BULLETIN OF MATHEMATICAL ANALYSIS AND APPLICATIONS
ISSN: 1821-1291, URL: HTTP://WWW.BMATHAA.ORG
VOLUME 16 ISSUE 4(2024), PAGES 130-142
HTTPS://DOI.ORG/10.54671/BMAA-2024-4-9

ON THE ELEMENTARY SOLUTION OF THE OPERATOR $\otimes^k$ AND THE FOURIER TRANSFORM OF THEIR CONVOLUTION

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ABSTRACT. In this paper, the operator $\otimes^k$ is introduced of the partial differential operator related to the diamond operator iterated k times and is defined by

$$\otimes^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \frac{m^2}{2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} - \frac{m^2}{2}\right)^2\right]^k,$$

where $p + q = n$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, k is a non-negative integer, m is a non-negative real number and n is the dimension of $\mathbb{R}^n$. In this work we study the elementary solution of the partial differential operator related to the diamond operator. Then, we study the Fourier transform of the elementary solution and also the Fourier transform of their convolution.

1. INTRODUCTION

The operator $\lozenge^k$ has been first introduced by Kananthai [4], is named as the diamond operator iterated k-times, and is defined by

$$\lozenge^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k, \quad p + q = n, \tag{1.1}$$

where n is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and k is a non-negative integer. The operator $\lozenge^k$ can be expressed in the form $\lozenge^k = \Box^k \triangle^k = \triangle^k \Box^k$, the operator $\triangle^k$ is the Laplace operator iterated k-times, which is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k \tag{1.2}$$

and the operator $\Box^k$ is the ultra-hyperbolic operator iterated k-times, which is defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 46F10.
Key words and phrases. fundamental solution; Fourier transform; diamond operator.

Submitted May 9, 2024. Accepted November 21, 2024. Published December 16, 2024.
Communicated by P.Wang.

By putting $p = 1$ and $x_1 = t$ (time) in (1.3), then we obtain the wave operator

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}. \tag{1.4}$$

In 1997, Kananthai [4] has showed that the convolution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is the elementary solution of the operator $\Diamond^k$, that is

$$\Diamond^k((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) = \delta, \tag{1.5}$$

where the function $R^H_{2k}(x)$ is defined by (2.11) and $R^e_{2k}(x)$ is defined by (2.10). The elementary solution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is called the *diamond kernel of Marcel Riesz*. Moreover, Kananthai, Suantai and Longani [6] have studied the elementary solution of the operator $\oplus^k$ and the weak solution of the equation $\oplus^k u(x) = f(x)$, where the operator $\oplus^k$ is defined by

$$\begin{aligned}
\oplus^k &= \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^4\right]^k \\
&= \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\
&= \diamond^k L_1^k L_2^k \\
&= \diamond^k L^k
\end{aligned} \tag{1.6}$$

where k is a non-negative integer, and $f(x)$ is a generalized function.

Next, Kananthai, Suantai and Longani [5] have studied the relationship between the operator $\oplus^k$ and the wave operator, and the relationship between the operator $\oplus^k$ and the Laplace operator. Moreover, the equation $\oplus^k K(x) = \delta$,

$$K(x) = [R^H_{2k}(x) * (-1)^k R^e_{2k}(x)] * S_{2k}(x) * T_{2k}(x)$$

is the elementary solution of the operator $\oplus^k$. Later, Kananthai [2] has studied the inversion of the kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator $\oplus^k$.

In 1988, Trione [18] has studied the elementary solution of the ultra-hyperbolic Klein–Gordon operator iterated k-times, which is defined by

$$(\Box + m^2)^k = \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2\right]^k. \tag{1.7}$$

Bupasiri [16] has studied the partial differential operator $\otimes^k$, iterated k-times is defined by

$$\otimes^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \frac{m^2}{2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} - \frac{m^2}{2}\right)^2\right], p + q = n, \tag{1.8}$$

$\otimes^k((-1)^k R^e_{2k}(x) * W_{2k}(x, m)) = \delta$, where $\otimes^k = \triangle^k(\Box + m^2)^k$, m is a non-negative real number. Later, Lunnaree and Nonlaopon [10] introduced the operator $(\Diamond +$

$m^2)^k$, that is named as the diamond Klein-Gordon operator iterated k-times, which is defined by

$$(\diamondsuit + m^2)^k = \left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 + m^2 \right)^k, \tag{1.9}$$

where $p + q = n$ is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, m is a non-negative real number and k is a non-negative integer, see [8, 9, 12, 13] for more details. Moreover, Kananthai [3] has studied the elementary solution for the $(\diamondsuit + m^4)^k$, which related to the Klein-Gordon operator.

In this paper, we study the elementary solution of the equation of the form

$$\otimes^k u(x) = \sum_{r=0}^{t} c_r \otimes^r \delta.$$

After that, we study the Fourier transform of the operator $\otimes^k$.

2. Preliminary Notes

Definition 2.1. [11] *Let $L(D)$ be a differential operator with constant coefficients. We say that a distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is the elementary solution of the differential operator $L(D)$ if E satisfies $L(D)E = \delta$ in $\mathcal{D}'(\mathbb{R}^n)$.*

Definition 2.2. *Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional space $\mathbb{R}^n$,*

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.1}$$

where $p + q = n$.

Define $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0$ and $u > 0\}$, which designates the interior of the forward cone and $\overline{\Gamma}_+$ designates its closure and the following functions introduce by Nozaki [21, p. 72], that

$$R_\alpha^H(x) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+; \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \tag{2.2}$$

is called the ultra-hyperbolic kernel of Marcel Riesz. Here, α is a complex parameter and n the dimension of the space. The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)} \tag{2.3}$$

and p is the number of positive terms of

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad p + q = n$$

and let supp $R_\alpha^H(x) \subset \overline{\Gamma}_+$. Now, $R_\alpha^H(x)$ is an ordinary function if Re $\alpha \geq n$ and is a distribution of α if Re $\alpha < n$. Now, if $p = 1$ then (2.2) reduces to the function $M_\alpha(u)$, and is defined by

$$M_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{if } x \in \Gamma_+; \\ 0, & \text{if } x \notin \Gamma_+, \end{cases} \tag{2.4}$$

where $u = x_1^2 - x_2^2 - \cdots - x_n^2$ and $H_n(\alpha) = \pi^{\frac{(n-1)}{2}} 2^{\alpha-1} \Gamma(\frac{\alpha-n+2}{2})$. The function $M_\alpha(u)$ is called the hyperbolic kernel of Marcel Riesz.

Definition 2.3. *Let* $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ *and*

$$v = x_1^2 + x_2^2 + \cdots + x_p^2 + x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2, \qquad p + q = n. \tag{2.5}$$

For any complex number β *, we define the function*

$$R_\beta^e(x) = 2^{-\beta}\pi^{-n/2}\Gamma\left(\frac{n-\beta}{2}\right)\frac{v^{(\beta-n)/2}}{\Gamma(\beta/2)}. \tag{2.6}$$

The function $R_\beta^e(x)$ *is called the elliptic kernel of Marcel Riesz. It is an ordinary function if Re* $\beta \geq n$ *and a distribution of* β *if Re* $\beta < n$.

Definition 2.4. *Let* $f(x) \in L_1(\mathbb{R}^n)$ *(the space of integrable function in* $\mathbb{R}^n$*). The Fourier transform of* $f(x)$ *is defined as*

$$\widehat{f(\xi)} = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx, \tag{2.7}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n), x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \xi\cdot x = (\xi_1 x_1, \xi_2 x_2, \dots, \xi_n x_n)$ *is the usual inner product in* $\mathbb{R}^n$ *and* $dx = dx_1 dx_2 \dots dx_n$. *The inverse of the Fourier transform is defined by*

$$f(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi\cdot x}\widehat{f(\xi)}d\xi. \tag{2.8}$$

If f *is a distribution with compact supports, by* [1, Theorem 7.4-3], *Equation* (2.8) *can be written as*

$$\widehat{f}(\xi) = \mathcal{F}f(x) = \frac{1}{(2\pi)^{n/2}}\left\langle f(x), e^{-i\xi\cdot x}\right\rangle. \tag{2.9}$$

Lemma 2.5. [4] *Given the equation* $\triangle^k u(x) = \delta$ *for* $x \in \mathbb{R}^n$, *where* $\triangle^k$ *is the Laplace operator iterated k-times, which is defined by* (1.2). *Then* $u(x) = (-1)^k R_{2k}^e(x)$ *is the elementary solution of the operator* $\triangle^k$, *where*

$$R_{2k}^e(x) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)}|x|^{2k-n}. \tag{2.10}$$

Lemma 2.6. [17] *If* $\Box^k u(x) = \delta$ *for* $x \in \Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$, *where* $\Box^k$*is the ultra-hyperbolic operator iterated k-times, which is defined by* (1.3). *Then* $u(x) = R_{2k}^H(x)$ *is the unique elementary solution of the operator* $\Box^k$, *where*

$$R_{2k}^H(x) = \frac{u^{\left(\frac{2k-n}{2}\right)}}{K_n(2k)} = \frac{(x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\left(\frac{2k-n}{2}\right)}}{K_n(2k)} \tag{2.11}$$

and

$$K_n(2k) = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2+2k-n}{2}\right)\Gamma\left(\frac{1-2k}{2}\right)\Gamma(2k)}{\Gamma\left(\frac{2+2k-p}{2}\right)\Gamma\left(\frac{p-2k}{2}\right)}. \tag{2.12}$$

Lemma 2.7. [4] *Given the equation* $\diamondsuit^k u(x) = \delta$ *for* $x \in \mathbb{R}^n$, *then* $u(x) = (-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ *is the unique elementary solution of the operator* $\diamondsuit^k$, *where* $\diamondsuit^k$ *is the diamond operator iterated k-times, which is defined by* (1.1), $R_{2k}^e(x)$ *and* $R_{2k}^H(x)$ *are defined by* (2.10) *and* (2.11), *respectively. Moreover,* $(-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ *is a tempered distribution.*

It is not difficult to show that $R_{-2k}^e(x) * R_{-2k}^H(x) = (-1)^k \diamondsuit^k \delta$, for k is a non-negative integer.

Definition 2.8. *Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$, the function $W_\alpha(x, m)$ is defined by*

$$W_\alpha(x,m) = \sum_{r=0}^{\infty} \binom{-\alpha/2}{r} (m^2)^r R^H_{\alpha+2r}(x), \tag{2.13}$$

where α is a complex parameter, m is a non-negative real number, $R^H_{\alpha+2r}(x)$ is defined by (2.11).

From the definition of $W_\alpha(x,m)$ and by putting $\alpha = -2k$, we have

$$W_{-2k}(x,m) = \sum_{r=0}^{\infty} \binom{k}{r} (m^2)^r R^H_{2(-k+r)}(x).$$

Since the operator $(\Box + m^2)^k$ defined in (1.9) is a linearly continuous and has $1-1$ mapping, then it has inverse. From Lemma 2.7, we obtain

$$\begin{aligned} W_{-2k}(x,m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \Box^{-k-r}\delta \\ &= (\Box + m^2)^k \delta. \end{aligned} \tag{2.14}$$

By putting $k = 0$ in (2.14), we have $W_0(x,m) = \delta$. By putting $\alpha = 2k$ into (2.13), we have

$$\begin{aligned} W_{2k}(x,m) = &\binom{-k}{0} (m^2)^0 R^H_{2k+0}(x) \\ &+ \sum_{r=1}^{\infty} \binom{-k}{r} (m^2)^r R^H_{2k+2r}(x). \end{aligned} \tag{2.15}$$

The second summand of the right-hand member of (2.15) vanishes for $m = 0$ and then, we have

$$W_{2k}(x, m = 0) = R^H_{2k}(x) \tag{2.16}$$

is the elementary solution of the ultra-hyperbolic operator $\Box^k$, iterated k-times.

Lemma 2.9. *The function $R^H_{-2k}(x)$ and $(-1)^k R^e_{-2k}(x)$ are the inverse in the convolution algebra of $R^H_{2k}(x)$ and $(-1)^k R^e_{2k}(x)$, respectively. That is,*

$$R^H_{-2k}(x) * R^H_{2k}(x) = R^H_{-2k+2k}(x) = R^H_0(x) = \delta$$

and

$$(-1)^k R^e_{-2k}(x) * (-1)^k R^e_{2k}(x) = (-1)^{2k} R^e_{-2k+2k}(x) = R^e_0(x) = \delta.$$

For the proof of the this Lemma is given in [19, 17].

Lemma 2.10. [20]*(Convolution of $R^e_\alpha(x)$ and $R^H_\alpha(x)$). If $R^e_\alpha(x)$ and $R^H_\alpha(x)$ are defined by* (2.10) *and* (2.11)*, respectively, then*

(i) $R^e_\alpha(x) * R^e_\beta(x) = R^e_{\alpha+\beta}(x)$*, where α and β are complex parameters;*
(ii) $R^H_\alpha(x) * R^H_\beta(x) = R^H_{\alpha+\beta}(x)$*, where α and β are both integers and except only the case both α and β are both integers.*

Lemma 2.11. *The function $W_{2k}(x,m)$ is the elementary solution of the operator $(\Box + m^2)^k$ where $(\Box + m^2)^k$ is the operator iterated k-times defined by (1.7) $W_{2k}(x,m)$ defined by (2.15).*

The proof of this Lemma is given in ([14], p 110).

Lemma 2.12. *The function $W_{-2k}(x,m)$ and $(-1)^k R^e_{-2k}(x)$ are the inverses in the convolution algebras of $W_{2k}(x,m)$ and $(-1)^k R^e_{2k}(x)$ respectively.*

Proof. We need to show that

$$W_{-2k}(x,m) * W_{2k}(x,m) = W_{-2k+2k}(x,m) = W_0(u,m) = \delta$$

and

$$(-1)^k R^e_{-2k}(x) * (-1)^k R^e_{2k}(x) = R^e_{-2k+2k}(x) = R^e_0(x) = \delta.$$

To prove these, see ([14],p 110), ([15],p 123) and ([19], p 118, p 158). □

Lemma 2.13. [16] *Given the equation*

$$\otimes^k K(x) = \triangle^k(\Box + m^2)^k K(x) = \delta, \tag{2.17}$$

where $\otimes^k$ is the operator iterated k-times, δ is the Dirac delta distribution, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and k is a non-negative integer. Then we obtain

$$K(x) = (-1)^k R^e_{2k}(x) * W_{2k}(x,m) \tag{2.18}$$

is an elementary solution of the equation (2.17), where $R^e_{2k}(x)$ and $W_{2k}(x,m)$ are defined by (2.6) and (2.15) respectively with $\beta = 2k$. Moreover, from (2.18)

$$(-1)^k R^e_{-2k}(x) * K(x) = W_{2k}(x,m) \tag{2.19}$$

as an elementary solution of the operator $(\Box + m^2)^k$ and in particular from (2.18) and (2.19) with $p = 1, q = n-1, k = 1, x_1 = t$ and $m = 0$, we obtain

$$(-1)^k R^e_{-2}(x) * K(x) = M_2(u) \tag{2.20}$$

as an elementary solution of the wave operator defined by (1.4) where $M_2(u)$ is defined by (2.4) with $\alpha = 2$. Also, for $q = 0$ and $m = 0$ then (2.17) become

$$\triangle_p^{2k} K(x) = \delta \tag{2.21}$$

and by (2.18) we obtain

$$K(x) = (-1)^k R^e_{2k}(x) * (-1)^k R^e_{2k}(x) = (-1)^{2k} R^e_{4k}(x) = R^e_{4k}(x) \tag{2.22}$$

is an elementary solution of (2.21) where $\triangle_p^{2k}$ is the Laplacian of p-dimension, iterated $2k$-times and $v = x_1^2 + x_2^2 + \cdots + x_p^2$.

Lemma 2.14. [16] *(The convolution of tempered distribution) $R^e_{2k}(x) * W_{2k}(x,m)$ exits and is a tempered distribution*

Lemma 2.15. *(The Fourier transform of $\otimes^k\delta$.) Let*

$$||\xi|| = \left(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2\right)^{1/2}$$

for $\xi \in \mathbb{R}^n$. Then

$$\left|\mathcal{F} \otimes^k \delta\right| = \left|\mathcal{F}\left(\triangle(\Box + m^2)\right)^k \delta\right| \leq \frac{1}{(2\pi)^{n/2}}(||\xi||^2 + m^2)^k ||\xi||^{2k}.$$

That is, $\mathcal{F} \otimes^k \delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Moreover, by the inverse Fourier transformation

$$\otimes^k \delta = \mathcal{F}^{-1} \frac{1}{(2\pi)^{n/2}} \left[\left((\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2) + \frac{m^2}{2}\right)^2 - \left((\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2) - \frac{m^2}{2}\right)^2\right]^k.$$

Proof. From the Fourier transform (2.7), we have

$$\mathcal{F}\left(\triangle(\square+m^2)\right)^k\delta=\frac{1}{(2\pi)^{n/2}}\left\langle\delta,\triangle^k(\square+m^2)^k e^{-i\xi\cdot x}\right\rangle$$

$$=\frac{1}{(2\pi)^{n/2}}\left\langle\delta,(\triangle^k\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2-\xi_{p+1}^2-\xi_{p+2}^2-\cdots-\xi_{p+q}^2+m^2\right)^k e^{-i\xi\cdot x}\right\rangle$$

$$=\frac{1}{(2\pi)^{n/2}}\langle\delta,(\xi_1^2+\xi_2^2+\cdots+\xi_p^2+\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)^k$$

$$\cdot\left(\xi_1^2+\xi_2^2+\cdots+\xi_p^2-\xi_{p+1}^2-\xi_{p+2}^2-\cdots-\xi_{p+q}^2+m^2\right)^k e^{-i\xi\cdot x}\Big\rangle$$

$$=\frac{1}{(2\pi)^{n/2}}\left\langle\delta,\left[\left(\sum_{i=1}^{p}\xi_i^2\right)+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)\right]^k\left[\left(\sum_{i=1}^{p}\xi_i^2\right)-\left(\sum_{j=p+1}^{p+q}\xi_j\right)+m^2\right]^k e^{-i\xi\cdot x}\right\rangle$$

$$=\frac{1}{(2\pi)^{n/2}}\left\langle\delta,\left[\left(\left(\sum_{i=1}^{p}\xi_i^2\right)+\frac{m^2}{2}\right)^2-\left(\left(\sum_{j=p+1}^{p+q}\xi_j\right)-\frac{m^2}{2}\right)^2\right]^k e^{-i\xi\cdot x}\right\rangle$$

$$=\frac{1}{(2\pi)^{n/2}}\left[\left(\left(\sum_{i=1}^{p}\xi_i^2\right)+\frac{m^2}{2}\right)^2-\left(\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)-\frac{m^2}{2}\right)^2\right]^k$$

$$=\frac{1}{(2\pi)^{n/2}}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+\frac{m^2}{2}\right)^2-\left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)-\frac{m^2}{2}\right)^2\right]^k.$$

Next, we consider the boundedness of $\mathcal{F}\otimes^k\delta$. Since

$$\otimes^k=\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+\frac{m^2}{2}\right)^2-\left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)-\frac{m^2}{2}\right)^2\right]^k$$

$$=\Big[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)-(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)+m^2\right)^k$$

$$\times\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)\right)^k\Big]$$

$$=\left[\left[(\xi_1^2+\xi_2^2+\cdots+\xi_{p+q}^2)\left(\xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2-\cdots-\xi_{p+q}^2+m^2\right)\right]^k\right],n=p+q$$

Thus

$$\mathcal{F}\otimes^k\delta=\frac{1}{(2\pi)^{n/2}}\left[(\xi_1^2+\xi_2^2+\cdots+\xi_{p+q}^2)(\xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2-\cdots-\xi_{p+q}^2+m^2)\right]^k,$$

$$\left|\mathcal{F}\otimes^k\delta\right|=\frac{1}{(2\pi)^{n/2}}\left(\left|\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right|\left|\xi_1^2+\cdots+\xi_p^2-\xi_{p+1}^2-\cdots-\xi_n^2\right|+m^2\right)^k$$

$$\leq\frac{1}{(2\pi)^{n/2}}\left(\left|\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right|+m^2\right)^k\left|\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right|^k$$

$$=\frac{1}{(2\pi)^{n/2}}(||\xi||^2+m^2)^k||\xi||^{2k},\ \ p+q=n$$

where $||\xi||=\left(\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right)^{1/2}$, $\xi_i(i=1,2,\ldots,n)\in\mathbb{R}$. Hence, we obtain $\mathcal{F}\otimes^k\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Since $\mathcal{F}$ is $1-1$ transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, then by (2.8), we have

$$\otimes^k \delta = \frac{1}{(2\pi)^{n/2}}\mathcal{F}^{-1}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+\frac{m^2}{2}\right)^2-\left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)-\frac{m^2}{2}\right)^2\right]^k.$$

□

3. Main Results

We now come to the proofs of our main result.

Theorem 3.1. *For $0<r<k$,*

$$\begin{aligned}&\otimes^k\left(W_{2k}(x,m)*(-1)^kR_{2k}^e(x)\right)\\&=W_{2(k-r)}(x,m)*\left((-1)^{(k-r)}R_{2(k-r)}^e(x)\right)\end{aligned}\tag{3.1}$$

and for $k\le t$,

$$\otimes^t\left(W_{2k}(x,m)*(-1)^kR_{2k}^e(x)\right)=\otimes^{t-k}\delta.\tag{3.2}$$

Proof. For $0<r<k$, by Lemma 2.13,

$$\otimes^k\left(W_{2k}(x,m)*(-1)^kR_{2k}^e(x)\right)=\delta.$$

Thus,

$$\otimes^{k-r}\otimes^r\left(W_{2k}(x,m)*(-1)^kR_{2k}^e(x)\right)=\delta$$

or

$$\otimes^{k-r}\delta*\otimes^r\left(W_{2k}(x,m)*(-1)^kR_{2k}^e(x)\right)=\delta.$$

Convolving both sides by $W_{2(k-r)}(x,m)*(-1)^{(k-r)}R_{2(k-r)}^e(x)$, we obtain

$$\begin{aligned}&\otimes^{k-r}\left(W_{2(k-r)}(x,m)*(-1)^{(k-r)}R_{2(k-r)}^e(x)\right)\\&*\otimes^r\left(W_{2k}(x,m)*(-1)^kR_{2k}^e(x)\right)\\&=W_{2(k-r)}(x,m)*(-1)^{(k-r)}R_{2(k-r)}^e(x)*\delta.\end{aligned}$$

By Lemma 2.13,

$$\begin{aligned}&\delta*\otimes^r\left(W_{2k}(x,m)*(-1)^kR_{2k}^e(x)\right)\\&=W_{2(k-r)}(x,m)*(-1)^{(k-r)}R_{2(k-r)}^e(x)*\delta.\end{aligned}$$

It follows that

$$\begin{aligned}&\otimes^r\left(W_{2k}(x,m)*(-1)^kR_{2k}^e(x)\right)\\&=W_{2(k-r)}(x,m)*(-1)^{(k-r)}R_{2(k-r)}^e(x)\end{aligned}$$

as required. For $k\le t$,

$$\begin{aligned}&\otimes^t\left(W_{2k}(x,m)*(-1)^kR_{2k}^e(x)\right)\\&=\otimes^{t-k}\otimes^k\left(W_{2k}(x,m)*(-1)^kR_{2k}^e(x)\right)\\&=\otimes^{t-k}\delta\end{aligned}$$

by Lemma 2.13. That completes the proofs. □

Theorem 3.2. *Consider the linear differential equation*

$$\otimes^k u(x) = \sum_{r=0}^{t} c_r \otimes^r \delta, \tag{3.3}$$

where $p+q=n$, $x \in \mathbb{R}^n$, c_r is a constant, δ is the Dirac-delta distribution and $\otimes^0\delta = \delta$. Then the type of solution to (3.3) *depends on the relationship between k and t, according to the following cases:*

(1) *If $t < k$ and $t = 0$, then* (3.3) *has the solution*

$$u(x) = W_{2k}(x,m) * c_0 \left((-1)^k R^e_{2k}(x)\right)$$

which is the elementary solution of the operator $\otimes^k$ in Lemma 2.13*, is an ordinary function when $2k \geq n$ and is a temper distribution when $2k < n$.*

(2) *If $t < k$ and $t = m = 0$, then* (3.3) *has the solution*

$$u(x) = c_0(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$$

which is the elementary solution of the diamond operator $\diamondsuit^k$, is an ordinary function when $2k \geq n$ and is a temper distribution when $2k < n$.

(3) *If $0 < t < k$ then the solution of* (3.3) *is*

$$u(x) = \sum_{r=1}^{t} W_{2(k-r)}(x,m) * c_r \left((-1)^{(k-r)} R^e_{2(k-r)}(x)\right)$$

which is an ordinary function when $2k - 2r \geq n$ and is a tempered distribution when $2k - 2r < n$.

(4) *If $t \geq k$ and $k \leq t \leq M$, then* (3.3) *has the solution*

$$u(x) = \sum_{r=k}^{M} c_r \otimes^{r-k} \delta$$

which is only a singular distribution.

Proof. (1) For $t = 0$, we have $\otimes^k u(x) = c_0\delta$, and by Lemma 2.13 we obtain

$$u(x) = W_{2k}(x,m) * c_0 \left((-1)^k R^e_{2k}(x)\right).$$

Now, $W_{2k}(x,m)$,$(-1)^k R^e_{2k}(x)$ are the analytic functions for $2k \geq n$ and also $W_{2k}(x,m) * (-1)^k R^e_{2k}(x)$ exits and is an analytic function by (2.18). It follows that $W_{2k}(x,m) * (-1)^k R^e_{2k}(x)$ is an ordinary function for $2k \geq n$. By Lemma 2.14, $W_{2k}(x,m) * (-1)^k R^e_{2k}(x)$ is the tempered distributions with $2k < n$.

(2) For $t = m = 0$, we have $\otimes^k u(x) = \diamondsuit^k u(x) = c_0\delta$, and by Lemma 2.13, Lemma 2.7 and (2.16) we obtain

$$\begin{aligned} u(x) &= W_{2k}(x,0) * c_0 \left((-1)^k R^e_{2k}(x)\right) \\ &= c_0 \left((-1)^k R^e_{2k}(x) * R^H_{2k}(x)\right). \end{aligned}$$

Now, $(-1)^k R^e_{2k}(x)$ and $R^H_{2k}(x)$ are the analytic functions for $2k \geq n$ and also $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ exits and is an analytic function by (2.18). It follows that $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is an ordinary function for $2k \geq n$. By Lemma 2.7, $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is a tempered distributions with $2k < n$.

(3) For the case $0 < t < k$, we have

$$\otimes^k u(x) = c_1 \otimes \delta + c_2 \otimes^2 \delta + \cdots + c_t \otimes^t \delta.$$

We convolved both sides of the above equation by $W_{2k}(x,m) * (-1)^k R^e_{2k}(x)$ to obtain

$$\begin{aligned} &\otimes^k W_{2k}(x,m) * \left((-1)^k R^e_k(x)\right) * u(x) \\ &= c_1 \otimes \left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right) \\ &+ c_2 \otimes^2 \left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right) \\ &+ \cdots + c_t \otimes^t \left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right). \end{aligned}$$

By Lemma 2.13 and Theorem 3.1, we obtain

$$\begin{aligned} u(x) &= c_1 \left(W_{2(k-1)}(x,m) * (-1)^{(k-1)} R^e_{2(k-1)}(x)\right) \\ &+ c_2 \left(W_{2(k-2)}(x,m) * (-1)^{(k-2)} R^e_{2(k-2)}(x)\right) \\ &+ \cdots + c_t \left(W_{2(k-t)}(x,m) * (-1)^{(k-t)} R^e_{2(k-t)}(x)\right). \end{aligned}$$

or

$$u(x) = \sum_{r=1}^{t} c_r \left(W_{2(k-r)}(x,m) * (-1)^{(k-r)} R^e_{2(k-r)}(x)\right).$$

Similarly, as in the case (1), $u(x)$ is an ordinary function for $2k - 2r \geq n$ and is a tempered distribution for $2k - 2r < n$.

(4) For the case $t \geq k$ and $k \leq t \leq M$, we have

$$\otimes^k u(x) = c_k \otimes^k \delta + c_{k+1} \otimes^{k+1} \delta + \cdots + c_M \otimes^M \delta.$$

Convolved both sides of the above equation by $W_{2k}(x,m) * (-1)^k R^e_{2k}(x)$ to obtain

$$\begin{aligned} &\otimes^k \left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right) * u(x) \\ &= c_k \otimes^k \left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right) \\ &+ c_{k+1} \otimes^{k+1} \left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right) \\ &+ \cdots + c_M \otimes^M \left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right). \end{aligned}$$

By Lemma 2.13 and Theorem 3.1 again, we obtain

$$\begin{aligned} u(x) &= c_k \delta + c_{k+1} \otimes \delta + c_{k+2} \otimes^2 \delta + \cdots + c_M \otimes^{M-k} \delta \\ &= \sum_{r=k}^{M} c_r \otimes^{r-k} \delta. \end{aligned}$$

Since $\otimes^{r-k}\delta$ is a singular distribution, hence $u(x)$ is only the singular distribution. That completes the proof.

□

Theorem 3.3.

$$\begin{aligned} \mathcal{F}\left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right) &= \frac{1}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+\frac{m^2}{2}\right)^2 - \left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)-\frac{m^2}{2}\right)^2\right]^k} \\ &= \left|\mathcal{F}\left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right)\right| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} N \end{aligned} \tag{3.4}$$

for a large $\xi_i \in \mathbb{R}$, where m is a non-negative real number and N is a constant. That is, $\mathcal{F}$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distributions.

Proof. By Lemma 2.13, we obtain

$$\otimes^k \left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right) = \delta$$

or

$$(\otimes^k \delta)\left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right) = \delta$$

Taking the Fourier transform on both sides of the above equation, we obtain

$$\mathcal{F}\left((\otimes^k \delta)\left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right)\right) = \mathcal{F}\delta = \frac{1}{(2\pi)^{n/2}}.$$

By (2.9), we have

$$\frac{1}{(2\pi)^{n/2}}\left\langle (\otimes^k \delta)\left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right), e^{-i(\xi\cdot x)}\right\rangle = \frac{1}{(2\pi)^{n/2}}.$$

By the definition of convolution

$$\begin{aligned}
\frac{1}{(2\pi)^{n/2}}\left\langle (\otimes^k \delta)\left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right), e^{-i\xi\cdot(x+r)}\right\rangle &= \frac{1}{(2\pi)^{n/2}},\\
\frac{1}{(2\pi)^{n/2}}\left\langle \left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right), e^{-i(\xi\cdot r)}\right\rangle \cdot \left\langle (\otimes^k \delta), e^{-i(\xi\cdot x)}\right\rangle &= \frac{1}{(2\pi)^{n/2}},\\
\mathcal{F}(W_{2k}(x,m) * (-1)^k R^e_{2k}(x))(2\pi)^{\frac{n}{2}}\mathcal{F}\left(\otimes^k \delta\right) &= \frac{1}{(2\pi)^{n/2}}.
\end{aligned}$$

By Lemma 2.15, we obtain

$$\begin{aligned}
&\mathcal{F}(W_{2k}(x,m) * (-1)^k R^e_{2k}(x))\\
&\quad \times \left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+\frac{m^2}{2}\right)^2 - \left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)-\frac{m^2}{2}\right)^2\right]^k\\
&= \frac{1}{(2\pi)^{n/2}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathcal{F}(W_{2k}(x,m) * (-1)^k R^e_{2k}(x))\\
&= \frac{1}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+\frac{m^2}{2}\right)^2 - \left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)-\frac{m^2}{2}\right)^2\right]^k}.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{1}{\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+\frac{m^2}{2}\right)^2 - \left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)-\frac{m^2}{2}\right)^2\right]}\\
&= \frac{1}{\left[(\xi_1^2+\xi_2^2+\cdots+\xi_n^2)\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2-\xi_{p+1}^2-\cdots-\xi_{p+q}^2)+m^2\right)\right]}. \qquad (3.5)
\end{aligned}$$

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \Gamma_+$ with Γ_+ defined by Definition 2.2. Then $(\xi_1^2+\xi_2^2+\cdots+\xi_p^2+\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2) > 0$ and for a large k, the right-hand side of (3.5) tend to zero. It follows that it is bounded by a positive constant N say, that is we obtain (3.4) as required and also by (3.4) $\mathcal{F}$ is continuous on the space $\mathcal{S}'$ of the tempered distribution. $\square$

Theorem 3.4.

$$\mathcal{F}\left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x) * (W_{2l}(x,m) * (-1)^l R^e_{2l}(x)\right)$$
$$= (2\pi)^{n/2}\mathcal{F}\left[W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right]\mathcal{F}\left[(W_{2l}(x,m) * (-1)^l R^e_{2l}(x)\right]$$
$$= \frac{1}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+\frac{m^2}{2}\right)^2 - \left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)-\frac{m^2}{2}\right)^2\right]^{k+l}},$$

where k and l are non-negative integers and $\mathcal{F}$ is bounded and continuous on the space $\mathcal{S}'$ of tempered distribution.

Proof. Since $R^e_{2k}(x)$ and $W_{2k}(x,m)$ are tempered distribution with compact support,

$$\left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right) * \left((W_{2l}(x,m) * (-1)^l R^e_{2l}(x)\right)$$
$$= \left[(-1)^{k+l} R^e_{2k}(x) * R^e_{2l}(x)\right] * \left[W_{2k}(x,m) * W_{2l}(x,m)\right]$$
$$= \left[(-1)^{k+l} R^e_{2(k+l)}(x)\right] * \left[W_{2(k+l)}(x,m)\right]$$

by ([1], p.156–159), Lemma 2.10 and [14]. Taking the Fourier transform on both sides and using Theorem 3.3, we obtain

$$\mathcal{F}\left[\left(W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right) * \left((W_{2l}(x,m) * (-1)^l R^e_{2l}(x)\right)\right]$$
$$= \frac{1}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+\frac{m^2}{2}\right)^2 - \left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)-\frac{m^2}{2}\right)^2\right]^{k+l}}$$
$$= \frac{1}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+\frac{m^2}{2}\right)^2 - \left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)-\frac{m^2}{2}\right)^2\right]^{k}}$$
$$\times \frac{(2\pi)^{n/2}}{(2\pi)^{n/2}\left[\left((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)+\frac{m^2}{2}\right)^2 - \left((\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)-\frac{m^2}{2}\right)^2\right]^{l}}$$
$$= (2\pi)^{n/2}\mathcal{F}\left[W_{2k}(x,m) * (-1)^k R^e_{2k}(x)\right]\mathcal{F}\left[(W_{2l}(x,m) * (-1)^l R^e_{2l}(x)\right].$$

Since $(-1)^{k+l} R^e_{2(k+l)}(x) * W_{2(k+l)}(x,m) \in \mathcal{S}'$, the space of tempered distribution and by Theorem 3.3, we obtain that $\mathcal{F}$ is bounded and continuous on $\mathcal{S}'$. □

Acknowledgments. The author would like to thank an anonymous referee who provided very useful comments and suggestions. This work is supported by the Sakon Nakhon Rajabhat University.

References

[1] A.H. Zemanian, *Distribution Theory and Transform Analysis*, McGraw-Hill, New York, 1965.

[2] A. Kananthai, *On the inversion of the kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator $\oplus^k$*, Indian Journal of Pure and Applied Mathematics, **34** (2003) 1419-1429.

[3] A. Kananthai, *On the Green function of the diamond operator related to the Klein-Gordon operator*, Bulletin of the Calcutta Mathematical Society, **93** (2001) 353-360.

[4] A. Kananthai, *On the solutions of the n-dimensional diamond operator*, Applied Mathematics and Computation, **88** (1997) 27-37.

[5] A. Kananthai, S. Suantai and V. Longani, *On the operator $\oplus^k$ related to the wave equation and Laplacian*, Applied Mathematics and Computation, **132** (2002) 219-229.

[6] A. Kananthai, S. Suantai and V. Longani, *On the weak solution of the equation related to the diamond operator*, Computational Technologies, **5** (2000) 42-48.

[7] A. Kananthai, *On the distribution related to the Ultra-hyperbolic equation*, Journal of Computational and Applied Mathematics, **84** (1997) 101-106.

[8] A. Liangprom and K. Nonlaopon, *On the convolution equation related to the Klein-Gordon operator*, International Journal of Pure and Applied Mathematics, **71** (2011) 67-82.

[9] A. Liangprom and K. Nonlaopon, *On the convolution equation related to the diamond Klein-Gordon operator*, Abstract and Applied Analysis, **2011** (2011) 1-14.

[10] A. Lunnaree and K. Nonlaopon, *On the Fourier transform of the diamond Klein-Gordon kernel*, International Journal of Pure and Applied Mathematics, **68** (2011) 85-97.

[11] Daniel Eceizabarrena Pérez, *Distribution Theory and Fundamental Solutions of Differential Operators*, Leioa, June 24th, 2015

[12] K. Nonlaopon, *On the inverse ultrahyperbolic Klein-Gordon kernel*, Mathematics, **7** (2019) 534.

[13] K. Nonlaopon, A. Lunnaree and A. Kananthai, *On the solution of the n-dimensional diamond Klein-Gordon operator and its convolution*, Far East Journal of Mathematical Sciences, **63** (2012) 203-220.

[14] M.A. Tellez, *The convolution product of $W_\alpha(u,m) * W_\beta(u,m)$* , Mathematicae Notae, **38** (1995-1996) 105-111.

[15] M.A. Tellez and S.E. Trione, *The distributional Riesz's ultra-hyperbolic kernel*, Resvista de la Union Mathematica Argentina, **39** (1995) 115-124.

[16] S. Bupasiri, *On the Operator $\diamondsuit_m^k$ Related to the Wave Equation and Laplacian*, Applied Mathematical Sciences, **8** (2014) 1127-1133.

[17] S.E. Trione, *On Marcel Rieszs ultra hyperbolic kernel*, Trabajos de Mathematica, 116(1987) 1 - 12.

[18] S.E. Trione, *On the elementary related, ultra-hyperbolic solution of the Klein-Gordon operator iterated k-times*, Studies in Applied Mathematics, **89** (1988) 121-141.

[19] W.F. Donoghue, *Distributions and Fourier Transforms*, Academic Press, 1969.

[20] W. Satsanit, *Green function and Fourier transform for o-plus operator*, Electronic Journal of Differential Equation, **2010** (2010) 1-14.

[21] Y. Nozaki, *On Riemann-Liouville integral of ultra-hyperbolic type*, Kodai Mathematical Seminar Reports, **6** (1964) 69-87.

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