Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 16 Issue 4(2024), Pages 116-129
https://doi.org/10.54671/BMAA-2024-4-8

MULTI-DIMENSIONAL MATRIX CHARACTERIZATION OF (ℓ_1, ℓ_1) AND MERCERIAN-TYPE THEOREM VIA MATRIX CONVOLUTION

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Abstract. This paper extends the study of infinite matrices to four-dimensional matrices in (ℓ_1, ℓ_1) under convolution operation. We characterize the space $(\ell_1, \ell_1; P)$ and establish the algebraic properties of (ℓ_1, ℓ_1), proving it forms a Banach algebra under convolution. The main result is a Mercerian-type theorem for four-dimensional matrices under convolution.

1. Introduction

Four-dimensional matrix transformations have grown as a key area of mathematical study over many decades. Robison [10] and Hamilton [5] laid the early foundation by studying matrix regularity in four dimensions. Later, Móricz and Rhoades [6] made important progress with their work on strongly regular matrices for double sequences. Zeltser et al. [13] added to this knowledge by studying almost conservative and almost regular four-dimensional matrices. A major step forward came when Başar and Savaşcı [1] wrote their detailed book, which covered various matrix types and presented new findings about Mercerian and Steinhaus type theorems, including both their own discoveries and those of others. The reader can also refer to the recent articles [2], [3], [4], [11] and [12] covering new approaches to double sequences. Mursaleen and Mohiuddine [7] developed crucial new methods for analyzing double sequence convergence, filling important gaps in the theory.

The space of infinite matrices mapping ℓ_1 into itself, denoted as (ℓ_1, ℓ_1), has been extensively studied in the two-dimensional setting. Seminal work by Natarajan in [8] established key results on the algebraic properties of this space. Our paper extends this analysis to the four-dimensional domain, introducing a four-dimensional matrix convolution product. This convolution operation allows us to explore the algebraic structure of four-dimensional matrices in (ℓ_1, ℓ_1) from a new perspective.

The structure of this paper is as follows: We begin with essential definitions and preliminary results, including a formal introduction of four-dimensional convolution product. We then discuss the algebraic properties of (ℓ_1, ℓ_1) and $(\ell_1, \ell_1; P)$ in

2010 *Mathematics Subject Classification.* 40B05, 40C05.
Key words and phrases. Double sequences, Four-dimensional matrices, Convolution, Mercerian-type theorem.

Submitted November 2, 2024. Accepted November 15, 2024. Published November 21, 2024.
Communicated by F. Basar.

this four-dimensional matrix context, with particular emphasis on their behavior under the convolution product. This paper presents a Mercerian-type theorem for four-dimensional matrices in (ℓ_1, ℓ_1), demonstrating how classical results extend to double sequences under convolution product framework.

2. Preliminaries and Definitions

Definition 2.1. *By Ω, we denote the space of all real or complex valued double sequences $x = (x_{k,l})_{k,l\geq 0}$, which forms a vector space with coordinatewise addition and scalar multiplication.*

The space of absolutely summable double sequences is defined as:

Definition 2.2. *The space of absolutely summable double sequences, denoted by ℓ_1, is defined as*

$$\ell_1 = \left\{ x = (x_{k,l})_{k,l\geq 0} \in \Omega : \sum_{k,l=0,0}^{\infty,\infty} |x_{k,l}| < \infty \right\}.$$

The space ℓ_1 is a Banach space with the norm $\|x\|_{\ell_1} = \sum_{k,l=0,0}^{\infty,\infty} |x_{k,l}|$.

The $A-$transform of a double sequence is defined as:

Definition 2.3. *Let $A = (a_{m,n,k,l})_{m,n,k,l\geq 0}$ be a four-dimensional infinite matrix. The A-transform of a double sequence $x = (x_{k,l})_{k,l\geq 0}$ is the double sequence $Ax = \{(Ax)_{m,n}\}$, where*

$$(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l}, \quad m,n \geq 0$$

assuming that the double series on the right exists.

The class of four-dimensional matrices that map ℓ_1 into itself is defined as:

Definition 2.4. *We write $A = (a_{m,n,k,l}) \in (\ell_1, \ell_1)$ if for every $x = (x_{k,l}) \in \ell_1$, the double sequence $Ax = \{(Ax)_{m,n}\}$ belongs to the space ℓ_1.*

A particularly important subclass of (ℓ_1, ℓ_1), denoted as $(\ell_1, \ell_1; P)$, which we define as follows:

Definition 2.5. *The class $(\ell_1, \ell_1; P)$ is defined as the set of all four-dimensional matrices $A = (a_{m,n,k,l}) \in (\ell_1, \ell_1)$ that satisfy the additional property*

$$\sum_{m,n=0,0}^{\infty,\infty} (Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} x_{k,l}$$

for all $x = (x_{k,l}) \in \ell_1$.

3. Characterization of $(\ell_1, \ell_1; P)$

Patterson's theorem in [9] lays the groundwork for our characterizing four-dimensional matrices that map ℓ_1 into itself. Extending this foundation, we develop a theorem that specifically characterizes the class $(\ell_1, \ell_1; P)$.

Theorem 3.1 (cf. [9, Theorem 6]). *A four-dimensional matrix $A = (a_{m,n,k,l}) \in (\ell_1, \ell_1)$ if and only if there exists a positive constant M_A such that for each k and l,*

$$\sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| < M_A.$$

Using Theorem 3.1, we now present a theorem that characterizes the class $(\ell_1, \ell_1; P)$.

Theorem 3.2. *Let $A = (a_{m,n,k,l})$ be a four-dimensional matrix. Then $A \in (\ell_1, \ell_1; P)$ if and only if*

(i) *there exists $M > 0$ such that $\sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| \leq M$, and*

(ii) *for all $k, l \geq 0$, $\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$.*

Proof. We will prove that conditions (i) and (ii) hold. Let $A \in (\ell_1, \ell_1; P)$. Then, $A \in (\ell_1, \ell_1)$ and there exists a positive constant M_A such that for each $k, l \geq 0$ by Theorem 3.1, $\sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| < M_A$. This directly implies condition (i) with $M = M_A$.

Since $A \in (\ell_1, \ell_1; P)$, by definition, it preserves the sum of any $(x_{k,l}) \in \ell_1$. For fixed $k, l \geq 0$, let $e^{(k,l)} = (e_{i,j})$ be the double sequence defined as

$$e_{i,j} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $e^{(k,l)} \in \ell_1$ for any $k, l \geq 0$. Applying the sum-preserving property to $e^{(k,l)}$ we get

$$\sum_{m,n=0,0}^{\infty,\infty} (Ae^{(k,l)})_{m,n} = \sum_{i,j=0,0}^{\infty,\infty} e_{i,j} = 1. \tag{1}$$

Expanding the left side in (1) and using the fact that $e_{i,j} = 1$ when $i = k$ and $j = l$, and 0 otherwise, we obtain

$$\sum_{m,n=0,0}^{\infty,\infty} (Ae^{(k,l)})_{m,n} = \sum_{m,n=0,0}^{\infty,\infty} \left(\sum_{i,j=0,0}^{\infty,\infty} a_{m,n,i,j} e_{i,j} \right) = \sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l}. \tag{2}$$

From (1) and (2), it follows that $\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$ for fixed k and l. This holds for all $k, l \geq 0$, proving condition (ii). Thus, we have shown that if $A \in (\ell_1, \ell_1; P)$, then conditions (i) and (ii) are satisfied, completing the proof of necessity.

For the sufficiency part, assume that both conditions (i) and (ii) hold and let $x = (x_{k,l}) \in \ell_1$. We show that $Ax \in \ell_1$ and that A preserves the sum of x. First, we show that $Ax \in \ell_1$. Since

$$\begin{aligned}\sum_{m,n=0,0}^{\infty,\infty} |(Ax)_{m,n}| &= \sum_{m,n=0,0}^{\infty,\infty} \left| \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \right| \\ &\leq \sum_{m,n=0,0,}^{\infty,\infty} \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}||x_{k,l}| = \sum_{k,l=0,0}^{\infty,\infty} |x_{k,l}| \left(\sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| \right) \\ &\leq M \sum_{k,l=0,0}^{\infty,\infty} |x_{k,l}| < \infty,\end{aligned}$$

where the last inequality follows from condition (i) and the fact that $x \in \ell_1$.

Now, we show that A preserves the sum of x. Since

$$\begin{aligned}\sum_{m,n=0,0}^{\infty,\infty} (Ax)_{m,n} &= \sum_{m,n=0,0}^{\infty,\infty} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \\ &= \sum_{k,l=0,0}^{\infty,\infty} x_{k,l} \left(\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} \right) = \sum_{k,l=0,0}^{\infty,\infty} x_{k,l}.\end{aligned}$$

The interchange of summation order is justified by the absolute convergence of the series involved, which follows from the boundedness condition (i) and the fact that $x \in \ell_1$. Therefore, $A \in (\ell_1, \ell_1; P)$. □

4. Algebraic Properties of (ℓ_1, ℓ_1) and $(\ell_1, \ell_1; P)$

In this section, we introduce a norm on (ℓ_1, ℓ_1) suitable for our four-dimensional context. We then demonstrate that (ℓ_1, ℓ_1), when equipped with this norm and the four-dimensional matrix convolution operation, forms a Banach algebra. Subsequently, we investigate the algebraic characteristics of $(\ell_1, \ell_1; P)$ as a subset of (ℓ_1, ℓ_1).

Theorem 4.1. *Let $A = (a_{m,n,k,l})$ be a four-dimensional matrix in (ℓ_1, ℓ_1). Define a function $\phi : (\ell_1, \ell_1) \to \mathbb{R}$ by*

$$\phi(A) = \sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}|.$$

Then ϕ is a norm on the space of four-dimensional matrices in (ℓ_1, ℓ_1).

Proof. The verification that ϕ satisfies the properties of a norm (positive definiteness, absolute homogeneity, and the triangle inequality) follows directly from its definition, utilizing properties of supremum and absolute value. □

Definition 4.2 (Norm for Four-Dimensional Matrices in (ℓ_1, ℓ_1))**.** *For any four-dimensional matrix A in (ℓ_1, ℓ_1), we define $\|A\| = \phi(A)$, where ϕ is the function proven to be a norm in Theorem 4.1. This $\|\cdot\|$ is adopted as the standard norm for the space of four-dimensional matrices in (ℓ_1, ℓ_1).*

Next, we define the convolution operation for four-dimensional matrices, followed by the concepts of identity matrix and matrix inverse in this convolution context.

Definition 4.3 (Convolution of Four-Dimensional Infinite Matrices)**.** *Let* $A = (a_{i,j,k,l})$ *and* $B = (b_{i,j,k,l})$ *be four-dimensional infinite matrices. Their convolution* $C = A * B = (c_{m,n,k,l})$ *is defined by*

$$c_{m,n,k,l} := \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j,k,l} b_{m-i,n-j,k,l}$$

for all $m, n, k, l \geq 0$.

Lemma 4.4 (Commutativity of Four-Dimensional Matrix Convolution)**.** *For any two four-dimensional matrices* $A = (a_{i,j,k,l})$ *and* $B = (b_{i,j,k,l})$*, the convolution operation is commutative. That is,* $A * B = B * A$.

Proof. Consider the (m, n, k, l)-th element of $B * A$

$$(B * A)_{m,n,k,l} = \sum_{i=0}^{m} \sum_{j=0}^{n} b_{i,j,k,l} a_{m-i,n-j,k,l}.$$

By making a change of variables: $i' = m - i$ and $j' = n - j$. As i goes from 0 to m, i' goes from m to 0. Similarly, as j goes from 0 to n, j' goes from n to 0. Applying this change of variables

$$(B * A)_{m,n,k,l} = \sum_{i'=0}^{m} \sum_{j'=0}^{n} b_{m-i',n-j',k,l} a_{i',j',k,l}.$$

Now, rearranging the summation terms and renaming i' back to i and j' back to j

$$\begin{aligned}(B * A)_{m,n,k,l} &= \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j,k,l} b_{m-i,n-j,k,l} \\ &= (A * B)_{m,n,k,l}.\end{aligned}$$

Since this equality holds for all $m, n, k, l \geq 0$, we have shown that $A * B = B * A$. □

Theorem 4.5 (Identity Element for Four-Dimensional Matrix Convolution)**.** *The identity element* $E = (e_{m,n,k,l})$ *for the convolution product of four-dimensional matrices is defined as*

$$e_{m,n,k,l} = \begin{cases} 1, & \text{if } m = n = 0, \text{ for all } k, l \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

That is, for any four-dimensional matrix $A = (a_{m,n,k,l})$*, we have* $A * E = E * A = A$. *Moreover,* $\|E\| = 1$ *and* $E \in (\ell_1, \ell_1; P)$.

Proof. Let $A = (a_{m,n,k,l})$ be any four-dimensional matrix. First, we will show that $A * E = A$. For any $m, n, k, l \geq 0$, we have

$$\begin{aligned}(A * E)_{m,n,k,l} &= \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j,k,l} e_{m-i,n-j,k,l} \\ &= a_{m,n,k,l} e_{0,0,k,l} + \sum_{(i,j) \neq (m,n)} a_{i,j,k,l} e_{m-i,n-j,k,l}.\end{aligned}$$

Now, by the definition of E, we have $e_{0,0,k,l} = 1$ for all $k, l \geq 0$ and $e_{m-i,n-j,k,l} = 0$ for all $(i, j) \neq (m, n)$. Therefore,

$$(A * E)_{m,n,k,l} = a_{m,n,k,l} \cdot 1 + 0 = a_{m,n,k,l}$$

which imply that $A * E = A$. Now $E * A = A$ follows by the commutativity of the convolution operation as proved in Lemma 4.4. Therefore, $A * E = E * A = A$ for all four-dimensional matrices A.

The norm of E is given by

$$\begin{aligned}\|E\| &= \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |e_{m,n,k,l}| \\ &= \sup_{k,l\geq 0} \left(|e_{0,0,k,l}| + \sum_{(m,n)\neq(0,0)} |e_{m,n,k,l}| \right) = 1.\end{aligned}$$

To show $E \in (\ell_1, \ell_1; P)$, we prove for all $x = (x_{k,l}) \in \ell_1$ that $\sum_{m,n=0,0}^{\infty,\infty} (Ex)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} x_{k,l}$. Let $x = (x_{k,l}) \in \ell_1$, then for any $m, n \geq 0$, we have

$$(Ex)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} e_{m,n,k,l} x_{k,l} = \begin{cases} \sum_{k,l=0,0}^{\infty,\infty} x_{k,l}, & \text{if } m = n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Summing over all m and n yields

$$\begin{aligned}\sum_{m,n=0,0}^{\infty,\infty} (Ex)_{m,n} &= (Ex)_{0,0} + \sum_{(m,n)\neq(0,0)} (Ex)_{m,n} \\ &= \sum_{k,l=0,0}^{\infty,\infty} x_{k,l} + 0 = \sum_{k,l=0,0}^{\infty,\infty} x_{k,l},\end{aligned}$$

and therefore, $E \in (\ell_1, \ell_1; P)$. □

Definition 4.6 (Inverse of a Four-Dimensional Matrix under Convolution)**.** *Let $A = (a_{m,n,k,l})$ be a four-dimensional matrix. If there exists a four-dimensional matrix $B = (b_{m,n,k,l})$ such that their convolution is equal to the identity element E, i.e., $A * B = B * A = E$, then the matrix B is called the inverse of the matrix A under convolution and is written $B = A^{-1}$.*

We shall now establish two critical lemmas concerning the convolution of transformations on four-dimensional matrices in (ℓ_1, ℓ_1) space which will be used in the subsequent theorems.

Lemma 4.7. *Let $A = (a_{m,n,k,l})$ and $B = (b_{m,n,k,l})$ be in (ℓ_1, ℓ_1). Then their convolution $A * B$ is also in (ℓ_1, ℓ_1). Moreover, the norm in (ℓ_1, ℓ_1) is submultiplicative under convolution, i.e., $\|A * B\| \leq \|A\|\|B\|$.*

Proof. Let $A = (a_{m,n,k,l})$ and $B = (b_{m,n,k,l})$ be in (ℓ_1, ℓ_1). Then, by Definition 4.2,

$$\|A\| = \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| < \infty \quad \text{and} \quad \|B\| = \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |b_{m,n,k,l}| < \infty.$$

Let $C = A * B = (c_{m,n,k,l})$. Then, by Definition 4.3 of convolution

$$c_{m,n,k,l} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j,k,l} b_{m-i,n-j,k,l}.$$

Now

$$\begin{aligned}
\sum_{m,n=0,0}^{\infty,\infty} |c_{m,n,k,l}| &= \sum_{m,n=0,0}^{\infty,\infty} \left| \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j,k,l} b_{m-i,n-j,k,l} \right| \\
&\leq \sum_{m,n=0,0}^{\infty,\infty} \sum_{i=0}^{m} \sum_{j=0}^{n} |a_{i,j,k,l}||b_{m-i,n-j,k,l}| \\
&= \sum_{i,j=0,0}^{\infty,\infty} |a_{i,j,k,l}| \sum_{m=i}^{\infty} \sum_{n=j}^{\infty} |b_{m-i,n-j,k,l}| \\
&= \sum_{i,j=0,0}^{\infty,\infty} |a_{i,j,k,l}| \sum_{m,n=0,0}^{\infty,\infty} |b_{m,n,k,l}| \leq \|A\|\|B\|.
\end{aligned}$$

Since this bound is independent of k and l, we have

$$\|C\| = \sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |c_{m,n,k,l}| \leq \|A\|\|B\| < \infty.$$

Therefore, $C = A * B \in (\ell_1, \ell_1)$ and the norm is submultiplicative under the convolution product. □

Lemma 4.8 (Completeness of (ℓ_1, ℓ_1))**.** *The space (ℓ_1, ℓ_1) of four-dimensional matrices $A = (a_{m,n,k,l})$ with the norm $\|A\| = \sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}|$ is complete.*

Proof. Let $(A^{(p)})_{p \in \mathbb{N}}$ be a Cauchy sequence in (ℓ_1, ℓ_1), where $A^{(p)} = (a^{(p)}_{m,n,k,l})$. We show that this sequence converges to an element in (ℓ_1, ℓ_1). By the definition of a Cauchy sequence, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $p, q > K$,

$$\|A^{(p)} - A^{(q)}\| = \sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a^{(p)}_{m,n,k,l} - a^{(q)}_{m,n,k,l}| < \varepsilon.$$

This implies that for each fixed m, n, k, l, $(a^{(p)}_{m,n,k,l})_{p \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, this sequence converges. We can thus define $\alpha_{m,n,k,l} = \lim_{p \to \infty} a^{(p)}_{m,n,k,l}$. Let $A = (\alpha_{m,n,k,l})_{m,n,k,l \geq 0}$. We show that $A \in (\ell_1, \ell_1)$. For any $\varepsilon > 0$, choose K as above. Then for any $p > K$ and any $k, l \geq 0$,

$$\begin{aligned}
\sum_{m,n=0,0}^{\infty,\infty} |\alpha_{m,n,k,l}| &= \sum_{m,n=0,0}^{\infty,\infty} \lim_{p \to \infty} |a^{(p)}_{m,n,k,l}| \\
&\leq \liminf_{p \to \infty} \sum_{m,n=0,0}^{\infty,\infty} |a^{(p)}_{m,n,k,l}| \leq \liminf_{p \to \infty} \|A^{(p)}\| < \infty,
\end{aligned}$$

where the first inequality follows from Fatou's lemma, and the last inequality holds because $(A^{(p)})$ is a Cauchy sequence in $\mathbb{C}$ and thus bounded, which proves that

$A \in (\ell_1, \ell_1)$. To prove that $A^{(p)} \to A$ in the (ℓ_1, ℓ_1) norm, observe that for any $p, q > K$,

$$
\begin{aligned}
\|A - A^{(p)}\| &= \sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |\alpha_{m,n,k,l} - a^{(p)}_{m,n,k,l}| \\
&= \sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} \lim_{q \to \infty} |a^{(q)}_{m,n,k,l} - a^{(p)}_{m,n,k,l}| \\
&\leq \liminf_{q \to \infty} \left(\sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a^{(q)}_{m,n,k,l} - a^{(p)}_{m,n,k,l}| \right) \\
&= \liminf_{q \to \infty} \|A^{(q)} - A^{(p)}\| \leq \varepsilon.
\end{aligned}
$$

As ε is arbitrary, we conclude that $A^{(p)} \to A$ in the (ℓ_1, ℓ_1) norm. Therefore, (ℓ_1, ℓ_1) is complete. □

With these properties established, we can now prove our main result in this section about the algebraic structure of (ℓ_1, ℓ_1).

Theorem 4.9. *Let $A = (a_{m,n,k,l}) \in (\ell_1, \ell_1)$. Then, the class (ℓ_1, ℓ_1) is a Banach algebra under the norm $\|A\| = \sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}|$, with the usual matrix addition, scalar multiplication, and convolution as the multiplication operation.*

Proof. We begin by observing that (ℓ_1, ℓ_1) forms a vector space over the complex field. The closure under addition and scalar multiplication follows from the properties of absolute convergence of the series defining the norm. The vector space axioms are readily verified.

In Definition 4.2, the norm on (ℓ_1, ℓ_1) has been defined and shown to satisfy all norm axioms in Theorem 4.1. Therefore, (ℓ_1, ℓ_1) is a normed linear space under this norm.

The completeness of (ℓ_1, ℓ_1) under this norm has been established in Lemma 4.8, demonstrating that (ℓ_1, ℓ_1) is a Banach space.

To show that (ℓ_1, ℓ_1) is an algebra, we need to demonstrate closure under convolution, and the continuity of this operation. The closure of (ℓ_1, ℓ_1) under convolution and the submultiplicativity of the norm under convolution have been established in Lemma 4.7. This lemma also demonstrates that the submultiplicativity property implies the continuity of the convolution operation, as required for a Banach algebra.

To show the associativity of convolution, let $A = (a_{m,n,k,l})$, $B = (b_{m,n,k,l})$ and $C = (c_{m,n,k,l}) \in (\ell_1, \ell_1)$, then for any $m, n, k, l \geq 0$ we have

$$\begin{aligned}
((A * B) * C)_{m,n,k,l} &= \sum_{i=0}^{m} \sum_{j=0}^{n} (A * B)_{i,j,k,l} c_{m-i,n-j,k,l} \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} \left(\sum_{p=0}^{i} \sum_{q=0}^{j} a_{p,q,k,l} b_{i-p,j-q,k,l} \right) c_{m-i,n-j,k,l} \\
&= \sum_{p=0}^{m} \sum_{q=0}^{n} a_{p,q,k,l} \sum_{i=p}^{m} \sum_{j=q}^{n} b_{i-p,j-q,k,l} c_{m-i,n-j,k,l} \\
&= \sum_{p=0}^{m} \sum_{q=0}^{n} a_{p,q,k,l} (B * C)_{m-p,n-q,k,l} \\
&= (A * (B * C))_{m,n,k,l}.
\end{aligned}$$

The distributivity of convolution over addition can be verified through similar computations as those used for associativity. Therefore, (ℓ_1, ℓ_1) satisfies all the requirements of a Banach algebra under convolution. □

We now examine the properties of the subclass $(\ell_1, \ell_1; P)$.

Theorem 4.10. *The class $(\ell_1, \ell_1; P)$, as a subset of (ℓ_1, ℓ_1), is a closed, convex semigroup with identity, the multiplication being the four-dimensional matrix convolution.*

Proof. Let $A = (a_{m,n,k,l})$ and $B = (b_{m,n,k,l}) \in (\ell_1, \ell_1; P)$. Then by definition, there exist $M_A > 0$ and $M_B > 0$ such that $\sup_{k,l \geq 0} \sum_{m,n=0}^{\infty,\infty} |a_{m,n,k,l}| \leq M_A$ and $\sup_{k,l \geq 0} \sum_{m,n=0}^{\infty,\infty} |b_{m,n,k,l}| \leq M_B$. Let $M = \max\{M_A, M_B\}$. Then, we have

$$\sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| \leq M \quad \text{and} \quad \sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |b_{m,n,k,l}| \leq M.$$

Let λ and μ be non-negative real numbers such that $\lambda + \mu = 1$. Then, one can see that

$$\begin{aligned}
&\sup_{k,l \geq 0} \sum_{m,n=0}^{\infty,\infty} |\lambda a_{m,n,k,l} + \mu b_{m,n,k,l}| \\
\leq \quad &\lambda \sup_{k,l \geq 0} \sum_{m,n=0}^{\infty,\infty} |a_{m,n,k,l}| + \mu \sup_{k,l \geq 0} \sum_{m,n=0}^{\infty,\infty} |b_{m,n,k,l}| \leq M.
\end{aligned}$$

Also, for all $k, l \geq 0$

$$\sum_{m,n=0}^{\infty,\infty} (\lambda a_{m,n,k,l} + \mu b_{m,n,k,l}) = \lambda \sum_{m,n=0}^{\infty,\infty} a_{m,n,k,l} + \mu \sum_{m,n=0}^{\infty,\infty} b_{m,n,k,l} = 1,$$

by using the fact that $\sum_{m,n=0}^{\infty,\infty} a_{m,n,k,l} = \sum_{m,n=0}^{\infty,\infty} b_{m,n,k,l} = 1$. This shows that $\lambda A + \mu B \in (\ell_1, \ell_1; P)$ and that $(\ell_1, \ell_1; P)$ is a convex subset of (ℓ_1, ℓ_1).

To show that $(\ell_1, \ell_1; P)$ is closed in (ℓ_1, ℓ_1), let $(A^{(r)})_{r\in\mathbb{N}}$ be a sequence in $(\ell_1, \ell_1; P)$ converging to some A in (ℓ_1, ℓ_1). We prove that $A \in (\ell_1, \ell_1; P)$. Given $\varepsilon > 0$, there exists a positive integer N such that

$$\|A^{(r)} - A\| < \varepsilon, \quad r > N$$

i.e.,

$$\sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a^{(r)}_{m,n,k,l} - a_{m,n,k,l}| < \varepsilon, \quad r > N.$$

Now, fix $r > N$. Since $A^{(r)} \in (\ell_1, \ell_1; P)$, we know that for all $k, l \geq 0$,

$$\sum_{m,n=0,0}^{\infty,\infty} a^{(r)}_{m,n,k,l} = 1.$$

For any fixed $k, l \geq 0$, we have

$$\begin{aligned}
\left| \sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} - 1 \right| &= \left| \sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} - \sum_{m,n=0,0}^{\infty,\infty} a^{(r)}_{m,n,k,l} \right| \\
&\leq \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l} - a^{(r)}_{m,n,k,l}| \\
&\leq \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l} - a^{(r)}_{m,n,k,l}| < \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we conclude that $\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$ for all $k, l \geq 0$. This, combined with the fact that $A \in (\ell_1, \ell_1)$, implies that $A \in (\ell_1, \ell_1; P)$. Therefore, $(\ell_1, \ell_1; P)$ is closed in (ℓ_1, ℓ_1).

As shown in Theorem 4.5, the identity element E is in $(\ell_1, \ell_1; P)$ for the four-dimensional matrix convolution.

To complete the proof of the theorem, it suffices to check closure under convolution. Let $A = (a_{m,n,k,l})$ and $B = (b_{m,n,k,l})$ be in $(\ell_1, \ell_1; P)$. We need to show that their convolution $C = A * B = (c_{m,n,k,l})$ is also in $(\ell_1, \ell_1; P)$. For fixed k, l,

$$\begin{aligned}
\sum_{m,n=0,0}^{\infty,\infty} c_{m,n,k,l} &= \sum_{m,n=0,0}^{\infty,\infty} \sum_{i,j=0,0}^{m,n} a_{i,j,k,l} b_{m-i,n-j,k,l} \\
&= \sum_{i,j=0,0}^{\infty,\infty} a_{i,j,k,l} \sum_{m=i,n=j}^{\infty,\infty} b_{m-i,n-j,k,l} \\
&= \sum_{i,j=0,0}^{\infty,\infty} a_{i,j,k,l} \sum_{m,n=0,0}^{\infty,\infty} b_{m,n,k,l} = 1.
\end{aligned}$$

This shows that $\sum_{m,n=0,0}^{\infty,\infty} c_{m,n,k,l} = 1$ for all $k, l \geq 0$, proving that $A * B \in (\ell_1, \ell_1; P)$. This completes the proof of the theorem. □

Remark. *The class $(\ell_1, \ell_1; P)$ is not an algebra since the sum of two elements of $(\ell_1, \ell_1; P)$ is not necessarily in $(\ell_1, \ell_1; P)$.*

To illustrate this last point, we provide the following example.

Example 4.11. *Define four-dimensional matrices $A = (a_{m,n,k,l})$ and $B = (b_{m,n,k,l})$ as*

$$a_{m,n,k,l} = \begin{cases} 1, & \text{if } m = n = k = l = 0; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$b_{m,n,k,l} = \begin{cases} 1, & \text{if } m = n = 0 \text{ and } k = l \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Both $A, B \in (\ell_1, \ell_1; P)$ since

$$\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} = \begin{cases} 1, & \text{if } k = l = 0; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\sum_{m,n=0,0}^{\infty,\infty} b_{m,n,k,l} = \begin{cases} 1, & \text{if } k = l \geq 1; \\ 0, & \text{if } k \neq l \text{ or } k = l = 0. \end{cases}$$

The sum $C = A + B = (c_{m,n,k,l})$ is given by

$$c_{m,n,k,l} = \begin{cases} 1, & \text{if } m = n = 0 \text{ and } k = l \geq 0; \\ 0, & \text{otherwise,} \end{cases}$$

with

$$\sum_{m,n=0,0}^{\infty,\infty} c_{m,n,k,l} = \begin{cases} 1, & \text{if } k = l \geq 0; \\ 0, & \text{if } k \neq l. \end{cases}$$

Since $\sum_{m,n=0,0}^{\infty,\infty} c_{m,n,k,l} \neq 1$ when $k \neq l$, we conclude $C \notin (\ell_1, \ell_1; P)$. Thus, $(\ell_1, \ell_1; P)$ is not closed under addition and does not form an algebra under usual matrix operations.

5. Mercerian-type Theorem for Four-Dimensional Matrices

We first establish a connection between double sequences in ℓ_1 and four-dimensional matrices in (ℓ_1, ℓ_1). We begin by defining a correspondence and then proving its properties in the following lemma.

Definition 5.1. *Define a correspondence ϕ between ℓ_1 and a subset of (ℓ_1, ℓ_1) as follows: For any $(z_{m,n}) \in \ell_1$, let $\phi((z_{m,n})) = Z = (z_{m,n,k,l})$ where*

$$z_{m,n,k,l} = \begin{cases} z_{m-k,n-l}, & \text{if } m \geq k \text{ and } n \geq l, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5.2. *The correspondence ϕ defined above is a bijection between ℓ_1 and a subset of (ℓ_1, ℓ_1). Moreover, this correspondence is norm-preserving, i.e.,*

$$\sum_{m,n=0,0}^{\infty,\infty} |z_{m,n}| = \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n,k,l}|$$

for all $(z_{m,n}) \in \ell_1$.

Proof. The well-definedness, injectivity, and norm-preserving properties of ϕ are immediate consequences of its definition and fundamental properties of summation and supremum. It remains to establish the surjectivity of ϕ. Given such a Z, define $(z_{m,n})$ by $z_{m,n} = z_{m,n,0,0}$ for all $m, n \geq 0$. We show that $(z_{m,n}) \in \ell_1$ and that $\phi((z_{m,n})) = Z$. First, $(z_{m,n}) \in \ell_1$ because

$$\sum_{m,n=0,0}^{\infty,\infty} |z_{m,n}| = \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n,0,0}| \leq \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n,k,l}| < \infty.$$

Now, we show that $\phi((z_{m,n})) = Z$. Let $\phi((z_{m,n})) = Y = (y_{m,n,k,l})$. By definition of ϕ,

$$y_{m,n,k,l} = \begin{cases} z_{m-k,n-l}, & \text{if } m \geq k \text{ and } n \geq l; \\ 0, & \text{otherwise.} \end{cases}$$

But this is exactly how Z is defined, so $Y = Z$. □

With this bijective correspondence established, we can now state and prove our main theorem.

Theorem 5.3 (Mercerian-type Theorem under Convolution)**.** *Let $(y_{m,n})$ and $(x_{m,n})$ be double sequences of complex numbers related by*

$$y_{m,n} = x_{m,n} + \lambda \sum_{k=0}^{m} \sum_{l=0}^{n} c^{m-k} d^{n-l} x_{k,l},$$

where λ, c, and d are complex numbers satisfying $|c| < 1$, $|d| < 1$, and $(y_{m,n}) \in \ell_1$. Then $(x_{m,n}) \in \ell_1$ provided $|\lambda| < (1 - |c|)(1 - |d|)$.

Proof. We represent the double sequences $(x_{m,n})$ and $(y_{m,n})$ as four-dimensional matrices using the correspondence established in Lemma 5.2

$$X_{m,n,k,l} = \begin{cases} x_{m-k,n-l}, & \text{if } m \geq k \text{ and } n \geq l, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$Y_{m,n,k,l} = \begin{cases} y_{m-k,n-l}, & \text{if } m \geq k \text{ and } n \geq l, \\ 0, & \text{otherwise.} \end{cases}$$

Define the four-dimensional matrix A as

$$A_{i,j,k,l} = \begin{cases} c^i d^j & \text{if } k = l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The (m,n,k,l)-th element of the convolution $A*X$ is given by

$$\begin{aligned}(A*X)_{m,n,k,l} &= \sum_{i=0}^{m}\sum_{j=0}^{n} A_{i,j,k,l}X_{m-i,n-j,k,l}\\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} c^i d^j X_{m-i,n-j,k,l} \quad (\text{when } k=l=0)\\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} c^i d^j x_{m-i-k,n-j-l} \quad (\text{when } m-i\geq k \text{ and } n-j\geq l)\\ &= \sum_{p=0}^{m-k}\sum_{q=0}^{n-l} c^{m-k-p}d^{n-l-q}x_{p,q}.\end{aligned}$$

The last step involves a change of variables: $p=m-i-k$ and $q=n-j-l$. Since the original sums are constrained by $m-i\geq k$ and $n-j\geq l$, the indices i and j only run up to $m-k$ and $n-l$, respectively. Therefore, p and q are guaranteed to be non-negative, with valid ranges $0\leq p\leq m-k$ and $0\leq q\leq n-l$. Now, when we set $k=l=0$, we get

$$(A*X)_{m,n,0,0} = \sum_{p=0}^{m}\sum_{q=0}^{n} c^{m-p}d^{n-q}x_{p,q},$$

and thus, we can express our original equation in terms of these four-dimensional matrices for the case $k=l=0$

$$Y_{m,n,0,0} = X_{m,n,0,0} + \lambda(A*X)_{m,n,0,0}.$$

Moreover, for $k>0$ or $l>0$, both sides of the equality are zero by the definition of our matrices Y, X, and A. Therefore, the equation

$$Y_{m,n,k,l} = X_{m,n,k,l} + \lambda(A*X)_{m,n,k,l}$$

holds for all $m,n,k,l\geq 0$, or concisely in the four-dimensional matrix form $Y = X+\lambda(A*X)$.

Now

$$\begin{aligned}\|A\| &= \sup_{k,l\geq 0}\sum_{i,j=0,0}^{\infty,\infty}|A_{i,j,k,l}| = \sum_{i,j=0,0}^{\infty,\infty}|c|^i|d|^j\\ &= \left(\sum_{i=0}^{\infty}|c|^i\right)\left(\sum_{j=0}^{\infty}|d|^j\right) = \frac{1}{(1-|c|)(1-|d|)}.\end{aligned}$$

We have previously established in Theorem 4.9 that (ℓ_1,ℓ_1) is a Banach algebra under four-dimensional matrix convolution. Therefore, if $|\lambda| < \|A\|^{-1} = (1-|c|)(1-|d|)$, then $I+\lambda A$ has an inverse in (ℓ_1,ℓ_1). When $|\lambda| < (1-|c|)(1-|d|)$, we can solve for X

$$X = (I+\lambda A)^{-1}*Y.$$

Since $Y\in(\ell_1,\ell_1)$ (as $(y_{m,n})\in\ell_1$) and $(I+\lambda A)^{-1}\in(\ell_1,\ell_1)$, we conclude that $X\in(\ell_1,\ell_1)$. By Lemma 5.2, this implies $(x_{m,n})\in\ell_1$, completing the proof. □

Acknowledgements. The authors gratefully acknowledge the anonymous reviewers for their insightful comments and suggestions.

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