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A COMMON FIXED POINT THEOREM IN FUZZY METRIC SPACE

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

IRSHAD AALAM, SUNEEL KUMAR AND B. D. PANT

Abstract. The purpose of this paper is to obtain a common fixed point theorem for weakly compatible self-maps satisfying the property (E.A) using implicit relation on a non complete fuzzy metric space. Also, our result does not require the continuity of the maps. Our result generalizes the results of B. Singh and S. Jain [Int. J. Math. Math. Sci. 2005 (16) (2005), 2617-2629]. An example is also given to illustrate the result.

1. Introduction

There has been always a tendency in mathematics to regard the concept of Probability as one of the basic mathematical concepts. In fact, the more general (i.e., not necessarily probabilistic in nature) concept of uncertainty is considered a basic ingredient of some basic mathematical structures. In 1965, the concept of fuzzy sets was introduced by Zadeh [12], which constitutes an example where the concept of uncertainty was introduced in the theory of sets, in a non probabilistic manner. Fuzzy set theory has applications in applied sciences such as mathematical programming, modeling theory, engineering sciences, image processing, control theory, communication etc. In 1975, Kramosil and Michalek [4] introduced the concept of fuzzy metric space (briefly, FM-space), which opened an avenue for further development of analysis in such spaces. Consequently in due course of time some metric fixed point results were generalized to FM-spaces by various authors viz George and Veeramani [2], Grabiec [3], Subrahmanyam [11] and others.

In 1994, Mishra, Sharma and Singh [5] introduced the notion of compatible maps under the name of asymptotically commuting maps in FM-space. Singh and Jain [10] studied the notions of weak compatibility and semi compabilty of maps in FM-spaces. However, the study of common fixed points of non compatible maps is also of great interest. Pant [6] initiated the study of common fixed points of

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non compatible maps in metric space. In 2002, Aamri and Moutawakil [1] studied a new property for pair of maps i.e. the so-called property (E. A), which is a generalization of the concept of non compatible maps in metric space. Recently, Pant and Pant [7] studied the common fixed points of a pair of non compatible maps and the property (E. A) in FM-space.

Singh and Jain [10] proved some common fixed point theorems for four self maps on a complete fuzzy metric space satisfying an implicit relation. These results have been proved for semicompatible, compatible and weakly compatible maps and being one of the maps continuous. Since the concept of weakly compatible maps is most general among all the commutativity concepts in this field as every pair of compatible maps is weakly compatible but reverse is not always true (see [10], Ex. 2.16) and similarly, every pair of semi compatible maps is weakly compatible but reverse is not always true (see [10], Ex. 2.14). In this paper, our objective is to prove a common fixed point theorem for weakly compatible maps on a non complete fuzzy metric space satisfying an implicit relation. Also, our result does not require the continuity of the maps. Our result generalizes the results of Singh and Jain ([10], Theorem 3.1, Corollary 3.2, Theorem 3.5, Corollary 3.6).

1.1. Preliminary.

Definition 1.1. [12] *Let X be any set. A fuzzy set A in X is a function with domain X and values in* $[0, 1]$.

Definition 1.2. [9] *A binary operation* $* : [0, 1] \times [0, 1] \to [0, 1]$ *is called a continuous t -norm if* $([0, 1], *)$ *is an abelian topological monoid with the unit 1 such that* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0, 1]$.

Definition 1.3. [4] *The triplet* $(X, M, *)$ *is a FM-space if X is an arbitrary set, $*$ is a continuous t -norm and M is a fuzzy set in* $X^2 \times [0, \infty)$ *satisfying the following conditions for all* $x, y, z \in X$ *and* $t, s > 0$,
(FM1) $M(x, y, t) = 1$ *for all* $t > 0$ *iff* $x = y$;
(FM2) $M(x, y, 0) = 0$;
(FM3) $M(x, y, t) = M(y, x, t)$;
(FM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(FM5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous.*

In the following example (see [2]), we know that every metric induces a fuzzy metric:

Example 1.1. *Let (X, d) be a metric space. Define* $a * b = ab$ *(or* $a * b = min(a, b)$*) for all* $x, y \in X$ *and* $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ *is a FM-space and the fuzzy metric M induced by the metric d is often referred to as the standard fuzzy metric.*

Definition 1.4. [5] *Let A and B maps from a FM-space* $(X, M, *)$ *into itself. The maps A and B are said to be compatible (or asymptotically commuting), if for all* $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1,$$

whenever $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ *for some* $z \in X$.

Definition 1.5. [10] *Let A and B be maps from a FM-space $(X, M, *)$ into itself. The maps are said to be weakly compatible if they commute at their coincidence points, that is, $Az = Bz$ implies that $ABz = BAz$.*

Definition 1.6. [10] *Let A and B maps from a FM-space $(X, M, *)$ into itself. The maps A and B are said to be semicompatible, if for all $t > 0$,*

$$\lim_{n\to\infty} M(ABx_n, Bz, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Note that the semicompatibility of the pair (A, B), need not imply the semicompatibility of (B, A).

Definition 1.7. [7] *Let A and B be two self-maps of a FM-space $(X, M, *)$. We say that A and B satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.*

Note that weakly compatible and property (E. A) are independent to each other (see [8], Ex.2.2).

Remark 1.1. *From Definition 1.4, it is inferred that two self maps A and B on FM-space $(X, M, *)$ are noncompatible iff there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$, but for some $t > 0$ either $\lim_{n\to\infty} M(ABx_n, BAx_n, t) \neq 1$, or the limit does not exist.Therefore, any two noncompatible self-maps of $(X, M, *)$ satisfy the property (E.A) from Definition 1.7. But the following example shows that two maps satisfying the property (E.A) need not be non compatible.*

Example 1.2. *Let $X = [2, 20]$ and d be the usual metric on X. For each $t \in [0, \infty)$, define*

$$M(x, y, t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, x, y \in X. \end{cases}$$

Clearly, $(X, M, *)$ is a fuzzy metric space, where $*$ is defined by $a * b = ab$. Let A and B be self-maps of X defined as:

$$A(x) = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5; \\ 6, & \text{if } 2 < x \leq 5. \end{cases}$$

$$B(x) = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5; \\ 12, & \text{if } 2 < x \leq 5; \\ \frac{(x+1)}{3}, & \text{if } x > 5. \end{cases}$$

Let sequence $\{x_n\}$ be defined as $x_n = 5 + \frac{1}{n}, n \geq 1$, then we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 2.$$

Hence, A and B satisfy the property (E.A).
Also, $\lim_{n\to\infty} M(ABx_n, BAx_n, t) = \frac{t}{t+|2-2|} = 1$. This shows that A and B are compatible.

Lemma 1.1. [5] *If for all $x, y \in X, t > 0$ and for a number $k \in (0, 1)$;*
$M(x, y, kt) \geq M(x, y, t)$,
then $x = y$.

Throughout this paper, $(X, M, *)$ is considered to be a FM-space with condition (FM6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

2. Implicit Relation

In our result, we deal with implicit relation used in [10]. Let Φ be the set of all real continuous functions $\phi : (R^+)^4 \to R$, nondecreasing in the first argument and satisfying the following conditions:

$$\text{For } u, v \geq 0, \phi(u, v, u, v) \geq 0 \text{ or } \phi(u, v, v, u) \geq 0 \text{ implies that } u \geq v. \tag{2.1}$$

$$\phi(u, u, 1, 1) \geq 0 \text{ implies that } u \geq 1. \tag{2.2}$$

Example 2.1. *Define* $\phi(t_1, t_2, t_3, t_4) = 14t_1 - 12t_2 + 6t_3 - 8t_4$. *Then* $\phi \in \Phi$.

3. Result

Theorem 3.1. *Let* A, B, S *and* T *be self-maps of a FM-space* $(X, M, *)$ *satisfying the following conditions:*

$$A(X) \subseteq T(X), B(X) \subseteq S(X); \tag{3.1}$$

$$(A, S) \text{ and } (B, T) \text{ are weakly compatible pairs;} \tag{3.2}$$

$$(A, S) \text{ or } (B, T) \text{ satisfies the property (E.A);} \tag{3.3}$$

For some $\phi \in \Phi$, *there exist* $k \in (0, 1)$ *such that for all* $x, y \in X, t > 0$

$$\phi(M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t)) \geq 0. \tag{3.4}$$

If the range of one of the maps A, B, S *or* T *is a complete subspace of* X, *then* A, B, S *and* T *have a unique common fixed point in* X.

Proof. If the pair (B, T) satisfies the property (E.A), then there exists a sequence $\{x_n\}$ such that $Bx_n \to z$ and $Tx_n \to z$, for some $z \in X$ as $n \to \infty$. Since $B(X) \subseteq S(X)$, there exists in X a sequence $\{y_n\}$ such that $Bx_n = Sy_n$. Hence, $Sy_n \to z$ as $n \to \infty$. Now we claim that $Ay_n \to z$ as $n \to \infty$. Suppose $Ay_n \to w(\neq z) \in X$, then by (3.4), we have

$$\phi(M(Ay_n, Bx_n, kt), M(Sy_n, Tx_n, t), M(Ay_n, Sy_n, t), M(Bx_n, Tx_n, t)) \geq 0,$$

that is,

$$\phi(M(Ay_n, Bx_n, kt), M(Bx_n, Tx_n, t), M(Ay_n, Bx_n, t), M(Bx_n, Tx_n, t)) \geq 0.$$

As ϕ is nondecreasing in the first argument, we have

$$\phi(M(Ay_n, Bx_n, kt), M(Bx_n, Tx_n, t), M(Ay_n, Bx_n, t), M(Bx_n, Tx_n, t)) \geq 0.$$

Using (2.1), we get $M(Ay_n, Bx_n, kt) \geq M(Bx_n, Tx_n, t)$

Letting $n \to \infty$,

$M(w, z, t) \geq 1$ for all $t > 0$. Hence, $M(w, z, t) = 1$ Thus $w = z$. This shows that $Ay_n \to z$ as $n \to \infty$.

Suppose that $S(X)$ is a complete subspace of X. Then, $z = Su$ for some $u \in X$. Subsequently, we have $Ay_n \to Su, Bx_n \to Su, Tx_n \to Su$ and $Sy_n \to Su$ as $n \to \infty$. By (3.4), we have

$$\phi(M(Au, Bx_n, kt), M(Su, Tx_n, t), M(Au, Su, t), M(Bx_n, Tx_n, t)) \geq 0.$$

Letting $n \to \infty$,

$$\phi(M(Au, Su, kt), 1, M(Au, Su, t), 1) \geq 0.$$

As ϕ is nondecreasing in the first argument, we have

$$\phi(M(Au, Su, t), 1, M(Au, Su, t), 1) \geq 0.$$

Using (2.1), we get $M(Au, Su, t) \geq 1$ for all $t > 0$. Hence, $M(Au, Su, t) = 1$. Thus, $Au = Su$. The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$.

On the other hand, since $A(X) \subseteq T(X)$, there exists a $v \in X$ such that $Au = Tv$. We show that $Tv = Bv$. By (3.4), we have

$$\phi(M(Au, Bv, kt), M(Su, Tv, t), M(Au, Su, t), M(Bv, Tv, t)) \geq 0,$$

that is, $\phi(M(Tv, Bv, kt), 1, 1, M(Bv, Tv, t)) \geq 0$.

As ϕ is nondecreasing in the first argument, we have

$$\phi(M(Tv, Bv, t), 1, 1, M(Bv, Tv, t)) \geq 0.$$

Using (2.1), we get $M(Tv, Bv, t) \geq 1$ for all $t > 0$. Hence, $M(Tv, Bv, t) = 1$. Thus, $Bv = Tv$. This implies $Au = Su = Tv = Bv$. The weak compatibility of B and T implies that $BTv = TBv$ and $TTv = TBv = BTv = BBv$.

Let us show that Au is a common fixed point of A, B, S and T. In view of (3.4), it follows

$$\phi(M(AAu, Bv, kt), M(SAu, Tv, t), M(AAu, SAu, t), M(Bv, Tv, t)) \geq 0,$$

that is, $\phi(M(AAu, Au, kt), M(AAu, Au, t), 1, 1) \geq 0$.

As ϕ is nondecreasing in the first argument, we have

$$\phi(M(AAu, Au, t), M(AAu, Au, t), 1, 1) \geq 0.$$

Using (2.2), we get $M(AAu, Au, t) \geq 1$ for all $t > 0$. Hence, $M(AAu, Au, t) = 1$. Thus, $AAu = Au$.

Therefore, $Au = AAu = SAu$ and Au is a common fixed point of A and S. Similarly, we prove that Bv is a common fixed point of B and T. Since $Au = Bv$, we conclude that Au is a common fixed point of A, B, S and T. The proof is similar when $T(X)$ is assumed to be a complete subspace of X. The cases in which $A(X)$ or $B(X)$ is a complete subspace of X are similar to the cases in which $T(X)$ or $S(X)$ respectively, is complete since $A(X) \subseteq T(X), B(X) \subseteq S(X)$. If $Au = Bu = Tu = Su = u$ and $Av = Bv = Sv = Tv = v$, then (3.4) gives

$$\phi(M(Au, Bv, kt), M(Su, Tv, t), M(Au, Su, t), M(Bv, Tv, t)) \geq 0,$$

that is, $\phi(M(u, v, kt), M(u, v, t), 1, 1) \geq 0$.

As ϕ is nondecreasing in the first argument, we have

$$\phi(M(u, v, t), M(u, v, t), 1, 1) \geq 0.$$

Using (2.2), we get $M(u, v, t) \geq 1$ for all $t > 0$. Hence, $M(u, v, t) = 1$. Thus, $u = v$. Therefore, $u = v$ and the common fixed point is unique. □

Remark 3.1. *Our result generalizes the results of Singh and Jain* [10]*, Theorem 3.1, Corollary 3.2, Theorem 3.5, Corollary 3.6) in the sense that the concept of weakly compatible maps is most general among all the commutativity concepts. Also, our result does not require either the completeness of the whole space or continuity of the maps.*

The following example illustrates our result:

Example 3.1. *Let* $X = [2, 20)$ *with the metric* $d(x, y) = |x - y|$ *and for each* $t \in [0, \infty)$*, define*

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|}, & \textit{if } t > 0; \\ 0, & \textit{if } t = 0, x, y \in X. \end{cases}$$

Clearly, $(X, M, *)$ is a noncomplete FM-space, where $*$ is defined by $a * b = ab$. Define $A, B, S, T : X \to X$ by

$$A(x) = \begin{cases} 2, & \text{if } x = 2; \\ 3, & \text{if } 2 < x < 20. \end{cases}$$

$$B(x) = \begin{cases} 2, & \text{if } x = 2; \\ 7, & \text{if } 2 < x < 20. \end{cases}$$

$$S(x) = \begin{cases} 2, & \text{if } x = 2; \\ 6, & \text{if } 2 < x \leq 10; \\ (x-7), & \text{if } 10 < x < 20. \end{cases}$$

$$T(x) = \begin{cases} 2, & \text{if } x = 2; \\ 3, & \text{if } 2 < x \leq 10; \\ (x-3), & \text{if } 10 < x < 20. \end{cases}$$

Then A, B, S and T satisfy all the conditions of Theorem with $k \in (0, 1)$ and have a unique common fixed point $x = 2$. Clearly, (A, S) and (B, T) are weakly compatible since they commute at their coincidence points. Let sequence $\{x_n\}$ be defined as $x_n = 10 + \frac{1}{n}, n \geq 1$, then we have $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = 7$. Hence, B and T satisfy the property (E.A). Also, $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = \frac{t}{t+|3-6|} \neq 1$. and $\lim_{n\to\infty} M(BTx_n, TBx_n, t) = \frac{t}{t+|7-3|} \neq 1$. This shows that (A, S) and (B, T) are noncompatible pairs. As well as all the maps A, B, S and T are discontinuous at the common fixed point $x = 2$.

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Irshad Aalam
Govt. H. Sec. School Nauda
P. O. Kotabagh (Nainital)
Uttarakhand India- 263159
E-mail address: aalam_math@rediffmail.com

Suneel Kumar
Goverment Higher Secondary School
Sanyasiyawala P. O. Jaspur
Uttarakhand India-244712
E-mail address: ksuneel_math@rediffmail.com

B. D. Pant
Govt. Degree College Champawat
Uttarakhand India.
E-mail address: bd_pant@rediffmail.com