag{5.160}$$

By analytic continuation, (5.159) holds $\forall z \in \mathbb{C}$ and (5.160), with its right-hand side given in the form of a multiplication product of distributions, holds $\forall z \in \mathbb{C}\backslash\mathbb{Z}_-$.

At $z = -k \in \mathbb{C}\backslash\mathbb{Z}_-$, (5.148) yields

$$D^n\left(\left(D_w^m(x \pm i0)^w\right)_{w=-k}\right) = -\frac{(\pm i\pi)^{m+1}}{m+1} \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1+n)} + D^n\left(x^{-k} \ln^m(x \pm i0)\right). \tag{5.161}$$

The last term in (5.161) can be worked out further using (5.146) and (5.46).

5.8.4. *Generalized convolution derivatives.* From (5.135) and (5.48) follows, $\forall n \in \mathbb{Z}_+$ and $\forall m \in \mathbb{N}$,

$$X^n\left((x \pm i0)^z \ln^m(x \pm i0)\right) = (x \pm i0)^{z+n} \ln^m(x \pm i0). \tag{5.162}$$

By analytic continuation, (5.162) holds $\forall z \in \mathbb{C}\backslash\mathbb{Z}_-$.

At $z = -k \in \mathbb{C}\backslash\mathbb{Z}_-$, (5.148) yields

$$\begin{aligned} & X^n\left(\left(D_w^m(x \pm i0)^w\right)_{w=-k}\right) \\ = \; & -\frac{(\pm i\pi)^{m+1}}{m+1} 1_{n \le k-1} \frac{(-1)^{k-1-n}\delta^{(k-1-n)}}{(k-1-n)!} + X^n\left(x^{-k} \ln^m(x \pm i0)\right), \end{aligned} \tag{5.163}$$

wherein we used (6.34). The last term in (5.163) can be worked out further using (5.146) and (5.49).

5.8.5. *Fourier transforms.* Inverting the result (6.36) below, gives for the Fourier transform of the distributions $(x \pm i0)^{z-1}$, $\forall z \in \mathbb{C}$,

$$\mathcal{F}\left[(x \pm i0)^{z}\right] = (2\pi)^{-z}\, e^{\mp i(\pi/2)(-z)} \Phi_{\pm}^{-z}. \tag{5.164}$$

In particular for $z = k \in \mathbb{N}$,

$$\mathcal{F}\left[x^{k}\right] = (-2\pi i)^{-k}\, \delta^{(k)}. \tag{5.165}$$

In terms of the distributions $(\chi \pm i0)^{-(z+1)}$, (5.164) becomes

$$\mathcal{F}\left[(x + i0)^{z}\right] = \frac{(2\pi)^{-z}}{\Gamma(-z)} e^{-i(\pi/2)(-z)} \frac{1}{2} \begin{pmatrix} (\chi + i0)^{-(z+1)} \\ +(\chi - i0)^{-(z+1)} \end{pmatrix}, \tag{5.166}$$

$$\mathcal{F}\left[(x - i0)^{z}\right] = \frac{-i}{\sin(\pi z)} \frac{(2\pi)^{-z}}{\Gamma(-z)} e^{+i(\pi/2)(-z)} \frac{1}{2} \begin{pmatrix} (\chi + i0)^{-(z+1)} \\ -(\chi - i0)^{-(z+1)} \end{pmatrix}. \tag{5.167}$$

More generally, we obtain by (4.18), $\forall z \in \mathbb{C}$ and $\forall m \in \mathbb{Z}_{+}$,

$$\mathcal{F}\left[D_{z}^{m}(x \pm i0)^{z}\right] = (-1)^{m}\,(2\pi)^{-z}\, e^{\mp i(\pi/2)(-z)} \sum_{p=0}^{m} \binom{m}{p} \left(\operatorname{Log}(\mp 2\pi i)\right)^{m-p} \left(D_{w}^{p} \Phi_{\pm}^{w}\right)_{w=-z}. \tag{5.168}$$

In (5.168), $\operatorname{Log}(\mp 2\pi i) = \ln(2\pi) \mp i\pi/2$.

We will call $(x \pm i0)^{z}$ *homogeneous complex kernels* and $D_{z}^{m}(x \pm i0)^{z}$, $\forall m \in \mathbb{Z}_{+}$, *associated homogeneous complex kernels*. The distributions $D_{z}^{m}(x \pm i0)^{z}$ are useful to study complex analytic extensions of AHDs based on the real axis, to functions on the complex plane.

6. Normalized associated homogeneous distributions

In addition to the basic AHDs considered in the previous section, one introduces normalizations of some of these distributions to make them entire in their degree and/or to give them another convenient property.

6.1. Normalized half-line kernels $\Phi_{\pm}^{z}$.

6.1.1. *Definition.* Let Γ denote the gamma function and $\psi \triangleq (d\Gamma)/\Gamma$ the digamma function, [1, p. 258, 6.3.1]. The normalized distributions $\Phi_{\pm}^{z} \in \mathcal{D}'$, defined by (e.g., [23, vol I, p. 43] or [15, p. 115])

$$\Phi_{\pm}^{z} \triangleq \frac{x_{\pm}^{z-1}}{\Gamma(z)}, \tag{6.1}$$

are entire functions of z. Notice that $\Phi_{\pm}^{z}$ are homogeneous distributions of degree $z-1$.

They take the special values, $\forall k, l \in \mathbb{N}$,

$$\Phi_{\pm}^{-k} = (\pm 1)^{k} \delta^{(k)} = (\pm \delta^{(1)})^{*k}, \tag{6.2}$$

$$\Phi_{\pm}^{l+1} = \frac{x_{\pm}^{l}}{l!} = (1_{\pm})^{*(l+1)}. \tag{6.3}$$

Herein is $f^{*0} \triangleq \delta$, $f^{*1} \triangleq f$, $f^{*2} \triangleq f * f$, etc.. The distributions $\Phi_{\pm}^{z+1}$ are linearly independent $\forall z \in \mathbb{C} \backslash \mathbb{Z}_{-}$ and linearly dependent $\forall z \in \mathbb{Z}_{-}$. The distributions $(\pm 1)^{k} \Phi_{\pm}^{-k}$ are the convolution kernels for the generalized derivative operators of degree $k \in \mathbb{N}$, $D^{k} : \mathcal{D}' \to \mathcal{D}'$ such that $f \mapsto D^{k} f = (\pm 1)^{k} \Phi_{\pm}^{-k} * f$. Also, the distributions $\Phi_{\pm}^{k}$ are the convolution kernels for the generalized primitive operators of degree

$k \in \mathbb{Z}_+$, $I^k_{-\infty} : \mathcal{D}'_R \to \mathcal{D}'$ and $I^k_{+\infty} : \mathcal{D}'_L \to \mathcal{D}'$ such that $f \mapsto I^k_{\mp\infty} f = (\pm 1)^k \Phi^k_{\pm} * f$. In particular we find that, e.g., with $\varphi \in \mathcal{D}$,

$$\left(I^k_{\mp\infty}\varphi\right)(x) = \int_{\mp\infty}^{x} \frac{(x-\chi)^{k-1}}{(k-1)!}\varphi(\chi)\, d\chi, \tag{6.4}$$

which shows that $I^k_{\mp\infty}$ are just the classical Riemann-Liouville operators for k-times iterated integration (from $\mp\infty$, respectively). We will call $I_{-\infty}$ (integrating from $-\infty$ to x) a "forward primitive" operator and $I_{+\infty}$ (integrating from $+\infty$ to x) a "backward primitive" operator.

At $z = l+1$, $\forall l \in \mathbb{N}$, the distributions $\Phi^z_{\pm}$ are related to the even and odd distributions, introduced in the previous section, by the convenient relations,

$$\Phi^{l+1}_{+} - (-1)^{l+1}\Phi^{l+1}_{-} = \frac{x^l}{l!}, \tag{6.5}$$

$$\Phi^{l+1}_{+} + (-1)^{l+1}\Phi^{l+1}_{-} = \frac{x^l \operatorname{sgn}}{l!}. \tag{6.6}$$

The $\Phi^z_{\pm}$ satisfy the following well-known properties, [15, p. 116],

$$\Phi^a_{\pm} * \Phi^b_{\pm} = \Phi^{a+b}_{\pm},\ \forall a, b \in \mathbb{C}, \tag{6.7}$$

which implies in particular, $\forall z \in \mathbb{C}$ and $\forall k \in \mathbb{N}$,

$$(\pm D)^k \Phi^z_{\pm} = \Phi^{z-k}_{\pm}, \tag{6.8}$$

$$(\pm I_{\mp})^k \Phi^z_{\pm} = \Phi^{z+k}_{\pm}. \tag{6.9}$$

The homogeneous normalized half-line kernels $\Phi^z_{\pm}$ are fundamental for distributionally defining complex degree Riemann-Liouville (so called "fractional") integration over half-lines, [23, vol II, p.30], [22, p. 145].

Further, it is a direct consequence of (3.42) that the convolution operators $\Phi^z_{\pm} *$ are mutual adjoints. More precisely, $\forall f_L \in \mathcal{D}'_L$, $\forall f_R \in \mathcal{D}'_R$, $\forall z \in \mathbb{C}$ and $\forall \varphi \in \mathcal{D}$, holds

$$\left\langle \Phi^z_{+} * f_R, \varphi \right\rangle = \left\langle f_R, \lambda_{RL}\left(\Phi^z_{-} * \varphi\right)\right\rangle, \tag{6.10}$$

$$\left\langle \Phi^z_{-} * f_L, \varphi \right\rangle = \left\langle f_L, \lambda_{LR}\left(\Phi^z_{+} * \varphi\right)\right\rangle. \tag{6.11}$$

Note that $\Phi^z_{-} * \varphi \in \mathcal{D}_L$ ($\Phi^z_{+} * \varphi \in \mathcal{D}_R$), hence $\operatorname{supp}(f_R) \cap \operatorname{supp}\left(\Phi^z_{-} * \varphi\right)$ ($\operatorname{supp}(f_L) \cap \operatorname{supp}\left(\Phi^z_{+} * \varphi\right)$) is finite. This makes it possible, to place in the right-hand side of (6.10) ((6.11)) any $\lambda_{RL} \in \mathcal{D}$ ($\lambda_{LR} \in \mathcal{D}$) that equals 1 over $\operatorname{supp}(f_R) \cap \operatorname{supp}\left(\Phi^z_{-} * \varphi\right)$ ($\operatorname{supp}(f_L) \cap \operatorname{supp}\left(\Phi^z_{+} * \varphi\right)$), without changing the numerical value of both functionals. The test function λ_{RL} (λ_{LR}) in (6.10) ((6.11)) is necessary in order to have $\lambda_{RL}\left(\Phi^z_{-} * \varphi\right) \in \mathcal{D}$ ($\lambda_{LR}\left(\Phi^z_{+} * \varphi\right) \in \mathcal{D}$), as required by the fact that $f_R \in \mathcal{D}'$ ($f_L \in \mathcal{D}'$).

6.1.2. *Associated distributions.* (i) The coefficients of the Taylor series of $\Phi^z_{\pm}$ about $z = -k$, denoted $\left(D^m_z \Phi^z_{\pm}\right)_{z=-k}$, are obtained from the Laurent series of $x^{z-1}_{\pm}$ (5.38) and the Taylor series of $1/\Gamma$ about $z = -k$. The result is, $\forall m \in \mathbb{Z}_+$ and $\forall k \in \mathbb{N}$,

$$\left(D^m_w \Phi^w_{\pm}\right)_{w=-k} = \frac{(\mp 1)^k}{k!}\frac{c_{m+1}(-k)}{m+1}\delta^{(k)} + \sum_{p=0}^{m-1} \binom{m}{p} c_{m-p}(-k)\ x^{-(k+1)}_{\pm,0} \ln^p |x|. \tag{6.12}$$

The functions c_q are defined by

$$\begin{aligned} c_q(z) &\triangleq \left(d^q(1/\Gamma)\right)(z), && (6.13)\\ &= \frac{(-1)^q}{2\pi i}\int_C \left((-t)^{-z}\ln^q(-t)\right)e^{-t}d(-t), && (6.14)\end{aligned}$$

and the contour C is as in [27, p. 245]. At integer values holds, $\forall k\in\mathbb{N}$,

$$\begin{aligned} c_q(-k) &= \frac{1}{\pi}(-1)^k k!\,(-1)^q\,\mathrm{Im}\left(\lim_{u\to k+1}\left(D_u+\psi(u)-i\pi\right)^q 1\right), && (6.15)\\ c_q(k+1) &= \frac{1}{k!}\lim_{u\to k+1}\left(D_u-\psi(u)\right)^q 1. && (6.16)\end{aligned}$$

Some convenient expressions for the constants c_q are

$$\begin{aligned} c_0(-k) &= 0 \text{ and } c_0(k+1)=\frac{1}{k!}, && (6.17)\\ c_1(-k) &= (-1)^k k! \text{ and } c_1(k+1) = -\frac{1}{k!}\psi(k+1), && (6.18)\\ c_2(-k) &= -2(-1)^k k!\psi(k+1) \text{ and } c_2(k+1)=\frac{1}{k!}\begin{pmatrix}\psi^2(k+1)\\ -\psi'(k+1)\end{pmatrix}. && (6.19)\end{aligned}$$

Interesting linear combinations are, $\forall m\in\mathbb{Z}_+$ and $\forall k\in\mathbb{N}$,

$$\left(D_w^m\Phi_+^w\right)_{w=-k}-(-1)^k\left(D_w^m\Phi_-^w\right)_{w=-k}=\sum_{p=0}^{m-1}\binom{m}{p}c_{m-p}(-k)\;x^{-(k+1)}\ln^p|x|, \quad (6.20)$$

$$\begin{aligned} &\left(D_w^m\Phi_+^w\right)_{w=-k}+(-1)^k\left(D_w^m\Phi_-^w\right)_{w=-k}\\ &= 2\frac{(-1)^k}{k!}\frac{c_{m+1}(-k)}{m+1}\delta^{(k)}+\sum_{p=0}^{m-1}\binom{m}{p}c_{m-p}(-k)\left(x^{-(k+1)}\operatorname{sgn}\right)_0\ln^p|x|. \quad (6.21)\end{aligned}$$

(ii) The distributions $D_z^m\Phi_\pm^z$, are given by, $\forall m\in\mathbb{N}$ and $\forall z\in\mathbb{C}\backslash(-\mathbb{N})$,

$$D_z^m\Phi_\pm^z=\sum_{p=0}^{m}\binom{m}{p}c_{m-p}(z)\;x_\pm^{z-1}\ln^p|x|. \quad (6.22)$$

Special linear combinations for $z=k\in\mathbb{Z}_+$ are

$$\begin{aligned} D_k^m\Phi_+^k-(-1)^kD_k^m\Phi_-^k &= \sum_{p=0}^{m}\binom{m}{p}c_{m-p}(k)\;x^{k-1}\ln^p|x|, && (6.23)\\ D_k^m\Phi_+^k+(-1)^kD_k^m\Phi_-^k &= \sum_{p=0}^{m}\binom{m}{p}c_{m-p}(k)\;x^{k-1}\operatorname{sgn}\ln^p|x|. && (6.24)\end{aligned}$$

The distributions $D_z^m\Phi_\pm^z$, $\forall m\in\mathbb{Z}_+$ and $\forall z\in\mathbb{C}$, are entire functions of z, AHDs of order m and degree $z-1$, and, since $D_z^m\Phi_+^z$ and $D_z^m\Phi_-^z$ have different support, are linearly independent. Notice that (6.20) are AHDs of order $m-1$, while (6.21) are of order m.

In particular for $m = 1$, and using $c_1(-k) = (-1)^k k!$ and $c_2(-k) = -2(-1)^k k!\,\psi(k+1)$, we get from (6.12),

$$\left(D_w \Phi_{\pm}^{w}\right)_{w=-k} = (\pm 1)^k \left(\gamma \delta^{(k)} + \pi \eta_{\pm,0}^{(k)}\right), \tag{6.25}$$

$$\left(D_w \Phi_{+}^{w}\right)_{w=-k} - (-1)^k \left(D_w \Phi_{-}^{w}\right)_{w=-k} = \pi \eta^{(k)}, \tag{6.26}$$

$$\left(D_w \Phi_{+}^{w}\right)_{w=-k} + (-1)^k \left(D_w \Phi_{-}^{w}\right)_{w=-k} = -2\psi(k+1)\delta^{(k)} + \pi \left(\eta^{(k)} \operatorname{sgn}\right)_0, \tag{6.27}$$

($\gamma = H_k - \psi(k+1)$ is the Euler-Mascheroni constant) and from (6.22), $\forall l \in \mathbb{N}$,

$$\left(D_w \Phi_{\pm}^{w}\right)_{w=l+1} = \frac{x_{\pm}^{l}}{l!} \left(\ln|x| - \psi(l+1)\right), \tag{6.28}$$

$$\left(D_w \Phi_{+}^{w}\right)_{w=l+1} - (-1)^{l+1} \left(D_w \Phi_{-}^{w}\right)_{w=l+1} = \frac{x^{l}}{l!} \left(\ln|x| - \psi(l+1)\right), \tag{6.29}$$

$$\left(D_w \Phi_{+}^{w}\right)_{w=l+1} + (-1)^{l+1} \left(D_w \Phi_{-}^{w}\right)_{w=l+1} = \frac{x^{l} \operatorname{sgn}}{l!} \left(\ln|x| - \psi(l+1)\right). \tag{6.30}$$

Another interesting combination is

$$\left(D_w^2 \Phi_{+}^{w}\right)_{w=-k} - (-1)^k \left(D_w^2 \Phi_{-}^{w}\right)_{w=-k} = 2\pi \eta^{(k)} \left(\ln|x| - \psi(k+1)\right). \tag{6.31}$$

6.1.3. *Generalized multiplication derivatives.* We have, $\forall k, m \in \mathbb{N}$ and $\forall z \in \mathbb{C}$,

$$(\pm D)^k \left(D_z^m \Phi_{\pm}^{z}\right) = D_z^m \left((\pm D)^k \Phi_{\pm}^{z}\right) = D_{z-k}^m \Phi_{\pm}^{z-k}. \tag{6.32}$$

6.1.4. *Generalized convolution derivatives.* We have, $\forall k \in \mathbb{N}$ and for $0 < \operatorname{Re}(z)$,

$$(\pm X)^k \Phi_{\pm}^{z} = z^{(k)} \Phi_{\pm}^{z+k}, \tag{6.33}$$

with $z^{(k)}$ given by (2.4). By analytic continuation, (6.33) holds $\forall z \in \mathbb{C}$. In particular for $z = -l \in -\mathbb{N}$, we retrieve a well-known formula,

$$(-X)^k \frac{\delta^{(l)}}{l!} = 1_{k \le l} \frac{\delta^{(l-k)}}{(l-k)!}. \tag{6.34}$$

Also, $\forall k, m \in \mathbb{N}$ and $\forall z \in \mathbb{C}$,

$$(\pm X)^k \left(D_z^m \Phi_{\pm}^{z}\right) = D_z^m \left((\pm X)^k \Phi_{\pm}^{z}\right) = D_z^m \left(z^{(k)} \Phi_{\pm}^{z+k}\right). \tag{6.35}$$

6.1.5. *Fourier transforms.* The Fourier transform of the distributions $\Phi_{\pm}^{z}$ was derived in [15, p. 172] and reads for our sign choice in (3.24), $\forall z \in \mathbb{C}$,

$$\mathcal{F}\left[\Phi_{\pm}^{z}\right] = e^{\pm i(\pi/2)(-z)} \left((2\pi\chi) \mp i0\right)^{-z}. \tag{6.36}$$

In particular, $\forall k \in \mathbb{N}$,

$$\mathcal{F}\left[\delta^{(k)}\right] = (2\pi i \chi)^k. \tag{6.37}$$

In terms of the distributions $\Phi_{\pm}^{z}$ themselves we get

$$\mathcal{F}\left[\Phi_{\pm}^{z}\right] = (2\pi)^{-z} \Gamma(1-z) \left(e^{\mp i(\pi/2)z} \Phi_{+}^{1-z} + e^{\pm i(\pi/2)z} \Phi_{-}^{1-z}\right). \tag{6.38}$$

More generally, $\forall z \in \mathbb{C}$ and $\forall m \in \mathbb{Z}_+$, we obtain from (6.36) and by the commutativity of the $\mathcal{F}$ and D_z operators,

$$\mathcal{F}\left[D_z^m \Phi_{\pm}^{z}\right] = (-1)^m e^{\pm i(\pi/2)(-z)} \sum_{p=0}^{m} \binom{m}{p} \left(\pm i(\pi/2)\right)^{m-p} \left(D_w^p \left((2\pi\chi) \mp i0\right)^w\right)_{w=-z}. \tag{6.39}$$

We will call the $\Phi_\pm^z$ *homogeneous normalized half-line kernels* and the $D_z^m \Phi_\pm^z$, $\forall m \in \mathbb{Z}_+$, *associated homogeneous normalized half-line kernels*. The distributions $D_z^m \Phi_\pm^z$ are convenient to easily calculate generalized derivatives and primitives of AHDs.

6.2. **Normalized parity kernels of the first kind $\Phi_{e,o}^z$.**

6.2.1. *Definition.* We will need the even and odd distributions

$$\Phi_e^z \triangleq \frac{1}{2}\frac{\Phi_+^z + \Phi_-^z}{\cos\left((\pi/2)z\right)}, \tag{6.40}$$

$$\Phi_o^z \triangleq \frac{1}{2}\frac{\Phi_+^z - \Phi_-^z}{\sin\left((\pi/2)z\right)}. \tag{6.41}$$

By construction, $\mathrm{supp}(\Phi_e^z) = \mathrm{supp}(\Phi_o^z) = R$.

Using the Taylor series (2.10)–(2.11), we obtain the following series expansions of $\Phi_{e,o}^z$ about $z = k \in \mathbb{Z}_+$,

$$\begin{aligned} \Phi_e^z = {} & \frac{o_k (-1)^{(k+1)/2} \frac{1}{\pi} \frac{x^{k-1}}{(k-1)!}}{z-k} + \sum_{m=0}^{+\infty} \frac{(i\pi/2)^m}{m!} \\ & \begin{pmatrix} e_k (-1)^{k/2} \overline{E}_m \left(\frac{D_k}{i\pi/2}\right) \frac{1}{2}\left(\Phi_+^k + \Phi_-^k\right) \\ + i o_k (-1)^{(k+1)/2} \overline{B}_{m+1} \left(\frac{D_k}{i\pi/2}\right) \frac{1}{2}\left(\Phi_+^k + \Phi_-^k\right) \end{pmatrix} (z-k)^m, \end{aligned} \tag{6.42}$$

$$\begin{aligned} \Phi_o^z = {} & \frac{e_k (-1)^{k/2} \frac{1}{\pi} \frac{x^{k-1}}{(k-1)!}}{z-k} + \sum_{m=0}^{+\infty} \frac{(i\pi/2)^m}{m!} \\ & \begin{pmatrix} o_k (-1)^{(k-1)/2} \overline{E}_m \left(\frac{D_k}{i\pi/2}\right) \frac{1}{2}\left(\Phi_+^k - \Phi_-^k\right) \\ + i e_k (-1)^{k/2} \overline{B}_{m+1} \left(\frac{D_k}{i\pi/2}\right) \frac{1}{2}\left(\Phi_+^k - \Phi_-^k\right) \end{pmatrix} (z-k)^m. \end{aligned} \tag{6.43}$$

The Taylor series of $\Phi_{e,o}^z$ about $z = -k \in -\mathbb{N}$ are obtained as

$$\begin{aligned} \Phi_e^z = {} & \sum_{m=0}^{+\infty} \frac{(i\pi/2)^m}{m!} \\ & \begin{pmatrix} e_k (-1)^{\frac{k}{2}} \left(\overline{E}_m \left(\frac{D_z}{i\pi/2}\right) \frac{1}{2}\left(\Phi_+^z + \Phi_-^z\right)\right)_{z=-k} \\ + i o_k (-1)^{\frac{k-1}{2}} \left(\overline{B}_{m+1} \left(\frac{D_z}{i\pi/2}\right) \frac{1}{2}\left(\Phi_+^z + \Phi_-^z\right)\right)_{z=-k} \end{pmatrix} (z+k)^m, \end{aligned} \tag{6.44}$$

$$\begin{aligned} \Phi_o^z = {} & \sum_{m=0}^{+\infty} \frac{(i\pi/2)^m}{m!} \\ & \begin{pmatrix} o_k (-1)^{\frac{k+1}{2}} \left(\overline{E}_m \left(\frac{D_z}{i\pi/2}\right) \frac{1}{2}\left(\Phi_+^z - \Phi_-^z\right)\right)_{z=-k} \\ + i e_k (-1)^{\frac{k}{2}} \left(\overline{B}_{m+1} \left(\frac{D_z}{i\pi/2}\right) \frac{1}{2}\left(\Phi_+^z - \Phi_-^z\right)\right)_{z=-k} \end{pmatrix} (z+k)^m. \end{aligned} \tag{6.45}$$

(i) We see that Φ_e^z is complex analytic $\forall z \in \mathbb{C}$, except for simple poles at $z = 2p+1 \in \mathbb{Z}_{o,+}$, with residues

$$\left(\Phi_e^{2p+1}\right)_{-1} = \frac{(-1)^{p+1}}{\pi} \frac{x^{2p}}{(2p)!}. \tag{6.46}$$

The analytic finite part at a pole is

$$\left(\Phi_e^{2p+1}\right)_0 = \frac{(-1)^{p+1}}{\pi}\Big(\left(D_z\Phi_+^z\right)_{z=2p+1} + \left(D_z\Phi_-^z\right)_{z=2p+1}\Big), \tag{6.47}$$

$$= \frac{(-1)^{p+1}}{\pi}\frac{x^{2p}}{(2p)!}\left(\ln|x| - \psi(2p+1)\right). \tag{6.48}$$

At the ordinary points $z = 2p+2 \in \mathbb{Z}_{e,+}$, is

$$\Phi_e^{2p+2} = \frac{(-1)^{p+1}}{2}\frac{x^{2p+1}\operatorname{sgn}}{(2p+1)!}. \tag{6.49}$$

In addition, Φ_e^z has removable singularities at $z \in \mathbb{Z}_{o,-}$. Using (6.2), (5.58), (5.99) and the properties of the gamma function, we get for Φ_e^z at $z = -k \in -\mathbb{N}$,

$$\Phi_e^{-k} = e_k(-1)^{k/2}\delta^{(k)} + o_k(-1)^{(k-1)/2}\eta^{(k)}. \tag{6.50}$$

(ii) Also, Φ_o^z is complex analytic $\forall z \in \mathbb{C}$, except for simple poles at $z = 2p+2$, $\forall p \in \mathbb{N}$, with residues

$$\left(\Phi_o^{2p+2}\right)_{-1} = \frac{(-1)^{p+1}}{\pi}\frac{x^{2p+1}}{(2p+1)!}. \tag{6.51}$$

The analytic finite part at a pole is

$$\left(\Phi_o^{2p+2}\right)_0 = \frac{(-1)^{p+1}}{\pi}\Big(\left(D_z\Phi_+^z\right)_{z=2p+2} - \left(D_z\Phi_-^z\right)_{z=2p+2}\Big), \tag{6.52}$$

$$= \frac{(-1)^{p+1}}{\pi}\frac{x^{2p+1}}{(2p+1)!}\left(\ln|x| - \psi(2p+2)\right). \tag{6.53}$$

At the ordinary points $z = 2p+1$, $\forall p \in \mathbb{N}$, is

$$\Phi_o^{2p+1} = -\frac{(-1)^{p+1}}{2}\frac{x^{2p}\operatorname{sgn}}{(2p)!}. \tag{6.54}$$

In addition, Φ_o^z has removable singularities at $z \in \mathbb{Z}_{e,-]}$. Using (6.2), (5.77), (5.99) and the properties of the gamma function, we get for Φ_o^z at $z = -k \in -\mathbb{N}$,

$$\Phi_o^{-k} = -o_k(-1)^{(k-1)/2}\delta^{(k)} + e_k(-1)^{k/2}\eta^{(k)}. \tag{6.55}$$

The distributions Φ_e^z and Φ_o^z are known in fractional calculus theory as the kernels of the convolution operators that generate the Riesz potential $\Phi_e^z * \varphi$ and conjugate Riesz potential $\Phi_o^z * \varphi$, of a function $\varphi \in \mathcal{D}$, [22, p. 214]. Hence, the Riesz potential and conjugate Riesz potential are just regularizations, in the sense of [28, p. 132], of the distributions Φ_e^z and Φ_o^z, respectively.

6.2.2. *Associated distributions.* At $z \in \mathbb{C}\backslash\mathbb{Z}$ and $\forall m \in \mathbb{Z}_+$, we obtain from (6.106) the associated distributions

$$D_z^m\Phi_e^z = D_z^m\frac{1}{2}\frac{\Phi_+^z + \Phi_-^z}{\cos\left((\pi/2)z\right)}, z \in \mathbb{C}\backslash\mathbb{Z}_{o,+}, \tag{6.56}$$

$$D_z^m\Phi_o^z = D_z^m\frac{1}{2}\frac{\Phi_+^z - \Phi_-^z}{\sin\left((\pi/2)z\right)}, z \in \mathbb{C}\backslash\mathbb{Z}_{e,+}. \tag{6.57}$$

From (6.42)–(6.43) we can read off, $\forall m,p\in\mathbb{N}$,

$$\left((D_w^m\Phi_e^w)_{w=2p+1}\right)_0 = (-1)^{p+1}(i\pi/2)^m i\left(\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w+\Phi_-^w\right)\right)_{w=2p+1}, \tag{6.58}$$

$$(D_w^m\Phi_o^w)_{w=2p+1} = (-1)^p\,(i\pi/2)^m\left(\overline{E}_m\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w-\Phi_-^w\right)\right)_{w=2p+1}, \tag{6.59}$$

and

$$(D_w^m\Phi_e^w)_{w=2p+2} = (-1)^{p+1}\,(i\pi/2)^m\left(\overline{E}_m\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w+\Phi_-^w\right)\right)_{w=2p+2}, \tag{6.60}$$

$$\left((D_w^m\Phi_o^w)_{w=2p+2}\right)_0 = (-1)^{p+1}(i\pi/2)^m i\left(\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w-\Phi_-^w\right)\right)_{w=2p+2}. \tag{6.61}$$

Herein are $\left((D_w^m\Phi_e^w)_{w=2p+1}\right)_0$ and $\left((D_w^m\Phi_o^w)_{w=2p+2}\right)_0$ the analytic finite parts of $D_z^m\Phi_e^z$ and $D_z^m\Phi_o^z$ at their respectively poles. Also from (6.44)–(6.45), we have at $z=-k\in-\mathbb{N}$,

$$\begin{aligned}&(D_w^m\Phi_e^w)_{w=-k}\\ =\;&(i\pi/2)^m\left(\begin{array}{c} e_k\,(-1)^{k/2}\left(\overline{E}_m\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w+\Phi_-^w\right)\right)_{w=-k}\\ +io_k\,(-1)^{(k-1)/2}\left(\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right)\right)\frac{1}{2}\left(\Phi_+^w+\Phi_-^w\right)\Big)_{w=-k}\end{array}\right),\end{aligned} \tag{6.62}$$

$$\begin{aligned}&(D_w^m\Phi_o^w)_{w=-k}\\ =\;&(i\pi/2)^m\left(\begin{array}{c} o_k\,(-1)^{(k+1)/2}\left(\overline{E}_m\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w-\Phi_-^w\right)\right)_{w=-k}\\ +ie_k\,(-1)^{k/2}\left(\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w-\Phi_-^w\right)\right)_{w=-k}\end{array}\right).\end{aligned} \tag{6.63}$$

Expressions (6.58)–(6.63) are new.

The distributions $D_z^m\Phi_{e,o}^z$ (or where they are not defined, $\left(D_z^m\Phi_{e,o}^z\right)_0$) are, $\forall m\in\mathbb{N}$ and $\forall z\in\mathbb{C}$, linearly independent, of degree $z-1$ and associated of order m $(m+1)$.

6.2.3. *Generalized multiplication derivatives.* From (6.40)–(6.41) and (6.8) follows, $\forall k\in\mathbb{N}$ and $\forall z\in\mathbb{C}\backslash\mathbb{Z}_+$,

$$D^k\Phi_e^z = e_k(-1)^{k/2}\Phi_e^{z-k}-o_k(-1)^{(k-1)/2}\Phi_o^{z-k}, \tag{6.64}$$

$$D^k\Phi_o^z = e_k(-1)^{k/2}\Phi_o^{z-k}+o_k(-1)^{(k-1)/2}\Phi_e^{z-k}. \tag{6.65}$$

Then, by the commutativity of the D^k and D_z^m operators, we get

$$D^k\left(D_z^m\Phi_e^z\right) = e_k(-1)^{k/2}\left(D_w^m\Phi_e^w\right)_{w=z-k}-o_k(-1)^{(k-1)/2}\left(D_w^m\Phi_o^w\right)_{w=z-k}, \tag{6.66}$$

$$D^k\left(D_z^m\Phi_o^z\right) = e_k(-1)^{k/2}\left(D_w^m\Phi_o^w\right)_{w=z-k}+o_k(-1)^{(k-1)/2}\left(D_w^m\Phi_e^w\right)_{w=z-k}. \tag{6.67}$$

Further, from (6.32), (6.8) and (6.58)–(6.63) follows, $\forall m,n,p\in\mathbb{N}$,

$$\begin{aligned}&D^n\left((D_w^m\Phi_e^w)_{w=2p+1}\right)_0\\ =\;&(-1)^{p+1}(i\pi/2)^m i\left(\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w+(-1)^n\,\Phi_-^w\right)\right)_{w=2p+1-n},\end{aligned} \tag{6.68}$$

$$\begin{aligned} & D^n\left(\left(D_w^m\Phi_0^w\right)_{w=2p+1}\right) \\ = & (-1)^p\,(i\pi/2)^m\left(\overline{E}_m\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w-(-1)^n\,\Phi_-^w\right)\right)_{w=2p+1-n}, \end{aligned} \tag{6.69}$$

$$\begin{aligned} & D^n\left(\left(D_w^m\Phi_e^w\right)_{w=2p+2}\right) \\ = & (-1)^{p+1}\,(i\pi/2)^m\left(\overline{E}_m\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w+(-1)^n\,\Phi_-^w\right)\right)_{w=2p+2-n}, \end{aligned} \tag{6.70}$$

$$\begin{aligned} & D^n\left(\left(D_w^m\Phi_o^w\right)_{w=2p+2}\right)_0 \\ = & (-1)^{p+1}(i\pi/2)^m i\left(\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w-(-1)^n\,\Phi_-^w\right)\right)_{w=2p+2-n}, \end{aligned} \tag{6.71}$$

and, $\forall m,n,k\in\mathbb{N}$,

$$\begin{aligned} & D^n\left(\left(D_w^m\Phi_e^w\right)_{w=-k}\right) \\ = & (i\pi/2)^m\begin{pmatrix} e_k(-1)^{k/2}\overline{E}_m\left(\frac{D_w}{i\pi/2}\right) \\ +io_k(-1)^{(k-1)/2}\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right)\end{pmatrix}\frac{1}{2}\left(\Phi_+^w+(-1)^n\,\Phi_-^w\right)_{w=-(k+n)}, \end{aligned} \tag{6.72}$$

$$\begin{aligned} & D^n\left(\left(D_w^m\Phi_o^w\right)_{w=-k}\right) \\ = & (i\pi/2)^m\begin{pmatrix} o_k(-1)^{(k+1)/2}\overline{E}_m\left(\frac{D_w}{i\pi/2}\right) \\ +ie_k(-1)^{k/2}\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right)\end{pmatrix}\frac{1}{2}\left(\Phi_+^w-(-1)^n\,\Phi_-^w\right)_{w=-(k+n)}. \end{aligned} \tag{6.73}$$

These are new results.

6.2.4. *Generalized convolution derivatives.* Combining (6.40)–(6.41) with (6.33) gives, $\forall k,m\in\mathbb{N}$ and $\forall z\in\mathbb{C}\backslash\mathbb{Z}_+$,

$$X^k\left(D_z^m\Phi_e^z\right) = D_z^m\left(z^{(k)}\left(e_k\,(-1)^{k/2}\,\Phi_e^{z+k}+o_k\,(-1)^{(k-1)/2}\,\Phi_o^{z+k}\right)\right), \tag{6.74}$$

$$X^k\left(D_z^m\Phi_o^z\right) = D_z^m\left(z^{(k)}\left(-o_k\,(-1)^{(k-1)/2}\,\Phi_e^{z+k}+e_k\,(-1)^{k/2}\,\Phi_o^{z+k}\right)\right). \tag{6.75}$$

From (6.58)–(6.61) and (6.35) together with (2.20)–(2.21), (6.46), (6.51) and (6.58)–(6.61) we get, $\forall k,m,p\in\mathbb{N}$,

$$\begin{aligned} & X\left(\left(D_w^m\Phi_e^w\right)_{w=2p+1}\right)_0 \\ = & 1_{m=0}\left(\Phi_o^{2p+2}\right)_{-1}+(2p+1)\left(\left(D_w^m\Phi_o^w\right)_{w=2p+2}\right)_0 \\ & +1_{0<m}m\left(\left(D_w^{m-1}\Phi_o^w\right)_{w=2p+2}\right)_0, \end{aligned} \tag{6.76}$$

$$\begin{aligned} & X\left(D_w^m\Phi_o^w\right)_{w=2p+1} \\ = & -(2p+1)\left(D_w^m\Phi_e^w\right)_{w=2p+2}-1_{0<m}m\left(D_w^{m-1}\Phi_e^w\right)_{w=2p+2}; \end{aligned} \tag{6.77}$$

and

$$\begin{aligned} & X\left(D_w^m\Phi_e^w\right)_{w=2p+2} \\ = & (2p+2)\left(D_w^m\Phi_o^w\right)_{w=2p+3}+1_{0<m}m\left(D_w^{m-1}\Phi_o^w\right)_{w=2p+3}, \end{aligned} \tag{6.78}$$

$$\begin{aligned} & X\left(\left(D_w^m \Phi_o^w\right)_{w=2p+2}\right)_0 \\ = \; & -1_{m=0}\left(\Phi_e^{2p+3}\right)_{-1}-(2p+2)\left(\left(D_w^m \Phi_e^w\right)_{w=2p+3}\right)_0 \\ & -1_{0<m} m\left(\left(D_w^{m-1} \Phi_e^w\right)_{w=2p+3}\right)_0 . \end{aligned} \tag{6.79}$$

6.2.5. *Fourier transforms.* We easily obtain from (6.101)–(6.102),

$$\mathcal{F}\left[\Phi_e^z\right] = |2\pi\chi|^{-z}, \forall z \in \mathbb{C}\backslash\mathbb{Z}_{o,+}, \tag{6.80}$$

$$\mathcal{F}\left[\Phi_o^z\right] = -i\,|2\pi\chi|^{-z} \operatorname{sgn}, \forall z \in \mathbb{C}\backslash\mathbb{Z}_{e,+}. \tag{6.81}$$

More generally, we obtain from (6.80)–(6.81) and by the commutativity of the $\mathcal{F}$ and D_z operators, $\forall m \in \mathbb{Z}_+$,

(i) $\forall z \in \mathbb{C}\backslash\mathbb{Z}_{o,+}$,

$$\mathcal{F}\left[D_z^m \Phi_e^z\right] = D_z^m\,|2\pi\chi|^{-z} = (-1)^m (2\pi)^{-z} \sum_{p=0}^{m} \binom{m}{p} \ln^{m-p}(2\pi)\left(D_w^p\,|\chi|^w\right)_{w=-z}, \tag{6.82}$$

(ii) $\forall z \in \mathbb{C}\backslash\mathbb{Z}_{e,+}$,

$$\begin{aligned} \mathcal{F}\left[D_z^m \Phi_o^z\right] &= -iD_z^m\left(|2\pi\chi|^{-z}\operatorname{sgn}\right) \\ &= -(-1)^m i\,(2\pi)^{-z} \sum_{p=0}^{m} \binom{m}{p} \ln^{m-p}(2\pi)\left(D_w^p\,|\chi|^w \operatorname{sgn}\right)_{w=-z} \end{aligned} \tag{6.83}$$

From (6.119) and (6.106) follows, $\forall m \in \mathbb{N}$,

$$\mathcal{F}\left[\left(D_k^m \Phi_e^k\right)_0\right] = (-1)^m \begin{pmatrix} \left((2\pi\chi)^{-k}\operatorname{sgn}\right)_0 \ln^m|2\pi\chi| \\ +2\frac{\ln^{m+1}(2\pi)}{m+1}\frac{(2\pi)^{-k}\delta^{(k-1)}}{(k-1)!} \end{pmatrix}, \forall k \in \mathbb{Z}_{o,+}, \tag{6.84}$$

and

$$\mathcal{F}\left[\left(D_k^m \Phi_o^k\right)_0\right] = -(-1)^m i \begin{pmatrix} \left((2\pi\chi)^{-k}\operatorname{sgn}\right)_0 \ln^m|2\pi\chi| \\ -2\frac{\ln^{m+1}(2\pi)}{m+1}\frac{(2\pi)^{-k}\delta^{(k-1)}}{(k-1)!} \end{pmatrix}, \forall k \in \mathbb{Z}_{e,+}. \tag{6.85}$$

Expressions (6.84)–(6.85) are new.

We will call the $\Phi_{e,o}^z$ *homogeneous normalized parity kernels (of the first kind)* and the $D_z^m\Phi_{e,o}^z$, $\forall m \in \mathbb{Z}_+$, *associated homogeneous normalized parity kernels (of the first kind).* The distributions $D_z^m\Phi_{e,o}^z$ play a fundamental role in the construction of a convolution algebra of AHDs, [12].

6.3. Normalized parity kernels of the second kind $\Phi_{\mathfrak{e},\mathfrak{o}}^z$.

6.3.1. *Definition.* The normalized distributions defined by

$$\Phi_{\mathfrak{e}}^z \triangleq \pi^{z/2}\frac{|x|^{z-1}}{\Gamma(z/2)}, \tag{6.86}$$

$$\Phi_{\mathfrak{o}}^z \triangleq \pi^{z/2}\frac{|x|^{z-1}\operatorname{sgn}}{\Gamma((z+1)/2)}, \tag{6.87}$$

are entire functions of z. They take the special values:

(i) at $z \in -\mathbb{N}$, $\forall p \in \mathbb{N}$,

$$\Phi_{\mathfrak{e}}^{-k} = \begin{cases} \pi^{-p-1/2} \frac{x^{-(2p+2)}}{\Gamma(-p-1/2)} & \text{if} \quad k = 2p+1, \\ \frac{(-1)^p}{\pi^p} \frac{p!}{(2p)!} \delta^{(2p)} & \text{if} \quad k = 2p, \end{cases} \tag{6.88}$$

$$\Phi_{\mathfrak{o}}^{-k} = \pi^{-1/2} \begin{cases} -\frac{(-1)^p}{\pi^p} \frac{p!}{(2p+1)!} \delta^{(2p+1)} & \text{if} \quad k = 2p+1, \\ \pi^{-p+1/2} \frac{x^{-(2p+1)}}{\Gamma(-p+1/2)} & \text{if} \quad k = 2p, \end{cases} \tag{6.89}$$

(ii) at $z = k \in \mathbb{Z}_+$, $\forall p \in \mathbb{N}$,

$$\Phi_{\mathfrak{e}}^{k} = \begin{cases} 2^{2p} \pi^p p! \frac{x^{2p}}{(2p)!} & \text{if} \quad k = 2p+1, \\ \pi^{p+1} \frac{x^{2p+1} \operatorname{sgn}}{p!} & \text{if} \quad k = 2p+2, \end{cases} \tag{6.90}$$

$$\Phi_{\mathfrak{o}}^{k} = \pi^{+1/2} \begin{cases} \pi^p \frac{x^{2p} \operatorname{sgn}}{p!} & \text{if} \quad k = 2p+1, \\ 2^{2p+1} \pi^p p! \frac{x^{2p+1}}{(2p+1)!} & \text{if} \quad k = 2p+2, \end{cases} \tag{6.91}$$

Notice that $\Phi_{\mathfrak{e},\mathfrak{o}}^{z}$ are homogeneous distributions of degree $z-1$ and are linearly independent $\forall z \in \mathbb{C}$.

6.3.2. *Associated distributions.* The distributions $D_z^m \Phi_{\mathfrak{e},\mathfrak{o}}^{z}$ are, $\forall m \in \mathbb{Z}_+$ and $\forall z \in \mathbb{C}$, AHDs of order m, linearly independent and entire in their degree $z-1$.

The coefficients of the Taylor series of $\Phi_{\mathfrak{e},\mathfrak{o}}^{z}$ about $z = -k$, $\forall k \in \mathbb{N}$, denoted $(D_w^m \Phi_{\mathfrak{e}}^{w})_{w=-k}$ $((D_w^m \Phi_{\mathfrak{o}}^{w})_{w=-k})$, are obtainable from the Laurent series (5.38) of $|x|^{z-1}(|x|^{z-1} \operatorname{sgn})$ and the Taylor series of $\pi^{z/2}/\Gamma(z/2)$ $(\pi^{z/2}/\Gamma((z+1)/2))$ about $z = -k$. The coefficients of the Taylor series of $\Phi_{\mathfrak{e},\mathfrak{o}}^{z}$ about $z = k$, $\forall k \in \mathbb{Z}_+$, denoted $D_k^m \Phi_{\mathfrak{e}}^{k}$ $(D_k^m \Phi_{\mathfrak{o}}^{k})$, are obtainable from the Taylor series coefficients (5.31) of $|x|^{z-1}(|x|^{z-1} \operatorname{sgn})$ and the Taylor series of $\pi^{z/2}/\Gamma(z/2)$ $(\pi^{z/2}/\Gamma((z+1)/2))$ about $z = k$. We will not need their explicit expressions.

6.3.3. *Generalized multiplication derivatives.* From (6.86)–(6.87), (5.65), (5.84) and the duplication formula for the gamma function follows, $\forall k \in \mathbb{N}$ and $\forall z \in \mathbb{C}$,

$$D^k \Phi_{\mathfrak{e}}^{z} = 2^k \pi^{k/2} \begin{pmatrix} e_k\left((z-1)/2\right)_{(k/2)} \Phi_{\mathfrak{e}}^{z-k} \\ +o_k\left((z-k)/2\right)\left((z-1)/2\right)_{((k-1)/2)} \Phi_{\mathfrak{o}}^{z-k} \end{pmatrix}, \tag{6.92}$$

$$D^k \Phi_{\mathfrak{o}}^{z} = 2^k \pi^{k/2} \begin{pmatrix} e_k\left(z/2-1\right)_{(k/2)} \Phi_{\mathfrak{o}}^{z-k} \\ +o_k(z/2-1)_{((k-1)/2)} \Phi_{\mathfrak{e}}^{z-k} \end{pmatrix}. \tag{6.93}$$

Also, $\forall m \in \mathbb{N}$ and $\forall z \in \mathbb{C}$,

$$D\left(D_z^m \Phi_{\mathfrak{e}}^{z}\right) = D_z^m\left(D\Phi_{\mathfrak{e}}^{z}\right) = \pi^{1/2}(z-1) D_z^m \Phi_{\mathfrak{o}}^{z-1} + m\pi^{1/2} D_z^{m-1} \Phi_{\mathfrak{o}}^{z-1}, \tag{6.94}$$

$$D\left(D_z^m \Phi_{\mathfrak{o}}^{z}\right) = D_z^m\left(D\Phi_{\mathfrak{o}}^{z}\right) = 2\pi^{1/2} D_z^m \Phi_{\mathfrak{e}}^{z-1}. \tag{6.95}$$

6.3.4. *Generalized convolution derivatives.* Combining (6.86)–(6.87) with (5.68) and (5.88) gives, $\forall k \in \mathbb{N}$ and $\forall z \in \mathbb{C}$,

$$X^k \Phi_{\mathfrak{e}}^{z} = \pi^{-k/2}\left(e_k\left(z/2\right)^{(k/2)} \Phi_{\mathfrak{e}}^{z+k} + o_k\left(z/2\right)^{((k+1)/2)} \Phi_{\mathfrak{o}}^{z+k}\right), \tag{6.96}$$

$$X^k \Phi_{\mathfrak{o}}^{z} = \pi^{-k/2}\left(e_k\left((z+1)/2\right)^{(k/2)} \Phi_{\mathfrak{o}}^{z+k} + o_k\left(z/2\right)^{((k-1)/2)} \Phi_{\mathfrak{e}}^{z+k}\right). \tag{6.97}$$

Also, $\forall m\in\mathbb{N}$ and $\forall z\in\mathbb{C}$,

$$\begin{aligned} X\left(D_z^m\Phi_{\mathfrak{e}}^z\right) &= D_z^m\left(X\Phi_{\mathfrak{e}}^z\right)=D_z^m\left(\left(z/2\right)\Phi_{\mathfrak{o}}^{z+1}\right) && (6.98)\\ &= \pi^{-1/2}\frac{1}{2}\left(zD_z^m\Phi_{\mathfrak{o}}^{z+1}+mD_z^{m-1}\Phi_{\mathfrak{o}}^{z+1}\right), && (6.99)\\ X\left(D_z^m\Phi_{\mathfrak{o}}^z\right) &= D_z^m\left(X\Phi_{\mathfrak{o}}^z\right)=\pi^{-1/2}D_z^m\Phi_{\mathfrak{e}}^{z+1}. && (6.100)\end{aligned}$$

6.3.5. *Fourier transforms.* Expanding the distributions $\Phi_{\mathfrak{e},\mathfrak{o}}^z$ in terms of $\Phi_\pm^z$ and using (6.36) leads to the very simple Fourier transformation rules, $\forall z\in\mathbb{C}$,

$$\begin{aligned} \mathcal{F}\left[\Phi_{\mathfrak{e}}^z\right] &= \Phi_{\mathfrak{e}}^{1-z}, && (6.101)\\ \mathcal{F}\left[\Phi_{\mathfrak{o}}^z\right] &= -i\Phi_{\mathfrak{o}}^{1-z}. && (6.102)\end{aligned}$$

More generally, $\forall z\in\mathbb{C}$ and $\forall m\in\mathbb{Z}_+$, we obtain from (6.101)–(6.102) and by the commutativity of the $\mathcal{F}$ and D_z operators,

$$\begin{aligned} \mathcal{F}\left[D_z^m\Phi_{\mathfrak{e}}^z\right] &= (-1)^m\left(D_w^m\Phi_{\mathfrak{e}}^w\right)(1-z), && (6.103)\\ \mathcal{F}\left[D_z^m\Phi_{\mathfrak{o}}^z\right] &= -(-1)^m\, i\left(D_w^m\Phi_{\mathfrak{o}}^w\right)(1-z). && (6.104)\end{aligned}$$

We will call the $\Phi_{\mathfrak{e},\mathfrak{o}}^z$ *homogeneous normalized parity kernels of the second kind* and the $D_z^m\Phi_{\mathfrak{e},\mathfrak{o}}^z$, $\forall m\in\mathbb{Z}_+$, *associated homogeneous normalized parity kernels of the second kind*. The distributions $D_z^m\Phi_{\mathfrak{e},\mathfrak{o}}^z$ play a fundamental role in the representation of AHDs based on R, which are complex analytic with respect to their degree in some region $\Omega\subseteq\mathbb{C}$, [9, Theorem 7]. Also, as the results (6.101)–(6.102) show, they are the preferred basis distributions to expand AHDs in, in order to trivially calculate Fourier transforms.

6.4. **Normalized complex kernels $\Phi_{x\pm i0}^z$.** We finally define a pair of normalized basis AHDs, which are very convenient to calculate convolutions of AHDs and also to trivially calculate the Hilbert transform of an AHD, and which were not considered in [15].

6.4.1. *Definition.* Define the distributions

$$\begin{aligned} \Phi_{x\pm i0}^z &\triangleq \frac{1}{2\pi}\Gamma(1-z)e^{\mp i(\pi/2)(z-1)}(x\pm i0)^{z-1}, && (6.105)\\ &= \frac{1}{2}\left(\Phi_e^z\pm i\Phi_o^z\right), && (6.106)\\ &= \frac{\pm i}{2}\frac{1}{\sin(\pi z)}\left(e^{\mp i(\pi/2)z}\Phi_+^z-e^{\pm i(\pi/2)z}\Phi_-^z\right). && (6.107)\end{aligned}$$

For $z=0$, $\Phi_{x\pm i0}^z$ reduce to the Heisenberg distributions. The $\Phi_{x\pm i0}^z$ are homogeneous distributions of degree $z-1$, complex analytic $\forall z\in\mathbb{C}$, except for simple poles at $z=k\in\mathbb{Z}_+$ with residues

$$\left(\Phi_{x\pm i0}^k\right)_{-1}=\frac{(\pm i)^{k+1}}{2\pi}\frac{x^{k-1}}{(k-1)!}, \tag{6.108}$$

and analytic finite parts

$$\left(\Phi_{x\pm i0}^k\right)_0=\frac{1}{2}(\pm i)^k\left(\frac{1}{2}\frac{x^{k-1}\operatorname{sgn}}{(k-1)!}+\frac{\pm i}{\pi}\frac{x^{k-1}}{(k-1)!}\left(\ln|x|-\psi(k)\right)\right). \tag{6.109}$$

At the ordinary points $z=-k\in-\mathbb{N}$, they take the values

$$\Phi_{x\pm i0}^{-k}=\frac{1}{2}(\pm i)^{-k}\left(\delta^{(k)}\pm i\eta^{(k)}\right). \tag{6.110}$$

6.4.2. *Associated distributions.* At $z \in \mathbb{C}\backslash\mathbb{Z}$ and $\forall m \in \mathbb{Z}_+$, we obtain from (6.106) the associated distributions

$$D_z^m \Phi_{x\pm i0}^z = \frac{1}{2}\left(D_z^m \Phi_e^z \pm i D_z^m \Phi_o^z\right). \tag{6.111}$$

At $z \in \mathbb{Z}$ and $\forall m \in \mathbb{Z}_+$, $D_z^m \Phi_{x\pm i0}^z$ are readily found by combining (6.106) with (6.58)–(6.63). We obtain, $\forall p, k \in \mathbb{N}$,

$$\begin{aligned} &\left(\left(D_w^m \Phi_{x\pm i0}^w\right)_{w=2p+1}\right)_0 \\ =\ & \frac{1}{2}\left(\begin{array}{c} (-1)^{p+1}(i\pi/2)^m i\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w + \Phi_-^w\right) \\ \pm i\,(-1)^p\,(i\pi/2)^m \overline{E}_m\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w - \Phi_-^w\right) \end{array}\right)_{w=2p+1}, \end{aligned} \tag{6.112}$$

$$\begin{aligned} &\left(\left(D_w^m \Phi_{x\pm i0}^w\right)_{w=2p+2}\right)_0 \\ =\ & \frac{1}{2}\left(\begin{array}{c} (-1)^{p+1}\,(i\pi/2)^m \overline{E}_m\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w + \Phi_-^w\right) \\ \pm i(-1)^{p+1}(i\pi/2)^m i\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w - \Phi_-^w\right) \end{array}\right)_{w=2p+2}; \end{aligned} \tag{6.113}$$

and

$$\begin{aligned} &\left(D_w^m \Phi_{x\pm i0}^w\right)_{w=-k} \\ =\ & (i\pi/2)^m\,\frac{1}{2}\left(\begin{array}{c} e_k\,(-1)^{k/2}\,\overline{E}_m\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w + \Phi_-^w\right) \\ \pm i o_k\,(-1)^{(k+1)/2}\,\overline{E}_m\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w - \Phi_-^w\right) \\ +o_k\,(-1)^{(k-1)/2}\,i\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right))\frac{1}{2}\left(\Phi_+^w + \Phi_-^w\right) \\ \pm i e_k\,(-1)^{k/2}\,i\overline{B}_{m+1}\left(\frac{D_w}{i\pi/2}\right)\frac{1}{2}\left(\Phi_+^w - \Phi_-^w\right) \end{array}\right)_{w=-k} \end{aligned} \tag{6.114}$$

6.4.3. *Generalized multiplication derivatives.* From (6.106) and (6.66)–(6.67) follows, $\forall z \in \mathbb{C}\backslash\mathbb{Z}_+$,

$$D^k\left(D_z^m \Phi_{x\pm i0}^z\right) = e^{\pm ik\pi/2}\left(D_w^m \Phi_{x\pm i0}^w\right)_{w=z-k}. \tag{6.115}$$

The distributions $D^n\left(D_k^m \Phi_{x\pm i0}^k\right)_0$, $\forall n, m \in \mathbb{N}$ and $\forall k \in \mathbb{Z}_+$, are readily obtained as linear combinations of (6.68)–(6.71).

6.4.4. *Generalized convolution derivatives.* From (6.105) and (6.74)–(6.75) follows, $\forall z \in \mathbb{C}\backslash\mathbb{Z}_+$,

$$X^k\left(D_z^m \Phi_{x\pm i0}^z\right) = e^{\mp ik\pi/2} D_z^m\left(z^{(k)}\Phi_{x\pm i0}^{z+k}\right). \tag{6.116}$$

The distributions $X^n\left(D_k^m \Phi_{x\pm i0}^k\right)_0$, $\forall n, m \in \mathbb{N}$ and $\forall k \in \mathbb{Z}_+$, are readily obtained as linear combinations of (6.76)–(6.79).

6.4.5. *Fourier transforms.* We easily obtain from (6.105) and (5.164), $\forall z \in \mathbb{C}\backslash\mathbb{Z}_+$,

$$\mathcal{F}\left[\Phi_{x\pm i0}^z\right] = (2\pi\chi)_\pm^{-z}. \tag{6.117}$$

More generally, $\forall m \in \mathbb{N}$ and $\forall z \in \mathbb{C}\backslash\mathbb{Z}_+$, we have by the commutativity of the $\mathcal{F}$ and D_z operators, that

$$\mathcal{F}\left[D_z^m \Phi_{x\pm i0}^z\right] = (-1)^m\,(2\pi)^{-z}\sum_{p=0}^{m}\binom{m}{p}\ln^{m-p}(2\pi)\left(D_w^p \chi_\pm^w\right)_{w=-z}. \tag{6.118}$$

The Fourier transforms of the distributions $\left(D_k^m \Phi_{x\pm i0}^k\right)_0$, $\forall m \in \mathbb{N}$ and $\forall k \in \mathbb{Z}_+$, are obtained as follows. Use (5.38), the Laurent series of the distributions $\Phi_{x\pm i0}^z$ about their simple pole $z = k$, the Taylor series of $(2\pi)^{-z}$ about the ordinary point $z = k$, and the Fourier transform pair $\mathcal{F}\left[(-2\pi i x)^{k-1}\right] = \delta^{(k-1)}$. This results in

$$\begin{aligned} & \mathcal{F}\left[\left(D_k^m \Phi_{x\pm i0}^k\right)_0\right] \\ = \quad & (-1)^m \left((2\pi\chi)_{\pm,0}^{-k} \ln^m |2\pi\chi| + (\mp 1)^{k-1} \frac{\ln^{m+1}(2\pi)}{m+1} \frac{(2\pi)^{-k} \delta^{(k-1)}}{(k-1)!} \right) \end{aligned} \tag{6.119}$$

We will call $\Phi_{x\pm i0}^z$ *homogeneous normalized complex kernels* and $D_z^m \Phi_{x\pm i0}^z$, $\forall m \in \mathbb{Z}_+$, *associated homogeneous normalized complex kernels*. The distributions $D_z^m \Phi_{x\pm i0}^z$ simplify the calculation of convolution products of AHDs (due to [10, Theorem 12] and [12, Corollary 5]) and the calculation of the Hilbert transform of an AHD (due to [10, eq. (60)] and [12, eq. (50)]).

Appendix A. Appendix

A.1. **Definitions of AHDs.**

A.1.1. *Gel'fand–Shilov.* The definition of (one-dimensional) AHDs, in the sense of Gel'fand and Shilov as given in [15, Ch. I, Section 4.1], arises in the following way.

(i) A HD f_0^z, of degree of homogeneity z, is defined to be an eigendistribution of any dilatation operator U_r, $r > 0$, with eigenvalue r^z, [15, p. 82].

(ii) Generalized functions $f_1^z, f_2^z, \ldots, f_k^z, \ldots$ are said to be associated with the eigendistribution f_0^z of U_r iff

$$U_r f_0^z = c f_0^z, \tag{A.1}$$
$$U_r f_k^z = c f_k^z + d f_{k-1}^z, \forall k \in \mathbb{Z}_+, \tag{A.2}$$

wherein c and d are functions of r but are independent of k. According to (i), choosing $c = r^z$ makes f_0^z in (A.1) a HD.

(iii) Now d is calculated for the case $k = 1$. It is proved in [15, p. 83] that, up to a constant factor, $d = r^z \ln r$.

(iv) By setting $d = r^z \ln r$ for all $k \in \mathbb{Z}_+$ in (A.2), eqs. (A.1)–(A.2) take the form

$$U_r f_0^z = r^z f_0^z, \tag{A.3}$$
$$U_r f_k^z = r^z f_k^z + r^z \ln r\, f_{k-1}^z, \forall k \in \mathbb{Z}_+. \tag{A.4}$$

(v) In [15], HDs are defined by (A.3) and AHDs for all orders of association $k \in \mathbb{Z}_+$ are defined by (A.4), [15, p. 84, eq. (3)]. Eqs. (A.3)–(A.4) were derived from eqs. (A.1)–(A.2) by calculation of c and d. Hence, Definition (A.3)–(A.4) is equivalent to Definition (A.1)–(A.2) with $c = r^z$ and $d = r^z \ln r$.

(vi) In [25, Section 4] it was shown that Definition (A.3)–(A.4) is self contradictory for $k \geq 2$. The conclusion is that an AHD of order of association k is reproduced by the dilatation operator U_r, $\forall r > 0$, up to an AHD of order of association $k - 1$, only for $k = 1$. Transferring the notion of associated eigenvector to AHDs, in the sense of Gel'fand and Shilov, is thus impossible for $k \geq 2$.

Other inconsistencies in the definition of AHDs have appeared in the literature. For a discussion, see [25].

A.1.2. *Shelkovich.* Since the set of AHDs, in the sense of Gel'fand and Shilov, does not contain any members with order of association $m \geq 2$, a more general definition was introduced in [25], resulting in the set of Quasi Associated Homogeneous Distributions (QAHDs). It was shown by Shelkovich that his set of QAHDs coincides with the set of distributions discussed in [15, Ch. I, Section 4.1], [25, Theorem 3.2].

(i) A distribution $f_m^z \in \mathcal{D}'(R)$ is said to be a QAHD of degree of homogeneity $z \in \mathbb{C}$ and order of association $m \in \mathbb{N}$ iff for any $r > 0$,

$$U_r f_m^z = r^z f_m^z + \sum_{k=1}^{m} h_k(r)\ f_{m-k}^z, \tag{A.5}$$

wherein f_{m-k}^z is a QAHD of degree of homogeneity z and order of association $m-k$ and h_k are differentiable functions. For $m = 0$, the sum is assumed to be empty. Definition (A.5) is a natural generalization of the notion of associated eigenvector (A.2).

(ii) Next, it was proved in [25, Theorems 3.1, 3.2 and eqs. (3.21)–(3.22)] that Definition (A.5) is equivalent to

$$U_r f_m^z = r^z f_m^z + \sum_{k=1}^{m} r^z \ln^k r\ f_{m-k}^z, \tag{A.6}$$

wherein f_{m-k}^z is an AHD, as discussed in [15, Ch. I, Section 4.1], of degree of homogeneity z and order of association $m-k$. The relation (A.6) can thus be used as an equivalent definition of QAHDs.

Definition (4.2), used in this paper, is equivalent to (A.6) and defines a set of distributions, which are also called Associated Homogeneous Distributions (AHDs). The here considered set of AHDs is thus coinciding with Shelkovich's set of QAHDs and, due to [25, Theorem 3.2], is also coinciding with the distributions discussed (but not properly defined) in [15, Ch. I, Section 4.1].

A.1.3. *von Grudzinski.* For completeness we give here the higher-dimensional definitions corresponding to HDs and AHDs, as given in [26].

A distribution $f^z \in \mathcal{D}'(R^n)$ is said to be *quasihomogeneous* of degree $z \in \mathbb{C}$ and of type $p \triangleq (p_1, \ldots, p_n) \in \mathbb{R}^n \backslash \{0\}$ iff for any $r > 0$, [26, p. 77],

$$\left\langle f^z, \varphi \circ M_{1/r} \right\rangle = r^{z+n} \left\langle f^z, \varphi \right\rangle, \forall \varphi \in \mathcal{D}(R^n), \tag{A.7}$$

wherein $M_r x \triangleq (r^{p_1} x_1, \ldots, r^{p_n} x_n)$, $x \in R^n$. A quasihomogeneous distribution of degree z and of type $p \triangleq (1, \ldots, 1)$ is called a *homogeneous* distribution of degree z.

A distribution $f_m^z \in \mathcal{D}'(R^n)$ is said to be *almost quasihomogeneous* of degree $z \in \mathbb{C}$, of type $p \triangleq (p_1, \ldots, p_n) \in \mathbb{R}^n \backslash \{0\}$ and of order $\leq m \in \mathbb{Z}_+$ iff for any $r > 0$ there exists distributions $f_{m-1}^z, \ldots, f_0^z$ such that, [26, p. 94 and eq. (1.37)],

$$\left\langle f_m^z, \varphi \circ M_{1/r} \right\rangle = r^{z+n} \left\langle f_m^z + \sum_{k=1}^{m} \frac{\ln^k r}{k!}\ f_{m-k}^z\,, \varphi \right\rangle, \forall \varphi \in \mathcal{D}(R^n). \tag{A.8}$$

The definitions (A.7)–(A.8) essentially reduce to (A.5)–(A.6) for $n = 1$.

A.2. **Conversion of basis AHDs.** For the seven pairs of special homogeneous distributions, $\Phi^z_\pm$, $(x \pm i0)^{z-1}$, $\Phi^z_{e,o}$, $\Phi^z_{\mathfrak{e},\mathfrak{o}}$, $\left(|x|^{z-1}, |x|^{z-1}\operatorname{sgn}\right)$, $\Phi^z_{x\pm i0}$ and $x^{z-1}_\pm$, it is convenient to have expressions to convert one set into another. This is particularly useful to convert from one structure theorem to another, see [9].

A.2.1. *Expressions in terms of* $\Phi^z_\pm$.

$$
\begin{bmatrix}
\begin{bmatrix}\Phi^z_+\\ \Phi^z_-\end{bmatrix}\\
\begin{bmatrix}(x+i0)^{z-1}\\ (x-i0)^{z-1}\end{bmatrix}\\
\begin{bmatrix}\Phi^z_e\\ \Phi^z_o\end{bmatrix}\\
\begin{bmatrix}\Phi^z_{\mathfrak{e}}\\ \Phi^z_{\mathfrak{o}}\end{bmatrix}\\
\begin{bmatrix}|x|^{z-1}\\ |x|^{z-1}\operatorname{sgn}\end{bmatrix}\\
\begin{bmatrix}\Phi^z_{x+i0}\\ \Phi^z_{x-i0}\end{bmatrix}\\
\begin{bmatrix}x^{z-1}_+\\ x^{z-1}_-\end{bmatrix}
\end{bmatrix}
=
\begin{bmatrix}
\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\
\Gamma(z)\begin{bmatrix}1 & -e^{+i\pi z}\\ 1 & -e^{-i\pi z}\end{bmatrix}\\
\frac{1}{2}\begin{bmatrix}\frac{1}{\cos\left(\frac{\pi}{2}z\right)} & \frac{1}{\cos\left(\frac{\pi}{2}z\right)}\\ +\frac{1}{\sin\left(\frac{\pi}{2}z\right)} & -\frac{1}{\sin\left(\frac{\pi}{2}z\right)}\end{bmatrix}\\
2^{z-1}\pi^{(z-1)/2}\begin{bmatrix}\Gamma\left(\frac{z+1}{2}\right) & \Gamma\left(\frac{z+1}{2}\right)\\ \Gamma\left(\frac{z}{2}\right) & -\Gamma\left(\frac{z}{2}\right)\end{bmatrix}\\
\Gamma(z)\begin{bmatrix}1 & +1\\ 1 & -1\end{bmatrix}\\
\frac{i}{2\sin(\pi z)}\begin{bmatrix}+e^{-i\frac{\pi}{2}z} & -e^{+i\frac{\pi}{2}z}\\ -e^{+i\frac{\pi}{2}z} & +e^{-i\frac{\pi}{2}z}\end{bmatrix}\\
\Gamma(z)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}\Phi^z_+\\ \Phi^z_-\end{bmatrix}.
$$

A.2.2. *Expressions in terms of* $(x \pm i0)^{z-1}$.

$$
\begin{bmatrix}
\begin{bmatrix}\Phi^z_+\\ \Phi^z_-\end{bmatrix}\\
\begin{bmatrix}(x+i0)^{z-1}\\ (x-i0)^{z-1}\end{bmatrix}\\
\begin{bmatrix}\Phi^z_e\\ \Phi^z_o\end{bmatrix}\\
\begin{bmatrix}\Phi^z_{\mathfrak{e}}\\ \Phi^z_{\mathfrak{o}}\end{bmatrix}\\
\begin{bmatrix}|x|^{z-1}\\ |x|^{z-1}\operatorname{sgn}\end{bmatrix}\\
\begin{bmatrix}\Phi^z_{x+i0}\\ \Phi^z_{x-i0}\end{bmatrix}\\
\begin{bmatrix}x^{z-1}_+\\ x^{z-1}_-\end{bmatrix}
\end{bmatrix}
=
\begin{bmatrix}
\frac{1}{2}\begin{bmatrix}\frac{1}{\Gamma(z)} & \frac{1}{\Gamma(z)}\\ -\frac{1}{\pi i}\Gamma(1-z) & +\frac{1}{\pi i}\Gamma(1-z)\end{bmatrix}\\
\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\
\frac{\Gamma(1-z)}{2\pi}\begin{bmatrix}+ie^{-i\frac{\pi}{2}z} & -ie^{+i\frac{\pi}{2}z}\\ e^{-i\frac{\pi}{2}z} & e^{+i\frac{\pi}{2}z}\end{bmatrix}\\
\frac{1}{2}\pi^{z/2}\begin{bmatrix}+\frac{ie^{-i\left(\frac{\pi}{2}z\right)}}{\Gamma\left(\frac{z}{2}\right)\sin\left(\frac{\pi}{2}z\right)} & -\frac{ie^{+i\left(\frac{\pi}{2}z\right)}}{\Gamma\left(\frac{z}{2}\right)\sin\left(\frac{\pi}{2}z\right)}\\ \frac{e^{-i\left(\frac{\pi}{2}z\right)}}{\Gamma\left(\frac{z+1}{2}\right)\cos\left(\frac{\pi}{2}z\right)} & \frac{e^{+i\left(\frac{\pi}{2}z\right)}}{\Gamma\left(\frac{z+1}{2}\right)\cos\left(\frac{\pi}{2}z\right)}\end{bmatrix}\\
\frac{1}{2}\begin{bmatrix}\frac{+ie^{-i\frac{\pi}{2}z}}{\sin\left(\frac{\pi}{2}z\right)} & \frac{-ie^{+i\frac{\pi}{2}z}}{\sin\left(\frac{\pi}{2}z\right)}\\ \frac{e^{-i\frac{\pi}{2}z}}{\cos\left(\frac{\pi}{2}z\right)} & \frac{e^{+i\frac{\pi}{2}z}}{\cos\left(\frac{\pi}{2}z\right)}\end{bmatrix}\\
\frac{\Gamma(1-z)}{2\pi i}\begin{bmatrix}-e^{-i\frac{\pi}{2}z} & 0\\ 0 & +e^{+i\frac{\pi}{2}z}\end{bmatrix}\\
\frac{1}{2i\sin(\pi z)}\begin{bmatrix}-e^{-i\pi z} & e^{+i\pi z}\\ -1 & 1\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}(x+i0)^{z-1}\\ (x-i0)^{z-1}\end{bmatrix}.
$$

A.2.3. *Expressions in terms of* $\Phi^z_{e,o}$.

$$
\begin{bmatrix}
\begin{bmatrix}\Phi^z_+\\ \Phi^z_-\end{bmatrix}\\
\begin{bmatrix}(x+i0)^{z-1}\\ (x-i0)^{z-1}\end{bmatrix}\\
\begin{bmatrix}\Phi^z_e\\ \Phi^z_o\end{bmatrix}\\
\begin{bmatrix}\Phi^z_{\mathfrak{e}}\\ \Phi^z_{\mathfrak{o}}\end{bmatrix}\\
\begin{bmatrix}|x|^{z-1}\\ |x|^{z-1}\operatorname{sgn}\end{bmatrix}\\
\begin{bmatrix}\Phi^z_{x+i0}\\ \Phi^z_{x-i0}\end{bmatrix}\\
\begin{bmatrix}x_+^{z-1}\\ x_-^{z-1}\end{bmatrix}
\end{bmatrix}
=
\begin{bmatrix}
\begin{bmatrix}\cos\left(\frac{\pi}{2}z\right) & +\sin\left(\frac{\pi}{2}z\right)\\ \cos\left(\frac{\pi}{2}z\right) & -\sin\left(\frac{\pi}{2}z\right)\end{bmatrix}\\
\frac{\pi}{\Gamma(1-z)}\begin{bmatrix}-ie^{+i\frac{\pi}{2}z} & e^{+i\frac{\pi}{2}z}\\ +ie^{-i\frac{\pi}{2}z} & e^{-i\frac{\pi}{2}z}\end{bmatrix}\\
\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\
2^z\pi^{(z+1)/2}\begin{bmatrix}\frac{1}{\Gamma\left(\frac{1-z}{2}\right)} & 0\\ 0 & \frac{1}{\Gamma\left(1-\frac{z}{2}\right)}\end{bmatrix}\\
2\Gamma(z)\begin{bmatrix}\cos\left(\frac{\pi}{2}z\right) & 0\\ 0 & \sin\left(\frac{\pi}{2}z\right)\end{bmatrix}\\
\frac{1}{2}\begin{bmatrix}1 & +i\\ 1 & -i\end{bmatrix}\\
\Gamma(z)\begin{bmatrix}\cos\left(\frac{\pi}{2}z\right) & +\sin\left(\frac{\pi}{2}z\right)\\ \cos\left(\frac{\pi}{2}z\right) & -\sin\left(\frac{\pi}{2}z\right)\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}\Phi^z_e\\ \Phi^z_o\end{bmatrix}.
$$

A.2.4. *Expressions in terms of* $\Phi^z_{\mathfrak{e},\mathfrak{o}}$.

$$
\begin{bmatrix}
\begin{bmatrix}\Phi^z_+\\ \Phi^z_-\end{bmatrix}\\
\begin{bmatrix}(x+i0)^{z-1}\\ (x-i0)^{z-1}\end{bmatrix}\\
\begin{bmatrix}\Phi^z_e\\ \Phi^z_o\end{bmatrix}\\
\begin{bmatrix}\Phi^z_{\mathfrak{e}}\\ \Phi^z_{\mathfrak{o}}\end{bmatrix}\\
\begin{bmatrix}|x|^{z-1}\\ |x|^{z-1}\operatorname{sgn}\end{bmatrix}\\
\begin{bmatrix}\Phi^z_{x+i0}\\ \Phi^z_{x-i0}\end{bmatrix}\\
\begin{bmatrix}x_+^{z-1}\\ x_-^{z-1}\end{bmatrix}
\end{bmatrix}
=
\begin{bmatrix}
2^{-z}\pi^{-(z-1)/2}\begin{bmatrix}\frac{1}{\Gamma\left(\frac{z+1}{2}\right)} & +\frac{1}{\Gamma\left(\frac{z}{2}\right)}\\ \frac{1}{\Gamma\left(\frac{z+1}{2}\right)} & -\frac{1}{\Gamma\left(\frac{z}{2}\right)}\end{bmatrix}\\
\pi^{-z/2}\begin{bmatrix}-ie^{+i\left(\frac{\pi}{2}z\right)}\Gamma\left(\frac{z}{2}\right)\sin\left(\frac{\pi}{2}z\right) & e^{+i\left(\frac{\pi}{2}z\right)}\Gamma\left(\frac{z+1}{2}\right)\cos\left(\frac{\pi}{2}z\right)\\ +ie^{-i\left(\frac{\pi}{2}z\right)}\Gamma\left(\frac{z}{2}\right)\sin\left(\frac{\pi}{2}z\right) & e^{-i\left(\frac{\pi}{2}z\right)}\Gamma\left(\frac{z+1}{2}\right)\cos\left(\frac{\pi}{2}z\right)\end{bmatrix}\\
2^{-z}\pi^{-(z+1)/2}\begin{bmatrix}\Gamma\left(\frac{1-z}{2}\right) & 0\\ 0 & \Gamma\left(1-\frac{z}{2}\right)\end{bmatrix}\\
\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\
\pi^{-z/2}\begin{bmatrix}\Gamma(z/2) & 0\\ 0 & \Gamma((z+1)/2)\end{bmatrix}\\
2^{-(z+1)}\pi^{-(z+1)/2}\begin{bmatrix}\Gamma\left(\frac{1-z}{2}\right) & i\Gamma\left(1-\frac{z}{2}\right)\\ \Gamma\left(\frac{1-z}{2}\right) & -i\Gamma\left(1-\frac{z}{2}\right)\end{bmatrix}\\
\frac{1}{2}\pi^{-z/2}\begin{bmatrix}\Gamma\left(z/2\right) & \Gamma\left((z+1)/2\right)\\ \Gamma\left(z/2\right) & -\Gamma\left((z+1)/2\right)\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}\Phi^z_{\mathfrak{e}}\\ \Phi^z_{\mathfrak{o}}\end{bmatrix}.
$$

A.2.5. *Expressions in terms of* $|x|^{z-1}$, $|x|^{z-1}\operatorname{sgn}$.

$$\begin{bmatrix} \begin{bmatrix}\Phi_+^z\\ \Phi_-^z\end{bmatrix} \\ \begin{bmatrix}(x+i0)^{z-1}\\(x-i0)^{z-1}\end{bmatrix} \\ \begin{bmatrix}\Phi_e^z\\ \Phi_o^z\end{bmatrix} \\ \begin{bmatrix}\Phi_{\mathfrak{e}}^z\\ \Phi_{\mathfrak{o}}^z\end{bmatrix} \\ \begin{bmatrix}|x|^{z-1}\\ |x|^{z-1}\operatorname{sgn}\end{bmatrix} \\ \begin{bmatrix}\Phi_{x+i0}^z\\ \Phi_{x-i0}^z\end{bmatrix} \\ \begin{bmatrix}x_+^{z-1}\\ x_-^{z-1}\end{bmatrix}\end{bmatrix} = \begin{bmatrix} \frac{1}{2\Gamma(z)}\begin{bmatrix}1 & +1\\ 1 & -1\end{bmatrix} \\ \begin{bmatrix}-ie^{+i\frac{\pi}{2}z}\sin\left(\frac{\pi}{2}z\right) & e^{+i\frac{\pi}{2}z}\cos\left(\frac{\pi}{2}z\right)\\ +ie^{-i\frac{\pi}{2}z}\sin\left(\frac{\pi}{2}z\right) & e^{-i\frac{\pi}{2}z}\cos\left(\frac{\pi}{2}z\right)\end{bmatrix} \\ \frac{1}{2\Gamma(z)}\begin{bmatrix}\frac{1}{\cos\left(\frac{\pi}{2}z\right)} & 0\\ 0 & \frac{1}{\sin\left(\frac{\pi}{2}z\right)}\end{bmatrix} \\ \pi^{z/2}\begin{bmatrix}\frac{1}{\Gamma(z/2)} & 0\\ 0 & \frac{1}{\Gamma((z+1)/2)}\end{bmatrix} \\ \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \\ \frac{1}{4\Gamma(z)}\begin{bmatrix}\frac{1}{\cos\left(\frac{\pi}{2}z\right)} & \frac{+i}{\sin\left(\frac{\pi}{2}z\right)}\\ \frac{1}{\cos\left(\frac{\pi}{2}z\right)} & \frac{-i}{\sin\left(\frac{\pi}{2}z\right)}\end{bmatrix} \\ \frac{1}{2}\begin{bmatrix}1 & +1\\ 1 & -1\end{bmatrix} \end{bmatrix} \begin{bmatrix}|x|^{z-1}\\ |x|^{z-1}\operatorname{sgn}\end{bmatrix}.$$

A.2.6. *Expressions in terms of* $\Phi_{x\pm i0}^z$.

$$\begin{bmatrix} \begin{bmatrix}\Phi_+^z\\ \Phi_-^z\end{bmatrix} \\ \begin{bmatrix}(x+i0)^{z-1}\\(x-i0)^{z-1}\end{bmatrix} \\ \begin{bmatrix}\Phi_e^z\\ \Phi_o^z\end{bmatrix} \\ \begin{bmatrix}\Phi_{\mathfrak{e}}^z\\ \Phi_{\mathfrak{o}}^z\end{bmatrix} \\ \begin{bmatrix}|x|^{z-1}\\ |x|^{z-1}\operatorname{sgn}\end{bmatrix} \\ \begin{bmatrix}\Phi_{x+i0}^z\\ \Phi_{x-i0}^z\end{bmatrix} \\ \begin{bmatrix}x_+^{z-1}\\ x_-^{z-1}\end{bmatrix}\end{bmatrix} = \begin{bmatrix} \begin{bmatrix}e^{-i\frac{\pi}{2}z} & e^{+i\frac{\pi}{2}z}\\ e^{+i\frac{\pi}{2}z} & e^{-i\frac{\pi}{2}z}\end{bmatrix} \\ -\frac{2\pi i}{\Gamma(1-z)}\begin{bmatrix}+e^{+i\frac{\pi}{2}z} & 0\\ 0 & -e^{-i\frac{\pi}{2}z}\end{bmatrix} \\ \begin{bmatrix}1 & 1\\ -i & +i\end{bmatrix} \\ 2^z\pi^{(z+1)/2}\begin{bmatrix}\frac{1}{\Gamma\left(\frac{1-z}{2}\right)} & \frac{1}{\Gamma\left(\frac{1-z}{2}\right)}\\ -\frac{i}{\Gamma\left(1-\frac{z}{2}\right)} & \frac{i}{\Gamma\left(1-\frac{z}{2}\right)}\end{bmatrix} \\ 2\Gamma(z)\begin{bmatrix}\cos\left(\frac{\pi}{2}z\right) & \cos\left(\frac{\pi}{2}z\right)\\ -i\sin\left(\frac{\pi}{2}z\right) & +i\sin\left(\frac{\pi}{2}z\right)\end{bmatrix} \\ \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \\ \Gamma(z)\begin{bmatrix}e^{-i\frac{\pi}{2}z} & e^{+i\frac{\pi}{2}z}\\ e^{+i\frac{\pi}{2}z} & e^{-i\frac{\pi}{2}z}\end{bmatrix} \end{bmatrix} \begin{bmatrix}\Phi_{x+i0}^z\\ \Phi_{x-i0}^z\end{bmatrix}.$$

A.2.7. *Expressions in terms of $x_\pm^{z-1}$.*

$$\begin{bmatrix} \begin{bmatrix} \Phi_+^z \\ \Phi_-^z \end{bmatrix} \\ \begin{bmatrix} (x+i0)^{z-1} \\ (x-i0)^{z-1} \end{bmatrix} \\ \begin{bmatrix} \Phi_e^z \\ \Phi_o^z \end{bmatrix} \\ \begin{bmatrix} \Phi_{\mathfrak{e}}^z \\ \Phi_{\mathfrak{o}}^z \end{bmatrix} \\ \begin{bmatrix} |x|^{z-1} \\ |x|^{z-1}\,\mathrm{sgn} \end{bmatrix} \\ \begin{bmatrix} \Phi_{x+i0}^z \\ \Phi_{x-i0}^z \end{bmatrix} \\ \begin{bmatrix} x_+^{z-1} \\ x_-^{z-1} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Gamma(z)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & -e^{+i\pi z} \\ 1 & -e^{-i\pi z} \end{bmatrix} \\ \frac{1}{2\Gamma(z)} \begin{bmatrix} \frac{1}{\cos\left(\frac{\pi}{2}z\right)} & \frac{1}{\cos\left(\frac{\pi}{2}z\right)} \\ +\frac{1}{\sin\left(\frac{\pi}{2}z\right)} & -\frac{1}{\sin\left(\frac{\pi}{2}z\right)} \end{bmatrix} \\ \pi^{z/2} \begin{bmatrix} \frac{1}{\Gamma(z/2)} & \frac{1}{\Gamma(z/2)} \\ \frac{1}{\Gamma((z+1)/2)} & -\frac{1}{\Gamma((z+1)/2)} \end{bmatrix} \\ \begin{bmatrix} 1 & +1 \\ 1 & -1 \end{bmatrix} \\ -\frac{\Gamma(1-z)}{2\pi i} \begin{bmatrix} +e^{-i\frac{\pi}{2}z} & -e^{+i\frac{\pi}{2}z} \\ -e^{+i\frac{\pi}{2}z} & +e^{-i\frac{\pi}{2}z} \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_+^{z-1} \\ x_-^{z-1} \end{bmatrix}.$$

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