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# TOTALLY UMBILICAL CR-LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS

## (DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. In this paper, we consider totally umbilical CR-lightlike submanifolds of indefinite Kaehler manifolds and give a classification theorem for this.

## 1. Introduction

Cauchy Riemann (CR) submanifolds of Kaehler manifolds with Riemannian metric were introduced by Bejancu in 1978, [1, 2]. Then, totally umbilical CR-submanifolds of Kaehler manifolds were studied by Bejancu [3], Blair-Chen [4]. Duggal [5, 6], introduced Lorentzian $CR$-submanifold and claimed a fruitful mutual interplay between $CR$-submanifolds and physical spacetime geometry. Recently, Duggal-Bejancu [7, 8], introduced CR-lightlike submanifolds of indefinite Kaehler manifolds and related the study with the physically important asymptotically fiat spacetimes which further lead to the Twistor theory of Penrose and the Heaven theory of Newman. The growing importance of lightlike submanifolds in mathematical physics, in particular, their use in relativity and many more, motivated the authors to study lightlike submanifolds extensively. In this direction, present paper deals with the study of totally umbilical CR-lightlike submanifolds of indefinite Kaehler manifolds.

## 2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [8] by Duggal-Bejancu.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index $q$ such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and $(M, g)$ be an $m$-dimensional submanifold of $\bar{M}$ and $g$ the induced metric of $\bar{g}$ on $M$. If $\bar{g}$ is degenerate on the

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tangent bundle $TM$ of $M$ then $M$ is called a lightlike submanifold of $\bar{M}$. For a degenerate metric $g$ on $M$

$$TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u,v) = 0, \forall v \in T_xM, x \in M\}, \tag{2.1}$$

is a degenerate $n$-dimensional subspace of $T_x\bar{M}$. Thus, both $T_xM$ and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$RadTM : x \in M \longrightarrow RadT_xM, \tag{2.2}$$

defines a smooth distribution on $M$ of rank $r > 0$ then the submanifold $M$ of $\bar{M}$ is called $r$-lightlike submanifold and $RadTM$ is called the radical distribution on $M$.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in $TM$, that is,

$$TM = RadTM \perp S(TM), \tag{2.3}$$

$S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to $TM$ in $T\bar{M}\mid_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$ respectively. Then, we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2.4}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.5}$$

Let $u$ be a local coordinate neighborhood of $M$ and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along $M$, on $u$ as $\{\xi_1, ..., \xi_r, W_{r+1}, ..., W_n, N_1, ..., N_r, X_{r+1}, ..., X_m\}$, where $\{\xi_1, ..., \xi_r\}, \{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(RadTM\mid_u)$, $\Gamma(ltr(TM)\mid_u)$ and $\{W_{r+1}, ..., W_n\}, \{X_{r+1}, ..., X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$ and $\Gamma(S(TM)\mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have

**Theorem 2.1.** [8] *Let $(M, g, S(TM), S(TM^{\perp}))$ be an $r$-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then, there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathrm{u}}$, where* u *is a coordinate neighborhood of $M$, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \textit{for any} \quad i, j \in \{1, 2, .., r\}, \tag{2.6}$$

*where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.*

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then, according to the decomposition (2.5), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \forall X, Y \in \Gamma(TM), \tag{2.7}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \forall X \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{2.8}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here $\nabla$ is a torsion-free linear connection on $M$, $h$ is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, $A_U$ is linear a operator on $M$, known as shape operator.

According to (2.4), considering the projection morphisms $L$ and $S$ of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, (2.7) and (2.8) give

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.9}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.10}$$

where we put $h^l(X,Y) = L(h(X,Y)), h^s(X,Y) = S(h(X,Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As $h^l$ and $h^s$ are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore, we call them the lightlike second fundamental form and the screen second fundamental form on $M$. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.12}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.By using (2.4)-(2.5) and (2.9)-(2.12), we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.13}$$

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0, \tag{2.14}$$

$$\bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0, \tag{2.15}$$

for any $\xi \in \Gamma(RadTM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(ltr(TM))$.

Let $P$ be the projection morphism of $TM$ on $S(TM)$. Then using (2.3), we can induce some new geometric objects on the screen distribution $S(TM)$ on $M$, as:

$$\nabla_X PY = \nabla^*_X PY + h^*(X,Y), \tag{2.16}$$

and

$$\nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X PY, A^*_{\xi} X\}$ and $\{h^*(X,Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$ respectively. $\nabla^*$ and $\nabla^{*t}$ are linear connections on complementary distributions $S(TM)$ and $RadTM$, respectively. $h^*$ and $A^*$ are $\Gamma(RadTM)$-valued and $\Gamma(S(TM))$-valued bilinear forms and called as second fundamental forms of distributions $S(TM)$ and $RadTM$, respectively.

The screen distribution $S(TM)$ is said to be totally geodesic if $h^*(X,Y) = 0$ for any $X, Y \in \Gamma(TM)$.

From the geometry of Riemannian submanifolds and non-degenerate submanifolds, it is known that the induced connection $\nabla$ on a non-degenerate submanifold is a metric connection. Unfortunately, this is not true for a lightlike submanifold. Indeed, considering $\bar{\nabla}$ a metric connection, we have

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y), \tag{2.18}$$

for any $X, Y, Z \in \Gamma(TM)$. From [8] page 171, using the properties of linear connection, we have

$$(\nabla_X h^l)(Y,Z) = \nabla^l_X(h^l(Y,Z)) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z), \tag{2.19}$$

$$(\nabla_X h^s)(Y,Z) = \nabla^s_X(h^l(Y,Z)) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z). \tag{2.20}$$

Barros-Romero [9], defined indefinite Kaehler manifolds as

**Definition**: Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite Kaehler manifold if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, that is,

$$(\bar{\nabla}_X \bar{J})Y = 0, \quad \forall \quad X, Y \in \Gamma(T\bar{M}). \tag{2.21}$$

## 3. CR-lightlike submanifolds

**Definition**: Let $(\bar{M}, \bar{J}, \bar{g})$ be a real $2m$-dimensional indefinite Kaehler manifold and $M$ ba an $n$-dimensional submanifold of $\bar{M}$. Then, $M$ is said to be a CR-lightlike submanifold if the following two conditions are fulfilled:

(A) $\bar{J}(RadTM)$ is distribution on $M$ such that

$$RadTM \cap \bar{J}(RadTM) = 0. \tag{3.1}$$

(B) There exist vector bundles $S(TM), S(TM^{\perp}), ltr(TM), D_0$ and $D'$ over $M$ such that

$$S(TM) = \{\bar{J}(RadTM) \oplus D'\} \perp D_0; \bar{J}(D_0) = D_0; \bar{J}(D') = L_1 \perp L_2, \tag{3.2}$$

where $D_0$ is a non-degenerate distribution on $M$, $L_1$ and $L_2$ are vector sub-bundles of $ltr(TM)$ and $S(TM^{\perp})$, respectively.

Clearly, the tangent bundle of a CR-lightlike submanifold is decomposed as

$$TM = D \oplus D', \tag{3.3}$$

where

$$D = RadTM \perp \bar{J}(RadTM) \perp D_0. \tag{3.4}$$

**Theorem 3.1.** [8]*: Let $M$ be a 1-lightlike submanifold of codimension 2 of a real 2m-dimensional indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ such that $\bar{J}(RadTM)$ is a distribution on $M$. Then $M$ is a CR-lightlike submanifold.*

**Lemma 3.2.** *Let $M$ be a CR-lightlike submanifold of an indefinite Kaehler manifold and screen distribution be totally geodesic. Then, $\nabla_X Y \in \Gamma(S(TM))$ for any $X, Y \in \Gamma(S(TM))$.*

*Proof.* For any $X, Y \in \Gamma(S(TM))$, $\bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X Y, N) = g(Y, A_N X) = \bar{g}(h^*(X, Y), N)$. Hence, using hypothesis with Theorem (2.1), the lemma follows. □

**Lemma 3.3.** *Let $M$ be a CR-lightlike submanifold of an indefinite Kaehler manifold. Then, $\nabla_X \bar{J}X = \bar{J}\nabla_X X$ for any $X \in \Gamma(D_0)$.*

*Proof.* Let $X, Y \in \Gamma(D_0)$, we have

$$\begin{aligned} g(\nabla_X \bar{J}X, Y) &= \bar{g}(\bar{\nabla}_X \bar{J}X - h(X, \bar{J}X), Y) \\ &= \bar{g}(\bar{\nabla}_X \bar{J}X, Y) = \bar{g}(\bar{J}\bar{\nabla}_X X, Y) \\ &= -\bar{g}(\bar{\nabla}_X X, \bar{J}Y) = -g(\nabla_X X, \bar{J}Y) \\ &= \bar{g}(\bar{J}\nabla_X X, Y), \end{aligned}$$

that is $g(\nabla_X \bar{J}X - \bar{J}\nabla_X X, Y) = 0$ then, non-degeneracy of $D_0$ implies the result. □

## 4. Totally Umbilical CR-lightlike submanifolds

**Definition:** A lightlike submanifold $(M, g)$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on $M$, called the transversal curvature vector field of $M$, such that for all $X, Y \in \Gamma(TM)$,

$$h(X, Y) = H\bar{g}(X, Y). \tag{4.1}$$

Using (2.9), it is easy to see that $M$ is a totally umbilical, if and only if, on each coordinate neighborhood $u$, there exists smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$, such that $h^l(X,Y) = H^l \bar{g}(X,Y), h^s(X,Y) = H^s \bar{g}(X,Y)$.

**Theorem 4.1.** *Let $M$ be a totally umbilical CR-lightlike submanifold of an indefinite Kaehler manifold and screen distribution be totally geodesic.. Then, the screen transversal curvature vector field $H^s$ of $M$ belongs to $L_2^{\perp}$.*

*Proof.* Let $W \in \Gamma(L_2)$ and $X \in \Gamma(D_0)$ then for a totally umbilical $CR$-lightlike submanifold, with lemmas (3.2) and (3.3), we have $\bar{g}(\bar{J}\bar{\nabla}_X X, \bar{J}W) = \bar{g}(\bar{\nabla}_X \bar{J}X, \bar{J}W) = \bar{g}(\nabla_X \bar{J}X, \bar{J}W) + \bar{g}(h^s(X, \bar{J}X), \bar{J}W) = \bar{g}(\bar{J}\nabla_X X, \bar{J}W) = \bar{g}(\nabla_X X, W) = 0$ and $\bar{g}(\bar{J}\bar{\nabla}_X X, \bar{J}W) = \bar{g}(\bar{\nabla}_X X, W) = \bar{g}(\nabla_X X, W) + \bar{g}(h^s(X,X), W) = g(X,X)\bar{g}(H^s, W)$. Hence, we have $g(X,X)\bar{g}(H^s, W) = 0$. Since $D_0$ is non-degenerate therefore $\bar{g}(H^s, W) = 0$, this implies that $H^s \in \Gamma(L_2^{\perp})$. □

**Theorem 4.2.** *Let $M$ be a totally umbilical CR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ and screen distribution be totally geodesic. Then, $A_{\bar{J}Z}W = A_{\bar{J}W}Z, \forall \quad W, Z \in \Gamma(D')$.*

*Proof.* Since, $\bar{M}$ be an indefinite Kaehler manifold, therefore, $\bar{J}\bar{\nabla}_Z W = \bar{\nabla}_Z \bar{J}W$, this gives

$$\bar{J}\nabla_Z W + \bar{J}h(Z,W) = -A_{\bar{J}W}Z + \nabla^t_Z \bar{J}W. \tag{4.2}$$

Interchange the role of $Z$ and $W$, in the above equation and subtract the resulting equation from it, we get

$$A_{\bar{J}Z}W - A_{\bar{J}W}Z = \bar{J}(\nabla_Z W - \nabla_W Z) + \nabla^t_W \bar{J}Z - \nabla^t_Z \bar{J}W. \tag{4.3}$$

Taking inner product with $X \in \Gamma(D_0)$, then we have

$$\bar{g}(A_{\bar{J}Z}W - A_{\bar{J}W}Z, X) = \bar{g}(\nabla_W Z, \bar{J}X) - \bar{g}(\nabla_Z W, \bar{J}X). \tag{4.4}$$

Now, $\bar{g}(\nabla_W Z, \bar{J}X) = \bar{g}(\bar{\nabla}_W Z, \bar{J}X) = \bar{g}(\bar{J}Z, \bar{\nabla}_W X) = \bar{g}(\bar{J}Z, \nabla_W X) + \bar{g}(\bar{J}Z, h^s(W,X)) + \bar{g}(\bar{J}Z, h^l(W,X))$. Since $M$ is a totally umbilical $CR$-lightlike submanifold, therefore for any $W \in \Gamma(D')$ and $X \in \Gamma(D_0)$, we have $h^s(W,X) = H^s \bar{g}(W,X) = 0, h^l(W,X) = H^l \bar{g}(W,X) = 0$. Thus, $\bar{g}(\nabla_W Z, \bar{J}X) = \bar{g}(\bar{J}Z, \nabla_W X)$, using Lemma (3.2), this gives $\bar{g}(\nabla_W Z, \bar{J}X) = 0$. Similarly, $\bar{g}(\nabla_Z W, \bar{J}X) = 0$, so (4.4), implies $\bar{g}(A_{\bar{J}Z}W - A_{\bar{J}W}Z, X) = 0$. Then, non-degeneracy of $D_0$, implies that $A_{\bar{J}Z}W = A_{\bar{J}W}Z$. □

**Theorem 4.3.** *([8]) Let $M$ be a $CR$-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the totally real distribution $D'$ is integrable, if and only if, the shape operator of $M$ satisties $A_{\bar{J}Z}W = A_{\bar{J}W}Z, \forall \quad W, Z \in \Gamma(D')$.*

Thus we have the following

**Corollary 4.4.** *Let $M$ be a totally umbilical CR-lightlike submanifold of an indefinite Kaehler manifold then, the totally real distribution $D'$ is integrable.*

**Definition:** For a CR-lightlike submanifold $M$, a plane $X \wedge Z$, with $X \in \Gamma(D_0)$ and $Z \in \Gamma(D')$ is called a CR-lightlike section. The sectional curvature $\bar{K}(\pi)$ of a CR-lightlike section $\pi$ is called CR-lightlike sectional curvature.

**Theorem 4.5.** *Let $M$ be a totally umbilical CR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then, the CR-lightlike sectional curvature of $M$ vanishes, that is, $\bar{K}(\pi) = 0$ for all CR lightlike sections $\pi$.*

*Proof.* Since, $M$ is a totally umbilical CR-lightlike submanifold of $\bar{M}$ then, (2.19) and (2.20) implies

$$(\nabla_X h^l)(Y,Z) = g(Y,Z)\nabla^l_X H^l - H^l\{(\nabla_X g)(Y,Z)\}, \tag{4.5}$$

$$(\nabla_X h^s)(Y,Z) = g(Y,Z)\nabla^s_X H^s - H^s\{(\nabla_X g)(Y,Z)\}. \tag{4.6}$$

For a $CR$-lightlike section $\pi = X \wedge Z, X \in \Gamma(D_0), Z \in \Gamma(D')$, (2.18) implies that $(\nabla_X g)(Y,Z) = 0$. Therefore, (4.5)-(4.6) give

$$(\nabla_X h^l)(Y,Z) = g(Y,Z)\nabla^l_X H^l, \tag{4.7}$$

$$(\nabla_X h^s)(Y,Z) = g(Y,Z)\nabla^s_X H^s. \tag{4.8}$$

Using (3.9) at page 171 of [8] with above equations, we have

$$\begin{aligned}\{\bar{R}(X,Y)Z\}^{tr} \;=\;& g(Y,Z)\nabla^l_X H^l - g(X,Z)\nabla^l_Y H^l + g(Y,Z)D^l(X,H^s)\\ &-g(X,Z)D^l(Y,H^s) + g(Y,Z)\nabla^s_X H^s - g(X,Z)\nabla^s_Y H^s\\ &+g(Y,Z)D^s(X,H^l) - g(X,Z)D^s(Y,H^l). \end{aligned}\tag{4.9}$$

For any $V \in \Gamma(tr(TM))$, above equation gives

$$\begin{aligned}\bar{R}(X,Y,Z,V) \;=\;& g(Y,Z)\bar{g}(\nabla^l_X H^l, V) - g(X,Z)\bar{g}(\nabla^l_Y H^l, V)\\ &+g(Y,Z)\bar{g}(D^l(X,H^s),V) - g(X,Z)\bar{g}(D^l(Y,H^s),V)\\ &+g(Y,Z)\bar{g}(\nabla^s_X H^s, V) - g(X,Z)\bar{g}(\nabla^s_Y H^s, V)\\ &+g(Y,Z)\bar{g}(D^s(X,H^l),V) - g(X,Z)\bar{g}(D^s(Y,H^l),V)\end{aligned}\tag{4.10}$$

then, in particular, for any unit vectors $X \in \Gamma(D)$ and $Z \in \Gamma(D')$, we have

$$\bar{R}(X,Z,\bar{J}X,\bar{J}Z) = \bar{R}(X,Z,X,Z) = 0. \tag{4.11}$$

Hence the result. □

**Lemma 4.6.** *Let $M$ be a totally umbilical CR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ and screen distribution be totally geodesic. Then, $A_W X = \bar{g}(H^s, W)X$ and $A_N X = \bar{g}(H^l, N)X$ for any $X \in \Gamma(TM)$.*

*Proof.* Let $Y \in \Gamma(D_0)$ then, for totally umbilical CR-lightlike submanifold, from (2.13), we have $g(X,Y)\bar{g}(H^s,W) = g(A_W X, Y)$, then non-degeneracy of $D_0$, gives the result. Since, $\bar{\nabla}$ is a metric connection on $\bar{M}$, then $(\bar{\nabla}_X \bar{g})(Y,N) = 0$, further this gives, $\bar{g}(\nabla_X Y, N) + \bar{g}(h^l(X,Y),N) = g(A_N X, Y)$. Then, by using Lemma (3.2), we get $\bar{g}(h^l(X,Y),N) = g(A_N X, Y)$, then using umbilicity of $M$, we get $g(X,Y)\bar{g}(H^l,N) = g(A_N X, Y)$, and again using non-degeneracy of $D_0$, second result follows.

From above lemma it is clear that $\bar{g}(H,V)X = A_V X, \forall \quad V \in \Gamma(tr(TM)), X \in \Gamma(D_0)$. □

**Lemma 4.7.** *Let $M$ be a totally umbilical CR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then, $\nabla^t$ is a metric linear connection on $tr(TM)$.*

*Proof.* By lemma (4.6), we have

$$A_W X = \bar{g}(H^s, W)X, \tag{4.12}$$

for any $W \in \Gamma(S(TM^\perp))$ and $X \in \Gamma(D_0)$. Now, taking into account that $\bar{\nabla}$ is a metric connection and using (2.8), we have

$$\bar{g}(A_W X, N) = \bar{g}(W, \nabla^t_X N), \tag{4.13}$$

where $W \in S(TM^{\perp}), X \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$. Since $X \in \Gamma(D_0)$ then, (4.12) and (4.13), give $\bar{g}(W, \nabla^t_X N) = \bar{g}(H^s, W)\bar{g}(X, N) = 0$. Hence, the lightlike transversal vector bundle $ltr(TM)$ is parallel with respect to $\nabla^t$, then with Theorem (2.3) at page 159 of [8], $\nabla^t$ is a metric connection on $tr(TM)$. □

**Theorem 4.8.** *Let $M$ be a totally umbilical CR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ and screen distribution be totally geodesic. If dimension$D' > 1$, then $M$ is a CR lightlike product or a totally real submanifold.*

*Proof.* Since, dimension $D' > 1$, so choose two non-parallel vectors $X, Y \in \Gamma(D')$ such that $\bar{J}\bar{\nabla}_X Y = \bar{\nabla}_X \bar{J}Y$. Then, lemma (4.6) gives $\bar{J}\nabla_X Y + \bar{g}(X, Y)\bar{J}H = -\bar{g}(\bar{J}Y, H)X + \nabla^t_X \bar{J}Y$. Taking inner product with $X$ and using lemma (3.2), we have

$$\bar{g}(H, \bar{J}Y)\|X\|^2 = g(X, Y)\bar{g}(H, \bar{J}X). \tag{4.14}$$

Interchanging the role of $X$ and $Y$ in above equation, we get

$$\bar{g}(H, \bar{J}X)\|Y\|^2 = g(X, Y)\bar{g}(H, \bar{J}Y). \tag{4.15}$$

Using (4.14), in above equation, we get

$$\bar{g}(H, \bar{J}Y) = g(X, Y)^2/(\|X\|^2\|Y\|^2)\bar{g}(H, \bar{J}Y). \tag{4.16}$$

The possible solutions of (4.16) are (a) $H = 0$ or (b) $H \perp \bar{J}Y$. Thus, we have (a) $M$ is totally geodesic or (b) $H \in (\bar{J}D')^{\perp}$. Combining (a) with lemma (4.6) and Theorem (3.5), at page 209 of [8], we have first part of the theorem.

Next, suppose that $H \neq 0$ and $H \in (\bar{J}D')^{\perp}$. We observe that for $N \in \Gamma(\bar{J}D')$ and $Z \in \Gamma(D_0)$, $\bar{\nabla}_Z \bar{J}N = \bar{J}\bar{\nabla}_Z N$ gives $\nabla_Z \bar{J}N = \bar{J}\nabla^t_Z N$. This implies that for $N \in \Gamma(\bar{J}D')$ and $Z \in \Gamma(D_0)$, $\nabla^t_Z N \in \bar{J}D'$, by Lemma (3.2). Also, $\bar{g}(N, H) = 0$, then by lemma (4.7), we have $\bar{g}(\nabla^t_Z N, H) = -\bar{g}(N, \nabla^t_Z H)$. This together with $\nabla^t_Z N \in \Gamma(\bar{J}D')$ gives $\bar{g}(N, \nabla^t_Z H) = 0$. Hence, $\nabla^t_Z H \in (\bar{J}D')^{\perp}$. Now, for $Z \in \Gamma(D_0)$, we have $\bar{\nabla}_Z \bar{J}H = \bar{J}\bar{\nabla}_Z H$, this gives

$$-\bar{g}(H^s, H)\bar{J}Z - \bar{g}(H^l, H)\bar{J}Z + \bar{J}\nabla^t_Z H = \nabla^t_Z \bar{J}H,$$

or

$$-\bar{g}(H, H)\bar{J}Z + \bar{J}\nabla^t_Z H = \nabla^t_Z \bar{J}H.$$

By virtue of lemma (4.7), it follows that $\bar{J}Z = 0$ for any $Z \in D_0$. Hence, $D_0 = \{0\}$, this proves the second part. □

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