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BELL-BASED PARTIALLY DEGENERATE GENOCCHI POLYNOMIALS AND THEIR APPLICATIONS

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Abstract. In this paper, firstly we introduce not only partially degenerate Bell-Genocchi polynomials, but also a new generalization of degenerate Bell-Genocchi polynomials. Secondly, we investigate some behaviors of these polynomials. Furthermore, we establish some implicit summation formulae and symmetry identities by making use of the generating function of partially degenerate Bell-Genocchi polynomials. Finally, some results obtained here extend well-known summations and identities which we stated in the paper.

1. Introduction

Special polynomials and numbers possess much importance in multifarious areas of science such as physics, mathematics, applied sciences, engineering and other related research fields covering differential equations, number theory, functional analysis, quantum mechanics, mathematical analysis, mathematical physics. Some of the most significant polynomials in the theory of special polynomials are the Bell, Euler, Bernoulli, Hermite, and Genocchi polynomials. Recently, many mathematicians namely Carlitz [4, 5], Nadeem *et al.* [26, 27], Khan *et al.* [10-18], and Muhiuddin *et al.* [25] have studied and introduced various degenerate versions of many special polynomials and numbers (like as degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Daehee polynomials, degenerate Fubini polynomials, degenerate Stirling numbers of the first and second kind etc). In this paper, we focus on partially degenerate Bell-Genocchi polynomial and numbers. The aim of this paper is to introduce a partially degenerate version of the Bell-Genocchi polynomials and numbers, the so called partially degenerate Bell-Genocchi polynomials and numbers, constructing from the degenerate exponential function. We derive some explicit expressions and identities for those numbers and polynomials.

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of

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algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|.|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $\bigcup D(\mathbb{Z}_p)$ be the space of $\mathbb{C}_p$-valued uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \bigcup D(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_0(x + p^N\mathbb{Z}_p)$$

$$= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \text{(see [7])}. \tag{1.1}$$

From (1.1), we note that

$$I_0(f_n) - I_0(f) = \sum_{l=0}^{n-1} f'(l), \ (n \in \mathbb{N}), \text{(see [7, 8,9])}. \tag{1.2}$$

For $n \geq 0$, the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \text{(see [19-24])}. \tag{1.3}$$

From (1.3), we see that

$$\frac{1}{r!}(e^t - 1)^r = \sum_{n=r}^{\infty} S_2(n,r)\frac{t^n}{n!}. \tag{1.4}$$

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, each of degree n, are defined, respectively, by the following generating functions (see [1, 2]):

$$\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \ , | \ t \ |< 2\pi \tag{1.5}$$

$$\frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \ , | \ t \ |< \pi \tag{1.6}$$

and

$$\frac{2t}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \ , | \ t \ |< \pi. \tag{1.7}$$

It is easy to see that

$$B_n(0) = B_n \ , \ E_n(0) = E_n, \ G_n(0) = G_n \ (n \in \mathbb{N}).$$

The Daehee polynomials are defined by the generating function

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \quad \text{(see [7])}. \tag{1.8}$$

When $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.

In (2016), Jang *et al.* [11] introduced the partially degenerate Genocchi polynomials which are defined by the generating function

$$\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.9}$$

When $x = 0$, $G_{n,\lambda} = G_{n,\lambda}(0)$ are called the partially degenerate Genocchi numbers.

The Bell polynomials $Bel_n(x)$ are defined by the generating function (see [3, 6])

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}. \tag{1.10}$$

When $x = 1$, $Bel_n = Bel_n(1), (n \geq 0)$ are called the Bell numbers. From (1.2) and (1.9), we note that

$$Bel_n(x) = \sum_{k=0}^{n} S_2(k,n)x^k, (n \geq 0). \tag{1.11}$$

Recently, Duran *et al.* [6] introduced the generalized Bell-Bernoulli polynomials for two variables ${}_{Bel}B_n^{(\alpha)}(x,y)$ defined by

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+y(e^t-1)} = \sum_{n=0}^{\infty} {}_{Bel}B_n^{(\alpha)}(x,y)\frac{t^n}{n!}. \tag{1.12}$$

When $x = y = 0$ in (1.12), ${}_{Bel}B_n^{(\alpha)} = {}_{Bel}B_n^{(\alpha)}(0,0)$ are called the generalized Bell-Bernoulli numbers.
From (1.12), we note that

$$\sum_{n=0}^{\infty} {}_{Bel}B_n^{(\alpha)}(x,y)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+y(e^t-1)}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_{n-m}^{(\alpha)} Bel_m(x,y)\right)\frac{t^n}{n!}. \tag{1.13}$$

Comparing the coefficients of above equation, we get

$${}_{Bel}B_n^{(\alpha)}(x,y) = \sum_{m=0}^{n} B_{n-m}^{(\alpha)} Bel_m(x,y).$$

For each $k \in \mathbb{N}_0$, $T_k(n)$ [14] defined by

$$T_k(n) = \sum_{j=0}^{n}(-1)^j j^k \tag{1.14}$$

is called the alternating sum. The exponential generating function for $T_k(n)$ is

$$\sum_{k=0}^{\infty} T_k(n)\frac{t^k}{k!} = \frac{1-(-e^t)^{(n+1)}}{e^t+1}. \tag{1.15}$$

Inspired and motivated by [6], in this paper, we introduce not only partially degenerate Bell-Genocchi polynomials but also a new generalization of partially degenerate Bell-Genocchi polynomials and then give some of their applications. We also derive some implicit summation formula and general symmetry identities. For obtaining implicit summation formula and general symmetry identities, we use the proof techniques of Khan *et al.* [13, 14].

2. Bell-based partially degenerate Genocchi polynomials

In this section, we assume that $\lambda, t \in \mathbb{C}_p$ with $\mid \lambda t \mid_p < p^{-\frac{1}{p-1}}$. The partially degenerate Bell-Genocchi polynomials are defined by the generating function as

$$\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{xt+y(e^t-1)} = \sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!}. \tag{2.1}$$

When $x = y = 0$ in (2.1), ${}_{Bel}G_{n,\lambda} = {}_{Bel}G_{n,\lambda}(0,0)$ are called the partially degenerate Bell-Genocchi numbers.

Theorem 2.1. For $n \geq 0$, we have

$$ {}_{Bel}G_{n,\lambda}(x,y) = \sum_{m=0}^{n}\binom{n}{m}G_{n-m,\lambda}Bel_m(x,y). \tag{2.2}$$

Proof. From (2.1), we note that

$$\sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{xt+y(e^t-1)}$$

$$= \left(\sum_{n=0}^{\infty} G_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} Bel_m(x,y)\frac{t^m}{m!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}G_{n-m,\lambda}Bel_m(x,y)\right)\frac{t^n}{n!}. \tag{2.3}$$

Comparing the coefficients of t, we obtain (2.2).

□

Theorem 2.2 For $n \geq 0$, we have

$$ {}_{Bel}G_{n,\lambda}(x,y) = \sum_{m=0}^{n}\binom{n}{m}\frac{(-\lambda)^m}{m+1}m!{}_{Bel}G_{n-m}(x,y). \tag{2.4}$$

Proof. From (2.1), we have

$$\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{xt+y(e^t-1)} = \frac{\log(1+\lambda t)}{\lambda t}\frac{2t}{e^t+1}e^{xt+y(e^t-1)}$$

$$= \left(\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}(\lambda t)^m\right)\left(\sum_{n=0}^{\infty} {}_{Bel}G_n(x,y)\frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\frac{(-\lambda)^m}{m+1}m!{}_{Bel}G_{n-m}(x,y)\right)\frac{t^n}{n!}. \tag{2.5}$$

In view of (2.1) and (2.4), we get the required theorem. □

Theorem 2.3. For $n \geq 0$, we have

$$ {}_{Bel}G_{n,\lambda}(x,y) = \sum_{m=0}^{n}\binom{n}{m}\lambda^m D_{m\,Bel}G_{n-m}(x,y). \tag{2.6}$$

Proof. From (1.7) and (2.1), we have

$$\begin{aligned}\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{xt+y(e^t-1)} &= \frac{\log(1+\lambda t)}{\lambda t}\frac{2t}{e^t+1}e^{xt+y(e^t-1)} \\ &= \left(\sum_{m=0}^{\infty} D_m \frac{(\lambda t)^m}{m!}\right)\left(\sum_{n=0}^{\infty} {}_{Bel}G_n(x,y)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\lambda^m D_m {}_{Bel}G_{n-m}(x,y)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.7}$$

By (2.1) and (2.7), we get the desired result. □

Theorem 2.4. For $n \geq 1$, we have

$$ {}_{Bel}G_{n,\lambda}(x,y) = n\sum_{m=0}^{n-1}\binom{n-1}{m}\lambda^m D_m {}_{Bel}E_{n-m-1}(x,y). \tag{2.8}$$

Proof. From (2.1), we observe that

$$\begin{aligned}\sum_{n=1}^{\infty} {}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!} &= t\frac{\log(1+\lambda t)}{\lambda t}\frac{2}{e^t+1}e^{xt+y(e^t-1)} \\ &= t\left(\sum_{m=0}^{\infty} D_m\frac{(\lambda t)^m}{m!}\right)\left(\sum_{n=0}^{\infty} {}_{Bel}E_n(x,y)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\lambda^m D_m {}_{Bel}E_{n-m}(x,y)\right)\frac{t^{n+1}}{n!} \\ &= \sum_{n=1}^{\infty}\left(\sum_{m=0}^{n-1} n\binom{n-1}{m}\lambda^m D_m {}_{Bel}E_{n-m-1}(x,y)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.9}$$

Therefore, by (2.1) and (2.9), we get the desired result. □

Theorem 2.5. For $n \geq 0$, we have

$$ {}_{Bel}G_{n,\lambda}(x+1,y) = \sum_{m=0}^{n}\binom{n}{m} {}_{Bel}G_{n-m,\lambda}(x,y). \tag{2.10}$$

Proof. Using the generating function (2.1), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(x+1,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!} \\ &= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{(x+1)t+y(e^t-1)} - \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{xt+y(e^t-1)} \\ &= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{1+e^t}e^{xt+y(e^t-1)}(e^t-1) \\ &= \sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{t^m}{m!} - \sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} {}_{Bel}G_{n-m,\lambda}(x,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!}. \end{aligned} \tag{2.11}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the result (2.10). □

Theorem 2.6. For $n \in \mathbb{N} \cup \{0\}$, we have

$$ {}_{Bel}G_{n,\lambda}(x,y) \;=\; \sum_{m=0}^{n} \binom{n}{m} G_{n-m} \sum_{k=0}^{m} \binom{m}{k} \lambda^{m-k} D_{m-k} Bel_k(x,y). \tag{2.12}$$

Proof. Rewriting (2.1) to get

$$\sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!} \;=\; \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} e^{xt+y(e^t-1)} \;=\; \left(\frac{2t}{e^t+1}\right)\left(\frac{\log(1+\lambda t)}{\lambda t}\right) e^{xt+y(e^t-1)}$$

$$= \left(\sum_{n=0}^{\infty} G_n \frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} D_m \frac{(\lambda t)^m}{m!}\right)\left(\sum_{k=0}^{\infty} Bel_k(x,y)\frac{t^k}{k!}\right).$$

An application of manipulation of series yields

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} G_{n-m} \sum_{k=0}^{m}\binom{m}{k} D_{m-k}\lambda^{m-k} Bel_k(x,y)\right)\frac{t^n}{n!}. \tag{2.13}$$

Equating the coefficients of $\frac{t^n}{n!}$ in above equation, we get the result (2.12). □

Theorem 2.7. For $n \geq 0$, we have

$$ {}_{Bel}G_{n,\lambda}(x,y) = d^{n-1}\sum_{a=0}^{d-1} {}_{Bel}G_{n,\frac{\lambda}{d}}\left(\frac{a+x}{d}, y\right). \tag{2.14}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} e^{xt+y(e^t-1)}$$

$$= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1} e^{y(e^t-1)} \sum_{a=0}^{d-1} e^{(a+x)t}$$

$$= \sum_{n=0}^{\infty}\left(d^{n-1}\sum_{a=0}^{d-1} {}_{Bel}G_{n,\frac{\lambda}{d}}\left(\frac{a+x}{d}, y\right)\right)\frac{t^n}{n!}. \tag{2.15}$$

Equating the coefficients of $\frac{t^n}{n!}$ in above equation, we get the result (2.14). □

3. Conclusion

We assume that $d \in \mathbb{N}$ with $d \equiv 1(mod 2)$, let χ be a Dirichlet character with conductor d. The Bell-based generalized partially degenerate Genocchi polynomials attached to χ, are given by the generating function

$$\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a \chi(a) e^{(a+x)t+y(e^t-1)} = \sum_{n=0}^{\infty} {}_{Bel}G_{n,\chi,\lambda}(x,y)\frac{t^n}{n!}. \tag{3.1}$$

When $x = y = 0$ in (3.1), ${}_{Bel}G_{n,\chi,\lambda} = {}_{Bel}G_{n,\chi,\lambda}(0,0)$ are called the generalized partially degenerate Bell-Genocchi numbers attached to χ.
Note that

$$\lim_{\lambda\to 0} {}_{Bel}G_{n,\chi,\lambda}(x,y) = {}_{Bel}G_{n,\chi}(x,y)\ ,\ (n \in \mathbb{N}\cup\{0\}).$$

Theorem 3.1. For $n \in \mathbb{N} \cup \{0\}$, we have

$$ {}_{Bel}G_{n,\chi,\lambda}(x,y) = \sum_{m=0}^{n} \binom{n}{m} \lambda^m D_m \, {}_{Bel}G_{n-m,\chi}(x,y). \tag{3.2}$$

Proof. From (3.1), we have

$$\sum_{n=0}^{\infty} {}_{Bel}G_{n,\chi,\lambda}(x,y)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{(a+x)t+y(e^t-1)}$$
$$= \left(\frac{\log(1+\lambda t)}{\lambda t}\right)\left(\frac{2t}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{(a+x)t+y(e^t-1)}\right)$$
$$= \left(\sum_{m=0}^{\infty} D_m \frac{\lambda^m t^m}{m!}\right)\left(\sum_{n=0}^{\infty} {}_{Bel}G_{n,\chi}(x,y)\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}D_m\lambda^m \, {}_{Bel}G_{n-m,\chi}(x,y)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. □

Theorem 3.2. For $n \in \mathbb{N} \cup \{0\}$, we have

$$ {}_{Bel}G_{n,\chi,\lambda}(x,y) = d^{n-1}\sum_{a=0}^{d-1}(-1)^a\chi(a) \, {}_{Bel}G_{n,\frac{\lambda}{d}}\left(\frac{a+x}{d},y\right). \tag{3.3}$$

Proof. From (3.1), we have

$$\sum_{n=0}^{\infty} {}_{Bel}G_{n,\chi,\lambda}(x,y)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{(a+x)t+y(e^t-1)}$$
$$= \frac{1}{d}\sum_{a=0}^{d-1}(-1)^a\chi(a)\frac{2\log(1+\lambda t)^{\frac{d}{\lambda}}}{e^{dt}+1}e^{\left(\frac{a+x}{d}\right)dt+y(e^t-1)}$$
$$= \frac{1}{d}\sum_{a=0}^{d-1}(-1)^a\chi(a)\sum_{n=0}^{\infty} {}_{Bel}G_{n,\frac{\lambda}{d}}\left(\frac{a+x}{d},y\right)\frac{(dt)^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(d^{n-1}\sum_{a=0}^{d-1}(-1)^a\chi(a) \, {}_{Bel}G_{n,\frac{\lambda}{d}}\left(\frac{a+x}{d},y\right)\right)\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the result (3.3). □

Theorem 3.3. For $n \in \mathbb{N} \cup \{0\}$, we have

$$ {}_{Bel}G_{n,\chi,\lambda}(x,y) = \sum_{m=0}^{n}\binom{n}{m}G_{n-m,\chi,\lambda}(x)Bel_m(y). \tag{3.4}$$

Proof. By (3.1), we observe that

$$\sum_{n=0}^{\infty} {}_{Bel}G_{n,\chi,\lambda}(x,y)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{(a+x)t+y(e^t-1)}$$

$$= \left(\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{(a+x)t}\right)e^{y(e^t-1)}$$

$$= \left(\sum_{n=0}^{\infty} G_{n,\chi,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} Bel_m(y)\frac{t^m}{m!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}G_{n-m,\chi,\lambda}(x)Bel_m(y)\right)\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the result (3.4). □

Theorem 3.4. For $n \in \mathbb{N}\cup\{0\}$, we have

$$ {}_{Bel}G_{n,\chi,\lambda}(x,y) = \sum_{m=0}^{n}\binom{n}{m}G_{n-m,\chi,\lambda}Bel_m(x,y). \tag{3.5}$$

Proof. Using (3.1), we see

$$\sum_{n=0}^{\infty} {}_{Bel}G_{n,\chi,\lambda}(x,y)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{(a+x)t+y(e^t-1)}$$

$$= \left(\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}\right)e^{xt+y(e^t-1)}$$

$$= \left(\sum_{l=0}^{\infty} G_{l,\chi,\lambda}\frac{t^l}{l!}\right)\left(\sum_{n=0}^{\infty} x^n\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} Bel_m(x,y)\frac{t^m}{m!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}G_{n-m,\chi,\lambda}Bel_m(x,y)\right)\frac{t^n}{n!}.$$

Equating the coefficients of t^n on both sides of the above equation, we get the result (3.5). □

4. Summation Formulae

In this section we give implicit formula of partially degenerate Bell-Genocchi polynomials by making use of generating function technique. We start following theorem as.

Theorem 4.1. The following implicit summation formulae for partially degenerate Bell-Genocchi polynomials ${}_{Bel}G_{n,\lambda}(x,y)$ holds true:

$$ {}_{Bel}G_{k+l,\lambda}(z,y) = \sum_{n,p=0}^{k,l}\binom{k}{n}\binom{l}{p}(z-x)^{n+p}\,{}_{Bel}G_{k+l-p-n,\lambda}(x,y). \tag{4.1}$$

Proof. We replace t by $t + u$ and rewrite the generating function (2.1) as

$$\frac{2\log(1+\lambda(t+u))^{\frac{1}{\lambda}}}{e^{(t+u)}+1}e^{y(e^{t+u}-1)} = e^{-x(t+u)}\sum_{k,l=0}^{\infty} {}_{Bel}G_{k+l,\lambda}(x,y)\frac{t^k u^l}{k!l!}, \quad \text{(see [18])}. \tag{4.2}$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$e^{(z-x)(t+u)}\sum_{k,l=0}^{\infty} {}_{Bel}G_{k+l,\lambda}(x,y)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_{Bel}G_{k+l,\lambda}(z,y)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{4.3}$$

On expanding exponential function, (4.3) gives

$$\sum_{N=0}^{\infty}\frac{[(z-x)(t+u)]^N}{N!}\sum_{k,l=0}^{\infty} {}_{Bel}G_{k+l,\lambda}(x,y)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_{Bel}G_{k+l,\lambda}(z,y)\frac{t^k}{k!}\frac{u^l}{l!}, \tag{4.4}$$

which on using formula [18]

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{4.5}$$

in the left hand side becomes

$$\sum_{n,p=0}^{\infty}\frac{(z-x)^{n+p}t^n u^p}{n!p!}\sum_{k,l=0}^{\infty} {}_{Bel}G_{k+l,\lambda}(x,y)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_{Bel}G_{k+l,\lambda}(z,y)\frac{t^k}{k!}\frac{u^l}{l!} \tag{4.6}$$

$$\sum_{n,p=0}^{\infty}\sum_{k,l=0}^{n,p}\frac{(z-x)^{n+p}}{n!p!} {}_{Bel}G_{k+l-n-p,\lambda}(x,y)\frac{t^k}{(k-n)!}\frac{u^l}{(l-p)!}$$
$$= \sum_{k,l=0}^{\infty} {}_{Bel}G_{k+l,\lambda}(z,y)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{4.7}$$

Finally, on equating the coefficients of the powers of t and u in the above equation, we get the required result. □

Remark 4.1. By taking $l = 0$ in equation (4.1), we immediately deduce the following result.

Corollary 4.1. The next implicit summation formulae for partially degenerate Bell-Genocchi polynomials ${}_{Bel}G_{n,\lambda}(x,y)$ holds true:

$${}_{Bel}G_{k,\lambda}(z,y) = \sum_{n=0}^{k}\binom{k}{n}(z-x)^n {}_{Bel}G_{k-n,\lambda}(x,y). \tag{4.8}$$

Remark 4.2. On replacing z by $z + x$ and setting $y = 0$ in Theorem 4.1, we immediately deduce the following result.

Corollary 4.2. The next implicit summation formulae involving partially degenerate Genocchi polynomials $G_{n,\lambda}(x)$ holds true:

$$G_{k+l,\lambda}(z+x) = \sum_{n,p=0}^{k,l}\binom{k}{n}\binom{l}{p}(z)^{n+p}G_{k+l-p-n,\lambda}(x) \tag{4.9}$$

whereas by setting $z = 0$ in Theorem 4.1, we get another result involving partially degenerate Genocchi polynomials of one and two variables.

$$G_{k+l,\lambda}(y) = \sum_{n,p=0}^{k,l} \binom{k}{n} \binom{l}{p} (-x)^{n+p} {}_{Bel}G_{k+l-p-n,\lambda}(x,y). \tag{4.10}$$

Remark 4.3. Along with the above results we will exploit extended forms of partially degenerate Genocchi polynomials by setting $y = 0$ in the Theorem 4.1 to get

$$G_{k+l,\lambda}(z) = \sum_{n,p=0}^{k,l} \binom{k}{n} \binom{l}{p} (z-x)^{n+p} G_{k+l-p-n,\lambda}(x). \tag{4.11}$$

Theorem 4.2. The following implicit summation formulae for partially degenerate Bell-Genocchi polynomials ${}_{Bel}G_{n,\lambda}(x,y)$ holds true:

$${}_{Bel}G_{n,\lambda}(x+z,y+u) = \sum_{m=0}^{n} \binom{n}{m} {}_{Bel}G_{n-m,\lambda}(z,u) Bel_m(x,y). \tag{4.12}$$

Proof. Replacing x by $x + z$ and y by $y + u$ in (2.1), we have

$$\sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(x+z,y+u)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} e^{(x+z)t+(y+u)(e^t-1)}$$

$$= \left(\sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(z,u)\frac{t^n}{n!}\right) \left(\sum_{m=0}^{\infty} Bel_m(x,y)\frac{t^m}{m!}\right).$$

Finally replacing n by $n - m$ and comparing the coefficients of $\frac{t^n}{n!}$, we get the desired result (4.12). □

Theorem 4.3. The following implicit summation formulae for partially degenerate Bell-Genocchi polynomials ${}_{Bel}G_{n,\lambda}(x,y)$ holds true:

$${}_{Bel}G_{n,\lambda}(y,x) = \sum_{l=0}^{n}\sum_{k=0}^{l} \binom{n}{l} G_{n-l,\lambda}(x) y^k S_2(l,k). \tag{4.13}$$

Proof. Using (2.1), we get

$$\sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} e^{xt+y(e^t-1)}$$

$$= \sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!} \sum_{l=0}^{\infty}\sum_{k=0}^{l} y^k S_2(l,k)\frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n}\sum_{k=0}^{l} \binom{n}{l} G_{n-l,\lambda}(x) y^k S_2(l,k)\right) \frac{t^n}{n!}.$$

On comparing the coefficients of t^n, we get (4.13). □

Theorem 4.4. The following implicit summation formulae for partially degenerate Bell-Genocchi polynomials ${}_{Bel}G_{n,\lambda}(x,y)$ holds true:

$${}_{Bel}G_{n,\lambda}(x,y) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m,\lambda}(x-z) Bel_m(z,y). \tag{4.14}$$

Proof. We exploit (2.1) and rewrite (2.1) as

$$\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{(x-z)t}e^{zt+y(e^t-1)}=\sum_{n=0}^{\infty}G_{n,\lambda}(x-z)\frac{t^n}{n!}\sum_{m=0}^{\infty}Bel_m(z,y)\frac{t^m}{m!} \quad (4.15)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}G_{n-m,\lambda}(x-z)Bel_m(z,y)\right)\frac{t^n}{n!}.$$

Finally comparing the coefficients of powers of t in above equation gives the result (4.14). □

Theorem 4.5. The following implicit summation formulae for partially degenerate Bell-Genocchi polynomials ${}_{Bel}G_{n,\lambda}(x,y)$ holds true:

$${}_{Bel}G_{n,\lambda}(x+1,y) = \sum_{m=0}^{n}\binom{n}{m}{}_{Bel}G_{n-m,\lambda}(x,y). \quad (4.16)$$

Proof. Replacing x by $x+1$ in (2.1), we get

$$\sum_{n=0}^{\infty}{}_{Bel}G_{n,\lambda}(x+1,y)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{1+e^t}e^{(x+1)t+y(e^t-1)}$$

$$=\left(\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{xt+y(e^t-1)}\right)e^t$$

$$=\sum_{n=0}^{\infty}{}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{t^m}{m!}.$$

Replacing n by $n-m$ in the above equation and then comparing the coefficients of $\frac{t^n}{n!}$, we get the desired result (4.16). □

Theorem 4.6. The following implicit summation formulae for partially degenerate Bell-Genocchi polynomials ${}_{Bel}G_{n,\lambda}(x,y)$ holds true:

$${}_{Bel}G_{n,\lambda}(x+1,y)+{}_{Bel}G_{n,\lambda}(x,y)=2n\sum_{m=0}^{n-1}\binom{n-1}{m}\frac{(-\lambda)^m m!}{m+1}Bel_{n-1-m}(x,y). \quad (4.17)$$

Proof. Using the generating function (2.1), we have

$$\sum_{n=0}^{\infty}{}_{Bel}G_{n,\lambda}(x+1,y)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_{Bel}G_{n,\lambda}(x,y)\frac{t^n}{n!}=\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{(x+1)t+y(e^t-1)}+\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{xt+y(e^t-1)}$$

$$=2\log(1+\lambda t)^{\frac{1}{\lambda}}e^{xt+y(e^t-1)}$$

$$=2t\left(\frac{\log(1+\lambda t)}{\lambda t}\right)e^{xt+y(e^t-1)}$$

$$=2t\left(\sum_{m=0}^{\infty}\frac{(-1)^m(\lambda t)^m}{m+1}\right)\left(\sum_{n=0}^{\infty}Bel_n(x,y)\frac{t^n}{n!}\right)$$

$$=2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{(-\lambda)^m m!}{m+1}Bel_{n-m}(x,y)\frac{t^{n+1}}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in above equation, we get the result (4.17). □

5. Symmetry identities

Recently Khan *et al.* [13, 14] have established some interesting symmetry identities for various polynomials. Here, we present certain symmetry identities for the partially degenerate Bell-Genocchi polynomials ${}_{Bel}G_{n,\lambda}(x,y)$ in the following form.

Theorem 5.1. For each pair of integers a and b and all integers $n \geq 0$, the following symmetry identity holds true:

$$\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} {}_{Bel}G_{n-m,\lambda}(bx,y) {}_{Bel}G_{m,\lambda}(ax,y)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} {}_{Bel}G_{n-m,\lambda}(ax,y) {}_{Bel}G_{m,\lambda}(bx,y). \tag{5.1}$$

Proof. Let

$$A(t) = \frac{(2\log(1+\lambda t)^{\frac{a}{\lambda}})(2\log(1+\lambda t)^{\frac{b}{\lambda}})}{(e^{at}+1)(e^{bt}+1)} e^{2abxt+y(e^{at}-1)+y(e^{bt}-1)}. \tag{5.2}$$

Then the expression $A(t)$ is symmetric in a and b and can be expressed into series in two ways to obtain

$$A(t) = \sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(bx,y) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_{Bel}G_{m,\lambda}(ax,y) \frac{(bt)^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} {}_{Bel}G_{n-m,\lambda}(bx,by) {}_{Bel}G_{m,\lambda}(ax,y) \right) \frac{t^n}{n!}. \tag{5.3}$$

Similarly, $A(t)$ can be written as

$$A(t) = \sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(ax,y) \frac{(bt)^n}{n!} \sum_{m=0}^{\infty} {}_{Bel}G_{m,\lambda}(bx,y) \frac{(at)^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} {}_{Bel}G_{n-m,\lambda}(ax,y) {}_{Bel}G_{m,\lambda}(bx,y) \right) \frac{t^n}{n!}. \tag{5.4}$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we get the identity (5.1). □

Remark 5.1. Replacing $y = 0$ in Theorem 5.1, we get

Corollary 5.1. Let $a, b > 0$ with $a \neq b$ and $x \in \mathbb{R}$ and $n \geq 0$, the following symmetry identity holds true:

$$\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} G_{n-m,\lambda}(bx) G_{m,\lambda}(ax)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} G_{n-m,\lambda}(ax) G_{m,\lambda}(bx). \tag{5.5}$$

Remark 5.2. Replacing $b = 1$ in Theorem 5.1, we get

Corollary 5.2. Let $a, b > 0$ with $a \neq b$ and $x, y \in \mathbb{R}$ and $n \geq 0$, the following symmetry identity holds true:

$$\sum_{m=0}^{n} \binom{n}{m} a^{n-m} {}_{Bel}G_{n-m,\lambda}(bx, y) {}_{Bel}G_{m,\lambda}(ax, y)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^{m} {}_{Bel}G_{n-m,\lambda}(ax, y) {}_{Bel}G_{m,\lambda}(x, y). \tag{5.6}$$

Theorem 5.2. For each pair of integers a and b and all integers $n \geq 1$, the following symmetry identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} {}_{Bel}G_{n-k,\lambda}\left(bx_1 + \frac{b}{a} i + j, y\right) {}_{Bel}G_{k,\lambda}(ax_2, y)$$

$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-1)^{i+j} {}_{Bel}G_{n-k,\lambda}\left(ax_1 + \frac{a}{b} i + j, y\right) {}_{Bel}G_{k,\lambda}(bx_2, y). \tag{5.7}$$

Proof. Consider

$$B(t) = \frac{(2\log(1+\lambda t)^{\frac{a}{\lambda}})(2\log(1+\lambda t)^{\frac{b}{\lambda}})(e^{abt}+1)^2}{(e^{at}+1)^2(e^{bt}+1)^2} e^{ab(x_1+x_2)t + y(e^{at}-1) + y(e^{bt}-1)} \tag{5.8}$$

$$= \frac{2\log(1+\lambda t)^{\frac{a}{\lambda}}}{e^{at}+1} e^{abx_1 t + y(e^{at}-1)} \left(\frac{e^{abt}+1}{e^{bt}+1}\right) \frac{2\log(1+\lambda t)^{\frac{b}{\lambda}}}{e^{bt}+1} e^{abx_2 t + y(e^{bt}-1)} \left(\frac{e^{abt}+1}{e^{at}+1}\right)$$

$$= \frac{2\log(1+\lambda t)^{\frac{a}{\lambda}}}{e^{at}+1} e^{abx_1 t + y(e^{at}-1)} \left(\sum_{i=0}^{a-1} (-1)^i e^{bti}\right) \frac{2\log(1+\lambda t)^{\frac{b}{\lambda}}}{e^{bt}+1} e^{abx_2 t + y(e^{bt}-1)} \left(\sum_{j=0}^{b-1} (-1)^j e^{atj}\right)$$

$$= \frac{2\log(1+\lambda t)^{\frac{a}{\lambda}}}{e^{at}+1} e^{abx_1 t + y(e^{at}-1)} \left(\sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} e^{(bx + \frac{b}{a} i + j)at}\right) \left(\sum_{k=0}^{\infty} {}_{Bel}G_{k,\lambda}(ay, y) \frac{(bt)^k}{k!}\right)$$

$$= \left(\sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} {}_{Bel}G_{n,\lambda}\left(bx_1 + \frac{b}{a} i + j, y\right) \frac{(at)^n}{n!}\right) \left(\sum_{k=0}^{\infty} {}_{Bel}G_{k,\lambda}(ax_2, y) \frac{(bt)^k}{(k)!}\right),$$

$$B(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} {}_{Bel}G_{n-k,\lambda}\left(bx_1 + \frac{b}{a} i + j, y\right) {}_{Bel}G_{k,\lambda}(ax_2, y)\right) \frac{t^n}{n!}. \tag{5.9}$$

On the other hand, we have

$$B(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-1)^{i+j} {}_{Bel}G_{n-k,\lambda}\left(ax_2 + \frac{a}{b} i + j, y\right) {}_{Bel}G_{k,\lambda}(bx_2, y)\right) \frac{t^n}{n!}. \tag{5.10}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result. □

Theorem 5.3. For each pair of integers a and b and all integers $n \geq 0$, the following symmetry identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} {}_{Bel}G_{n-k,\lambda} \left(bx_1 + \frac{b}{a} i, y \right) {}_{Bel}G_{k,\lambda} \left(ax_2 + \frac{a}{b} j, y \right)$$
$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-1)^{i+j} {}_{Bel}G_{n-k,\lambda} \left(ax_1 + \frac{a}{b} i, y \right) {}_{Bel}G_{k,\lambda} \left(bx_2 + \frac{b}{a} j, y \right). \tag{5.11}$$

Proof. The proof is analogous to Theorem 5.2 but we need to write equation (5.8) in the form

$$C(t) = \left(\frac{2\log(1+\lambda t)^{\frac{a}{\lambda}}}{e^{at}+1} e^{abx_1 t + y(e^{at}-1)} \sum_{i=0}^{a-1} (-1)^i e^{bti} \right) \left(\frac{2\log(1+\lambda t)^{\frac{b}{\lambda}}}{e^{bt}+1} e^{abx_2 t + y(e^{bt}-1)} \sum_{j=0}^{b-1} (-1)^j e^{atj} \right)$$

$$C(t) = \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} \sum_{n=0}^{\infty} a^n {}_{Bel}G_{n,\lambda} \left(bx_1 + \frac{b}{a} i, y \right) \frac{t^n}{n!} \sum_{k=0}^{\infty} b^k {}_{Bel}G_{k,\lambda} \left(ax_2 + \frac{a}{b} j, y \right) \frac{t^k}{k!}. \tag{5.12}$$

On the other hand "equation" (5.8) can be shown equal to

$$C(t) = \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-1)^{i+j} \sum_{n=0}^{\infty} b^n {}_{Bel}G_{n,\lambda} \left(ax_1 + \frac{a}{b} i, y \right) \frac{t^n}{n!} \sum_{k=0}^{\infty} a^k {}_{Bel}G_{k,\lambda} \left(bx_2 + \frac{b}{a} j, y \right) \frac{t^k}{k!}. \tag{5.13}$$

Next making change of index and by equating the coefficients of $\frac{t^n}{n!}$ to zero in (5.12) and (5.13), we get the result (5.11). □

Theorem 5.4. For each pair of integers a and b and all integers $n \geq 0$, the following symmetry identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k} {}_{Bel}G_{n-k,\lambda} (bx_1, y) \sum_{i=0}^{k} \binom{k}{i} T_i(a-1) {}_{Bel}G_{k-i,\lambda}(ax_2, y)$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^{k} b^{n-k} {}_{Bel}G_{n-k,\lambda} (ax_1, y) \sum_{i=0}^{k} \binom{k}{i} T_i(b-1) {}_{Bel}G_{k-i,\lambda}(bx_2, y), \tag{5.14}$$

where the sum of alternative integer powers $T_k(n)$ is given by (1.15).

Proof. We now use

$$D(t) = \frac{(2\log(1+\lambda t)^{\frac{a}{\lambda}})(2\log(1+\lambda t)^{\frac{b}{\lambda}})(1-(-e^{bt})^a) e^{ab(x_1+x_2)t + y(e^{at}-1) + y(e^{bt}-1)}}{(e^{at}+1)(e^{bt}+1)^2},$$

to find that

$$D(t) = \frac{(2\log(1+\lambda t)^{\frac{a}{\lambda}})}{e^{at}+1} e^{abx_1 t + y(e^{at}-1)} \left(\frac{1-(-e^{bt})^a}{e^{bt}+1} \right) \frac{(2\log(1+\lambda t)^{\frac{b}{\lambda}})}{e^{bt}+1} e^{abx_2 t + y(e^{bt}-1)}$$
$$= \left(\sum_{n=0}^{\infty} {}_{Bel}G_{n,\lambda}(bx_1, y) \frac{(at)^n}{n!} \right) \left(\sum_{i=0}^{\infty} T_i(a-1) \frac{(bt)^i}{i!} \right) \left(\sum_{k=0}^{\infty} {}_{Bel}G_{k,\lambda}(ax_2, y) \frac{(bt)^k}{k!} \right).$$

Using a similar plan, we get

$$D(t)=\left(\sum_{n=0}^{\infty}{}_{Bel}G_{n,\lambda}(ax_1,y)\frac{(bt)^n}{n!}\right)\left(\sum_{i=0}^{\infty}T_i(b-1)\frac{(at)^i}{i!}\right)\left(\sum_{k=0}^{\infty}{}_{Bel}G_{k,\lambda}(bx_2,y)\frac{(at)^k}{k!}\right).$$

Finally (5.14) follows after an appropriate change of summation index and comparison of the coefficients of $\frac{t^n}{n!}$. □

6. Conclusion

Motivated by importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis and other fields of applied mathematics, various special numbers and polynomials, and their variants and generalizations have been extensively investigated (for example, see the references here and those cited therein). The results presented here, being very general, can be specialized to yield a large number of identities involving known or new simpler numbers and polynomials. For example, the case $y=0$ of the results presented here give the corresponding ones for the generalized partially degenerate Genocchi polynomials [8].

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References

[1] N. Alam, W. A. Khan, C. S. Ryoo, *A note on Bell-based Apostol-type Frobenius-Euler polynomials of a complex variable with its certain applications*, Mathematics, **2022**, 10, 2109. https:// doi.org/10.3390/math10122109.

[2] A. Al E'damat, W. A. Khan, C. S. Ryoo, *Certain properties on Bell based Apostol-Frobenius-Genocchi polynomials of complex variables*, J. Math. Comp. Sci., **2024**, 33(3), 326-338.

[3] E. T. Bell, *Exponential polynomials*, Ann. of Math., **1934**, 35, 258-277.

[4] L. Carlitz, *Degenerate Stirling Bernoulli and Eulerian numbers*, Util. Math., **1979**, 15, 51-88.

[5] L. Carlitz, *A degenerate Staud-Clausen theorem*, Arch. Math., **1956**, 7, 28-33.

[6] U. Duran, S. Araci, M. Acikgoz, *Bell-based Bernoulli polynomials with applications*, Axioms, **2021**, 10, 29, 1-23.

[7] H. Haroon, W. A. Khan, *Degenerate Bernoulli numbers and polynomials associated with degenerate Hermite polynomials*, Commun. Korean Math. Soc., **2018**, 33, 651-669.

[8] W. A. Khan and M. Ahmad , *Partially degenerate poly-Bernoulli polynomials associated with Hermite polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **2018**, 28(3), 487-496.

[9] W. A. Khan, *A new class of degenerate Frobenius-Euler-Hermite polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **2018**, 28(4), 567-576.

[10] W. A. Khan, M. Acikgoz, U. Duran, *Note on the type 2 degenerate multi-poly-Euler polynomials*, Symmetry. **2020**, 12, 1691; doi:10.3390/sym12101691.

[11] L-.C. Jang, H. I. Kwon, J. G. Lee, C. S. Ryoo, *On the generalized partially degenerate Genocchi polynomials*, Global J. Pure Appl. Mathe., **2015**, vol. 11, no. 6, pp. 4789-4799.

[12] W. A. Khan, W. A. M. A. Pathan, *On generalized Lagrange Hermite-Bernoulli and related polynomials*, Acta Comm. Univ. Tart. Math., **2019**, 23, no. 2, 211-224.

[13] W. A. Khan, *On generalized Lagrange-based Apostol type and related polynomials*, Kragujevac J. Math., **2022**, 46, no. 6, 865-882.

[14] W. A. Khan, K. S. Nisar, M. Acikgoz, U. Duran, U, A. H. Abusufian, *On unified Gould-Hopper based Apostol-type polynomials*, J. Math. Comput. Sci., **2022**, vol. 24, no. 4, pp. 287-298.

[15] W. A. Khan, M. S. Alatawi, U. Duran, U, *Applications, and properties of bivariate Bell-based Frobenius-type Eulerian polynomials.*, J. Funct. Spaces, **2023**. Volume 2023, Article ID 5205867, 10 pages.

[16] W. A. Khan, M. S. Alatawi, *A note on modified degenerate Changhee-Genocchi polynomials of the second kind*, Symmetry, **2023**, 15, 136. https://doi.org/10.3390/sym15010136.

[17] W. A. Khan, M. S. Alatawi, *Analytical properties of degenerate Genocchi polynomials of the second kind and some of their applications*, Symmetry, **2022**, 14, 1500. https://doi.org/10.3390/sym14081500.

[18] W. A. Khan, V. Yadav, *A study on q-analogue of degenerate Changhee numbers and polynomials*, Southeast Asian Journal of Mathematics and Mathematical Sciences, **2023**, 19(1), 29-42.

[19] W. A. Khan, *A note on q-analogues of degenerate Catalan-Daehee numbers and polynomials*, Journal of Mathematics, **2022**, Volume 2022, Article ID 9486880, 9 pages.

[20] W. A. Khan, *A note on q-analogue of degenerate Catalan numbers associated p-adic integral on $\mathbb{Z}_p$*, Symmetry, **2022**, 14, 1119. https://doi.org/10.3390/sym14061119.

[21] W. A. Khan, *A study on q-analogue of degenerate $\frac{1}{2}$-Changhee numbers and polynomials*, Southeast Asian Journal of Mathematics and Mathematical Sciences, **2022**, 18(2), 1-12.

[22] W. A. Khan, H. Haroon, *Higher-order degenerate Hermite-Bernoulli arising from p-adic integral on $\mathbb{Z}_p$*, Iranian Journal of Mathematical Sciences and Informatics, **2022**, 17(2), 171-189.

[23] W. A. Khan, J. Younis, U. Duran, A. Iqbal, *The higher-order type Daehee polynomials associated with p-adic integrals on $\mathbb{Z}_p$*, Applied Mathematics in Science and Engineering, **2022**, 30(1), 573-582.

[24] W. A. Khan, M. Kamarujjama, Daud,, *Construction of partially degenerate Bell-Bernoulli polynomials of the first kind*, Analysis (Germany), **2022**, 43(3), 171-184.

[25] G. Muhiuddin, W. A. Khan, J. Younis, *Construction of type 2 poly-Changhee polynomials and its applications*, Journal of Mathematics, **2021**, Vol. 2021, Article ID 7167633, 9 pages.

[26] M. Nadeem, W. A. Khan, *Symmetric identities for degenerate q-Genocchi numbers and polynomials*, Southeast Asian Journal of Mathematics and Mathematical Sciences, **2023**, 19(1), 17-28.

[27] M. Nadeem, W. A. Khan, K. A. H. Alzobydi, C. S. Ryoo, M. Shadab, R. Ali, *Certain properties on Bell-based Apostol-type Frobenius-Genocchi polynomials and its applications*, Adv. Math. Models Appl., **2023**, vol. 1, no. 8, 92-107.

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