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SOME RESULTS ON WOVEN FRAMES IN QUATERNIONIC HILBERT SPACES

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Abstract. In this paper, we obtain some new properties of weaving frames and present some conditions under which a family of frames is woven in quaternionic Hilbert spaces. Some characterizations of weaving frames in terms of operators are given. We also give a condition associated with synthesis operators of frames such that the sequence of frames is woven. Finally, for a family of woven frames, we show that they are stable under invertible operators and small perturbations.

1. Introduction

Frames [9] in Hilbert spaces were introduced in 1952 while studying the non-harmonic Fourier Series. But their potential was realized by the researchers after the work done by Daubechies et al.[8], due to its vast applications in various fields like signal and image processing, sigma-delta quantization, filter bank theory, sampling theory and wireless communication, for details one may refer to [7]. Over the past few years, many generalizations of frames were introduced and studied, fusion frames were defined and studied by P. Casazza and G. Kutyniok [5] and K-frames were introduced and studied by *Găvruta* [12]. X. C. Xiao, M. L. Ding and Yu Can Zhu [16] have defined and studied the K-fusion frames in Hilbert spaces. In [4] Bemrose et al. have defined and studied the properties of weaving frames in Hilbert spaces which are used in distributed signal processing. For details regarding fusion frames and woven fusion frames one may refer to [6, 11] and for details regarding weaving K-frames and their properties one may refer to [2, 22, 23]. Hamilton discovered the field of quaternion which is a generalization of complex numbers. It is a four-dimensional non-commutative real algebra. Quaternions are used to study rotation in the higher dimensional Euclidean spaces, theory of relativity, Newtonian and quantum mechanics and general relativity in which Lorentz transformation is given in terms of quaternions. For details regarding quaternions see [1]. In [13] R. Ghiloni et al. have extended the continuous functional calculus in the

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case of quaternionic Hilbert spaces. They also observed that every separable right quaternionic Hilbert space has an orthonormal basis (see Proposition 2.6 in [13]).

Khokulan et al. [14] have defined and studied frames in finite dimensional quaternionic Hilbert spaces. In [17], Sharma and Goel have studied frames in separable right quaternionic Hilbert spaces. In [10], H. Ellouz studied K-frames in right quaternionic Hilbert spaces. Recently in [3] Ruchi et al. have defined OPV-frames in right quaternionic Hilbert spaces and woven frames in right quaternionic Hilbert spaces were studied in [18]. For details regarding woven fusion frames, woven K-frames and woven K-fusion frames in quaternionic Hilbert spaces, one may refer to [19, 20, 21].

In this paper, we propose certain conditions under which a family of frames is woven in quaternionic Hilbert spaces and get some novel features of weaving frames. Also, few operator-based characterizations of weaving frames are given. Additionally, we provide a prerequisite linked to the synthesis operators of frames, guaranteeing that the series of frames is interwoven. Lastly, we demonstrate the stability of a family of woven frames under small perturbations and invertible operators.

Outline of the Paper: The paper is divided into three sections. Section 1 deals with introduction of the topic whereas in Section 2, we reviewed the basic definitions of frames and woven frames in the right quaternionic Hilbert spaces. Also, we stated some known results which will be used throughout the paper. Section 3 consists of the main results. In this section, some characterization of woven frames are given. Further, for the stability of woven frames systems, we proved some results related to small perturbation of given families of sequences in $\mathbb{H}^R(\mathfrak{Q})$.

2. Preliminaries

Throughout this paper, $\mathfrak{Q}$ will denote the non-commutative field of quaternions, $\mathcal{J}$ a countable set, and $[m]$ the collection of the first m natural numbers. $\mathbb{H}^R(\mathfrak{Q})$ a separable right quaternionic Hilbert space. By the term "right linear operator", we mean a "right $\mathfrak{Q}$-linear operator (i.e., a right linear operator T on $\mathbb{H}^R(\mathfrak{Q})$ is a mapping $T : \mathbb{H}^R(\mathfrak{Q}) \to \mathbb{H}^R(\mathfrak{Q})$ such that $T(uq_1 + vq_2) = T(u)q_1 + T(v)q_2,\ u, v \in \mathbb{H}^R(\mathfrak{Q}),\ q_1, q_2 \in \mathfrak{Q}$)". $\mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ denotes the set of all bounded (right $\mathfrak{Q}$-linear) operators on $\mathbb{H}^R(\mathfrak{Q})$. Quaternions are four dimensional non-commutative real algebra generated by $1, i, j, k$ where i, j, k are called imaginary units. They are an extension of the complex number $\mathbb{C}$ and operations on $\mathbb{C}$ are those of $\mathfrak{Q}$ restricted over $\mathbb{C}$, for operation and various properties of quaternions see [13].

For the definition of a right quaternionic Hilbert space one may refer to [13], for the definition of frame and Bessel sequence in a separable right quaternionic Hilbert space see [17], and for terminology related to woven frames and related concepts in right quaternionic Hilbert space one may consult [18].

For details concerning frame operators, perturbation and dual frames see [17].

Let $\mathfrak{F} = \{\{u_{ij}\}_{j\in\mathcal{J}} :\ i \in [m]\}$ be a family of frames for $\mathbb{H}^R(\mathfrak{Q})$. Then $\mathfrak{F}$ is said to be *woven frame* if there are positive real numbers r_1 and r_2 so that for every partition $P = \{\sigma_i\}_{i\in[m]}$ of $\mathcal{J}$, the family $\mathfrak{F}_P = \{u_{ij}\}_{j\in\sigma_i, i\in[m]}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper frame bounds r_1 and r_2 respectively. Each family $\mathfrak{F}_P$ is called weaving frame. If every weaving sequence is a Bessel sequence, then the family $\mathfrak{F}$ is called a woven Bessel sequence of $\mathbb{H}^R(\mathfrak{Q})$. For details regarding analysis, synthesis and frame operator corresponding to a woven frame see [18].

In the following result, Sharma et al. [18] proved that a family of Bessel sequences collectively forms a woven Bessel sequence.

Proposition 2.1. [18] *Consider a family of Bessel sequences* $\mathfrak{F} = \{\{u_{ij}\}_{j\in\mathcal{J}} : i \in [m]\}$ *of* $\mathbb{H}^R(\mathfrak{Q})$ *with bounds* r_{2i}. *Then for any partition* $P = \{\sigma_i\}_{i\in[m]}$ *of* $\mathcal{J}$, *the family* $\{u_{ij}\}_{j\in\sigma_i, i\in[m]}$ *is a Bessel sequence with bound* $\sum\limits_{i\in[m]} r_{2i}$.

The concept of woven frames is inspired by the principles of distributed signal processing and holds significant potential for applications in wireless sensor networks, particularly in scenarios requiring distributed processing across multiple frames. Additionally, woven frames are relevant for the pre-processing of signals using Gabor frames.

Consider a scenario in image processing where two sets of filters, $F = \{f_j\}_{j\in\mathcal{J}}$ and $G = \{g_j\}_{j\in\mathcal{J}}$, are used for feature extraction. For each image, we can apply either filter f_j or filter g_j to extract features. The extracted features are then represented by the set $\{\langle f|f_j\rangle\}_{j\in\sigma} \cup \{\langle f|g_j\rangle\}_{j\in\sigma^c}$ for some subset $\sigma \subset \mathcal{J}$. If the combined set of filters $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ still provides a comprehensive feature set, then the image f can be accurately reconstructed from these features. In this context, we describe the filter sets F and G as woven filters. Some interesting results on weaving frames are proved by D. Li [15] .

Extending concepts such as frames and woven frames to quaternionic Hilbert spaces is motivated by the unique mathematical properties and applications of quaternions, which differ from those of real or complex numbers. Quaternions are widely used in areas such as quantum mechanics, signal processing, and computer graphics because they provide a more compact and stable representation of rotations and orientations in three-dimensional space. Extending frame theory to quaternionic Hilbert spaces allows for the direct application of these concepts to problems involving rotational symmetries and other phenomena naturally modeled by quaternions. Woven frames in quaternionic Hilbert spaces could lead to more robust methods for signal recovery, particularly in environments where noise and other perturbations are quaternionic in nature.

3. Main Results

We begin this section with the following result that provide a necessary and sufficient condition for the weaving of two woven frames to be a tight frame in a right quaternionic Hilbert space.

Proposition 3.1. *Let two frames* $F = \{f_j\}_{j\in\mathcal{J}}$ *and* $G = \{g_j\}_{j\in\mathcal{J}}$ *be woven with synthesis operators* T_F *and* T_G, *respectively. For any* $\sigma \subset \mathcal{J}$, *a weaving* $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ *is an A-tight frame for* $\mathbb{H}^R(\mathfrak{Q})$ *if and only if* $T_F D_\sigma T_F^* + T_G D_{\sigma^c} T_G^* = AI_{\mathbb{H}^R(\mathfrak{Q})}$, *where* D_σ *is a* $|\mathcal{J}| \times |\mathcal{J}|$ *diagonal matrix with* $d_{jj} = 1$ *for* $j \in \sigma$ *and* 0 *otherwise,* D_{σ^c} *is a* $|\mathcal{J}| \times |\mathcal{J}|$ *diagonal matrix with* $d_{jj} = 1$ *for* $j \in \sigma^c$ *and* 0 *otherwise* .

Proof. For any $\sigma \subset \mathcal{J}$, then the synthesis operator of the weaving frame $W = \{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ is $T_F D_\sigma + T_G D_{\sigma^c}$. Then the frame operator

$$\begin{aligned} S_W =& (T_F D_\sigma + T_G D_{\sigma^c})(T_F D_\sigma + T_G D_{\sigma^c})^* \\ =& T_F D_\sigma T_F^* + T_F D_\sigma D_{\sigma^c} T_G^* + T_G D_{\sigma^c} D_\sigma T_F^* + T_G D_{\sigma^c} T_G^* \\ =& T_F D_\sigma T_F^* + T_G D_{\sigma^c} T_G^*. \end{aligned}$$ □

We can equivalently express D_σ in Proposition 3.1 as a mapping $D_\sigma : l^2(\mathfrak{Q}) \to l^2(\mathfrak{Q})$, where it acts on a sequence by mapping each component to itself if it belongs to σ, and to zero otherwise.

Next, we prove that under certain conditions, the canonical dual frames associated with woven frames also form woven frames.

Proposition 3.2. *Let two frames $F = \{f_j\}_{j\in\mathcal{J}}$ and $G = \{g_j\}_{j\in\mathcal{J}}$ be a woven frame of $\mathbb{H}^R(\mathfrak{Q})$ with universal constants A, B and frame operators S_F and S_G, respectively. If $||S_F||\,||S_F^{-1} - S_G^{-1}|| < \sqrt{\frac{A}{B}}$ (or $||S_G||\,||S_F^{-1} - S_G^{-1}|| < \sqrt{\frac{A}{B}}$), then $S_F^{-1}F = \{S_F^{-1}f_j\}_{j\in\mathcal{J}}$ and $S_G^{-1}G = \{S_G^{-1}g_j\}_{j\in\mathcal{J}}$ forms a woven frame for $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. We only consider the case of $||S_F||\,||S_F^{-1} - S_G^{-1}|| < \sqrt{\frac{A}{B}}$. Now for every $\sigma \subset \mathcal{J}$ and each $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned} &\Big(\sum_{j\in\sigma} |\langle S_F^{-1} f_j | f\rangle|^2 + \sum_{j\in\sigma^c} |\langle S_G^{-1} g_j | f\rangle|^2\Big)^{\frac{1}{2}} \\ &= \Big(\sum_{j\in\sigma} |\langle f_j | S_F^{-1} f\rangle|^2 + \sum_{j\in\sigma^c} |\langle g_j | S_G^{-1} f\rangle|^2\Big)^{\frac{1}{2}} \\ &= \Big(\sum_{j\in\sigma} |\langle f_j | S_F^{-1} f\rangle|^2 + \sum_{j\in\sigma^c} |\langle g_j | S_F^{-1} f + (S_G^{-1} - S_F^{-1}) f\rangle|^2\Big)^{\frac{1}{2}} \\ &\geq \Big(\sum_{j\in\sigma} |\langle f_j | S_F^{-1} f\rangle|^2 + \sum_{j\in\sigma^c} |\langle g_j | S_F^{-1} f\rangle|^2\Big)^{\frac{1}{2}} - \Big(\sum_{j\in\sigma^c} |\langle g_j | (S_G^{-1} - S_F^{-1}) f\rangle|^2\Big)^{\frac{1}{2}} \\ &\geq \sqrt{A}||S_F^{-1} f|| - \Big(\sum_{j\in\sigma^c} |\langle g_j | (S_G^{-1} - S_F^{-1}) f\rangle|^2\Big)^{\frac{1}{2}} \\ &\geq \sqrt{A}||S_F^{-1} f|| - \sqrt{B}||S_G^{-1} - S_F^{-1}||\,||f|| \\ &\geq \Big(\frac{\sqrt{A}}{||S_F||} - \sqrt{B}||S_G^{-1} - S_F^{-1}||\Big)||f|| \end{aligned}$$

and

$$\begin{aligned} \sum_{j\in\sigma} |\langle S_F^{-1} f_j | f\rangle|^2 + \sum_{j\in\sigma^c} |\langle S_G^{-1} g_j | f\rangle|^2 \quad &\leq \quad \sum_{j\in\mathcal{J}} |\langle S_F^{-1} f_j | f\rangle|^2 + \sum_{j\in\mathcal{J}} |\langle S_G^{-1} g_j | f\rangle|^2 \\ &\leq \quad B\Big(||S_F^{-1}||^2 + ||S_G^{-1}||^2\Big)||f||^2. \end{aligned}$$ □

Note that in Proposition 3.2, it is established that for any subset $\sigma \subset \mathcal{J}$, the collection $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ forms a frame for $\mathbb{H}^R(\mathfrak{Q})$. Similarly, the set $\{S_F^{-1}f_j\}_{j\in\sigma} \cup \{S_G^{-1}g_j\}_{j\in\sigma^c}$ also constitutes a frame. However, it is important to observe that the set $\{S_F^{-1}f_j\}_{j\in\sigma} \cup \{S_G^{-1}g_j\}_{j\in\sigma^c}$ does not generally serve as a dual frame for $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$. In this direction we have the following example:

Example 3.3. *Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space with an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$. Define*

$$\begin{aligned} F &= \{f_j\}_{j\in\mathbb{N}} = \{e_1,\ e_2,\ e_1+e_2,\ e_3,\ e_4,\cdots\} \quad \text{and} \\ G &= \{g_j\}_{j\in\mathbb{N}} = \{e_1,\ e_1+e_2,\ e_1-e_2, e_3, e_4,\cdots\}, \end{aligned}$$

then F and G constitute a woven frame for $\mathbb{H}^R(\mathfrak{Q})$. The corresponding frame operators and their inverse operators are given by

$$\begin{cases} S_F(e_1) &= 2e_1+e_2, \\ S_F(e_2) &= e_1+2e_2, \\ S_F(e_j) &= e_j,\ j\geq 3, \end{cases} \qquad \begin{cases} S_F^{-1}(e_1) &= \frac{2}{3}e_1-\frac{e_2}{3}, \\ S_F^{-1}(e_2) &= \frac{-e_1}{3}+\frac{2e_2}{3}, \\ S_F^{-1}(e_j) &= e_j,\ j\geq 3. \end{cases}$$

$$\begin{cases} S_G(e_1) &= 3e_1, \\ S_G(e_2) &= 2e_2, \\ S_G(e_j) &= e_j,\ \ j\geq 3, \end{cases} \quad \text{and} \quad \begin{cases} S_G^{-1}(e_1) &= \frac{e_1}{3}, \\ S_G^{-1}(e_2) &= \frac{e_2}{2}, \\ S_G^{-1}(e_j) &= e_j,\ j\geq 3. \end{cases}$$

The dual frames

$$\begin{aligned} S_F^{-1}F &= \left\{\frac{2}{3}e_1-\frac{1}{3}e_2,\ 1\frac{1}{3}e_1+\frac{2}{3}e_2, \frac{1}{3}e_1+\frac{1}{3}e_2,\ e_3, e_4,\cdots\right\} \quad \text{and} \\ S_G^{-1}G &= \left\{\frac{e_1}{3}, \frac{e_1}{3}+\frac{e_2}{2}, \frac{e_1}{3}-\frac{e_2}{2}, e_3, e_4,\cdots\right\} \end{aligned}$$

of F and G, respectively, are woven frames.

Further, corresponding to the partition $\{\sigma_1=\{1,2\},\ \sigma_2=\mathbb{N}\setminus\sigma_1\}$ of $\mathbb{N}$, the weaving frame W of F and G, and the weaving $\widetilde{W}$ of $S_F^{-1}F$ and $S_G^{-1}G$ given by

$$\begin{aligned} W &= \{f_1,\ f_2,\ g_3,\ g_4,\ g_5,\cdots\} = \{e_1,\ e_2,\ e_1-e_2,\ e_3, e_4,\cdots\} \quad \text{and} \\ \widetilde{W} &= \{S_F^{-1}f_1,\ S_F^{-1}f_2,\ S_G^{-1}g_3,\ S_G^{-1}g_4,\ S_G^{-1}g_5,\cdots\} \\ &= \left\{\frac{2}{3}e_1-\frac{1}{3}e_2,\ -\frac{1}{3}e_1+\frac{2}{3}e_2,\ \frac{1}{3}e_1-\frac{1}{2}e_2,\ e_3,\ e_4,\cdots\right\} \end{aligned}$$

forms frame for $\mathbb{H}^R(\mathfrak{Q})$. Observe that, dual frame corresponding to W is

$$S_W^{-1}W = \left\{\frac{2}{3}e_1+\frac{1}{3}e_2,\ \frac{1}{3}e_1+\frac{2}{3}e_2,\ \frac{1}{3}e_1-\frac{1}{3}e_2,\ e_3,\ e_4,\cdots\right\} \neq \widetilde{W}.$$

The following result is related to the dual frames of a woven frame.

Proposition 3.4. *Suppose that the family of frames $\left\{F_i=\{f_{ij}\}_{j\in\mathcal{J}} : i\in[m]\right\}$ is a woven frame for $\mathbb{H}^R(\mathfrak{Q})$. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathcal{J}$. Then the synthesis operator of dual frame of the weaving frame $W=\{f_{ij}\}_{j\in\sigma_i, i\in[m]}$ is given by $S_W^{-1}T_W+U$, where $\sum\limits_{i\in[m]} T_{F_i}D_{\sigma_i}U^*=0$.*

Proof. Straight forward. □

Let $P=\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathcal{J}$, define the space

$$\bigoplus_{i\in[m]} \ell^2(\mathfrak{Q})(\sigma_i) = \left\{ \{q_{ij}\}_{j\in\sigma_i,\ i\in[m]} \subset \mathfrak{Q} :\ \sum_{i\in[m]}\sum_{j\in\sigma_i} |q_{ij}|^2 < \infty \right\}.$$

Then, $\bigoplus\limits_{i\in[m]} \ell^2(\mathfrak{Q})(\sigma_i)$ is a right quaternionic Hilbert space with the quaternionic inner product

$$\left\langle \{p_{ij}\}_{j\in\sigma_i,\ i\in[m]} \middle| \{q_{ij}\}_{j\in\sigma_i,\ i\in[m]} \right\rangle = \sum_{i\in[m]} \sum_{j\in\sigma_i} \overline{p_{ij}} q_{ij}.$$

Next, we give a characterization of weaving frames in terms of bounded right linear operators.

Theorem 3.5. *For $i \in [m]$, let $F_i = \{f_{ij}\}_{j\in\mathcal{J}}$ be a sequence in $\mathbb{H}^R(\mathfrak{Q})$. Then the following assertions are equivalent:*

(i) The family of sequences $\{F_i\}_{i\in[m]}$ is woven frame for $\mathbb{H}^R(\mathfrak{Q})$.

(ii) For any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathcal{J}$, there exists $A > 0$ and a bounded right linear operator $T : \bigoplus\limits_{i\in[m]} \ell^2(\mathfrak{Q})(\sigma_i) \to \mathbb{H}^R(\mathfrak{Q})$ such that $T(u_{ij}) = f_{ij}$, for all $j \in \sigma_i, i \in [m]$, and $AI_{\mathbb{H}^R(\mathfrak{Q})} \leqslant TT^*$, where $\{u_{ij}\}_{j\in\sigma_i, i\in[m]}$ is the standard orthonormal basis for $\bigoplus\limits_{i\in[m]} \ell^2(\mathfrak{Q})(\sigma_i)$.

Proof. (i) $\Rightarrow$ (ii): Choose A to be the universal lower frame bound for the family of sequences $\{F_i\}_{i\in[m]}$. Let T_W be the synthesis operator associated with $W = \{f_{ij}\}_{j\in\sigma_i, i\in[m]}$. Take $T = T_W$. Then

$$T(u_{ij}) = T_W(u_{ij}) = \sum_{i\in[m]} T_{F_i} D_{\sigma_i}(u_{ij}) = f_{ij}, \quad j \in \sigma_i, i \in [m],$$

where $\{u_{ij}\}_{j\in\sigma_i, i\in[m]}$ is the standard orthonormal basis for $\bigoplus\limits_{i\in[m]} \ell^2(\mathfrak{Q})(\sigma_i)$. Further, for all $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$A\langle f|f\rangle = A\|f\|^2 \leqslant \sum_{j\in\sigma_i, i\in[m]} |\langle f_{ij}|f\rangle|^2 = \|T_W^*(f)\|^2 = \|T^*(f)\|^2 = \langle TT^*f|f\rangle$$

This gives $AI_{\mathbb{H}^R(\mathfrak{Q})} \leqslant TT^*$.

(ii) $\Rightarrow$ (i): For any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathcal{J}$, for $\{c_{ij}\} \in \bigoplus\limits_{i\in[m]} \ell^2(\mathfrak{Q})(\sigma_i)$ and $T : \bigoplus\limits_{i\in[m]} \ell^2(\mathfrak{Q})(\sigma_i) \to \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned} \langle T(\{c_{ij}\})|f\rangle &= \left\langle T\left(\sum_{j\in\sigma_i, i\in[m]} u_{ij}c_{ij}\right) \middle| f \right\rangle = \left\langle \sum_{j\in\sigma_i, i\in[m]} Tu_{ij}c_{ij} \middle| f \right\rangle \\ &= \left\langle \sum_{j\in\sigma_i, i\in[m]} f_{ij}c_{ij} \middle| f \right\rangle = \sum_{j\in\sigma_i, i\in[m]} \overline{c_{ij}} \langle f_{ij}| f\rangle \end{aligned}$$

This gives

$$T^*(f) = \langle f_{ij}|f\rangle_{j\in\sigma_i, i\in[m]}, \forall f \in \mathbb{H}^R(\mathfrak{Q}) \tag{3.1}$$

Since $AI_{\mathbb{H}^R(\mathfrak{Q})} \leqslant TT^*$, using (3.1), we obtain

$$A\|f\|^2 \leqslant \langle TT^*f|f\rangle = \|T^*(f)\|^2 = \sum_{i\in[m]} \sum_{j\in\sigma_i} |\langle f_{ij}|f\rangle|^2.$$

On the other hand, for any $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle f_{ij}|\, f\rangle|^2 = \|T^* f\|^2 \leqslant \|T^*\|^2 \, \|f\|^2.$$

Hence, $\{f_{ij}\}_{j\in\sigma_i, i\in[m]}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$ and the family of sequences $\{F_i\}_{i\in[m]}$ is woven frame for $\mathbb{H}^R(\mathfrak{Q})$. $\square$

Recall that if $F = \{f_j\}_{j\in\mathcal{J}}$ is a Bessel sequence in $\mathbb{H}^R(\mathfrak{Q})$ with synthesis operator(analysis operator) $T_F(T_F^*)$, then for any subset σ of $\mathcal{J}$ we defined the truncated form $T_F^\sigma(T_F^{\sigma *})$ of $T_F(T_F^*)$ as

$$T_F^\sigma\left(\{q_j\}_{j\in\mathcal{J}}\right) = \sum_{j\in\sigma} f_j q_j \left(T_F^{\sigma *}(u) = \{\langle f_j|u\rangle\}_{j\in\sigma}\right).$$

In the following result, we obtain sufficient condition for two Bessel sequences in terms of their synthesis operators to form a pair of woven frames.

Theorem 3.6. *Let $F = \{f_j\}_{j\in\mathcal{J}}$ and $G = \{g_j\}_{j\in\mathcal{J}}$ be two Bessel sequences for $\mathbb{H}^R(\mathfrak{Q})$ with synthesis operators T_F and T_G, respectively. For any $\sigma \subset \mathcal{J}$, if $I_{\mathbb{H}^R(\mathfrak{Q})} = T_F T_G^* = T_G T_F^*$ and $T_F^\sigma T_G^{\sigma *} = T_G^\sigma T_F^{\sigma *}$, then F and G are woven frames for $\mathbb{H}^R(\mathfrak{Q})$, where T_F^σ and T_G^σ are truncated form of T_F and T_G for $\sigma \subset \mathcal{J}$, respectively.*

Proof. Let B_1 and B_2 be Bessel bounds for F and G, respectively. For any $f \in \mathbb{H}^R(\mathfrak{Q})$ and $\sigma \subset \mathcal{J}$, we have $f = \sum\limits_{j\in\mathcal{J}} g_j \langle f_j|f\rangle = \sum\limits_{j\in\mathcal{J}} f_j \langle g_j|f\rangle$ and $\sum\limits_{j\in\sigma} g_j \langle f_j|f\rangle =$

$\sum\limits_{j\in\sigma} f_j \langle g_j|f\rangle$. By using $(a+b)^2 \leqslant 2a^2+2b^2$, we compute

$$\begin{aligned}
\|f\|^4 &= \left|\left\langle \sum_{j\in\mathcal{J}} g_j \langle f_j|f\rangle \,|f \right\rangle\right|^2 \\
&= \left|\left\langle \sum_{j\in\sigma} g_j \langle f_j|f\rangle + \sum_{j\in\sigma^c} g_j \langle f_j|f\rangle \,|f \right\rangle\right|^2 \\
&\leqslant 2\left|\left\langle \sum_{j\in\sigma} g_j \langle f_j|f\rangle \,|f \right\rangle\right|^2 + 2\left|\left\langle \sum_{j\in\sigma^c} g_j \langle f_j|f\rangle \,|f \right\rangle\right|^2 \\
&= 2\left|\left\langle \sum_{j\in\sigma} g_j \langle f_j|f\rangle \,|f \right\rangle\right|^2 + 2\left|\left\langle \sum_{j\in\sigma^c} f_j \langle g_j|f\rangle \,|f \right\rangle\right|^2 \\
&= 2\left|\sum_{j\in\sigma} \overline{\langle f_j|f\rangle}\, \langle g_j|f\rangle\right|^2 + 2\left|\sum_{j\in\sigma^c} \overline{\langle g_j|f\rangle}\, \langle f_j|f\rangle\right|^2 \\
&\leqslant 2\sum_{j\in\sigma} |\langle f_j|f\rangle|^2 \sum_{j\in\sigma} |\langle g_j|f\rangle|^2 + 2\sum_{j\in\sigma^c} |\langle g_j|f\rangle|^2 \sum_{j\in\sigma^c} |\langle f_j|f\rangle|^2 \\
&\leqslant 2B_2\|f\|^2 \sum_{j\in\sigma} |\langle f_j|f\rangle|^2 + 2B_1\|f\|^2 \sum_{j\in\sigma^c} |\langle g_j|f\rangle|^2 \\
&\leqslant 2\max\{B_1,B_2\}\,\|f\|^2 \left(\sum_{j\in\sigma} |\langle f_j|f\rangle|^2 + \sum_{j\in\sigma^c} |\langle g_j|f\rangle|^2\right).
\end{aligned}$$

Therefore, for all $f\in\mathbb{H}^R(\mathfrak{Q})$, we have

$$\frac{1}{2\max\{B_1,B_2\}}\|f\|^2 \leqslant \sum_{j\in\sigma} |\langle f_j|f\rangle|^2 + \sum_{j\in\sigma^c} |\langle g_j|f\rangle|^2 \leqslant (B_1+B_2)\,\|f\|^2$$

Hence, F and G are woven frames. □

In the following result, we give a sufficient condition for a family of frames that are image of a frame under a sequence of operators to be woven frames.

Theorem 3.7. *Let $F=\{f_j\}_{j\in\mathcal{J}}$ be a frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds A, B, and $\{U_i\}_{i\in[m]}\subset\mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$. If U_k has a left inverse $V\in\mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ and $\max_{i\neq k}\|U_k-U_i\| < \sqrt{\frac{A}{(m-1)B}}\,\frac{1}{\|V\|}$, then the family of frames $\{U_iF\}_{i\in[m]}$ is a woven frame.*

Proof. Since $VU_k = I_{\mathbb{H}^R(\mathfrak{Q})}$, we get

$$||I_{\mathbb{H}^R(\mathfrak{Q})} - VU_i|| = \|V\,(U_k-U_i)\| < \sqrt{\frac{A}{(m-1)B}} < \sqrt{\frac{A}{B}} < 1.$$

Therefore, VU_i is invertible for $i\in[m]$. Consequently, U_i has a left inverse. Hence, for any $i\in[m], U_i$ is bounded below by λ, i.e., $\lambda\|f\|\leqslant\|U_if\|$ for any $f\in\mathbb{H}^R(\mathfrak{Q})$. Since $\lambda^2\cdot I_{\mathbb{H}^R(\mathfrak{Q})}\leqslant U_i^*U_i$ and $A\cdot I_{\mathbb{H}^R(\mathfrak{Q})}\leqslant S_F\leqslant B\cdot I_{\mathbb{H}^R(\mathfrak{Q})}$, we get

$$A\lambda^2\cdot I_{\mathbb{H}^R(\mathfrak{Q})} \leqslant A\cdot U_i^*U_i \leqslant S_{U_iF} \leqslant B\cdot U_i^*U_i \leqslant B\,\|U_i\|^2\cdot I_{\mathbb{H}^R(\mathfrak{Q})}.$$

Hence $\{U_i f_j\}_{j\in\mathcal{J}}$ is a frame for any $i \in [m]$. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathcal{J}$. Then, for every $f \in \mathbb{H}^R(\mathfrak{Q})$ we have

$$\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle U_i f_j|f\rangle|^2 = \sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle f_j|U_i^* f\rangle|^2 \leqslant \sum_{i\in[m]}\sum_{j\in\mathcal{J}}|\langle U_i f_j|f\rangle|^2 \leqslant B\sum_{i\in[m]}\|U_i\|^2\|f\|^2.$$

On the other hand, we have

$$\begin{aligned}
\sum_{i\in[m]}\sum_{j\in\sigma_i}|\langle U_i f_j|f\rangle|^2 &= \sum_{j\in\sigma_1}|\langle U_1 f_j|f\rangle|^2 + \cdots + \sum_{j\in\sigma_i}|\langle U_i f_j|f\rangle|^2 + \cdots + \sum_{j\in\sigma_k}|\langle U_k f_j|f\rangle|^2 \\
&\quad + \cdots + \sum_{j\in\sigma_m}|\langle U_m f_j|f\rangle|^2 \\
&= \sum_{j\in\sigma_1}|\langle (U_1 - U_k + U_k)\, f_j|\, f\rangle|^2 + \cdots + \sum_{j\in\sigma_i}|\langle (U_i - U_k + U_k)\, f_j|\, f\rangle|^2 \\
&\quad + \cdots + \sum_{j\in\sigma_k}|\langle U_k f_j|\, f\rangle|^2 + \cdots + \sum_{j\in\sigma_m}|\langle (U_m - U_k + U_k)\, f_j|\, f\rangle|^2 \\
&\geqslant \sum_{j\in\sigma_k}|\langle U_k f_j|f\rangle|^2 + \sum_{j\in\sigma_i}|\langle U_k f_j|f\rangle|^2 - \sum_{j\in\sigma_i}|\langle (U_i - U_k)\, f_j|\, f\rangle|^2 \\
&\quad + \sum_{j\in\sigma_1}|\langle U_k f_j|\, f\rangle|^2 - \sum_{j\in\sigma_1}|\langle (U_1 - U_k)\, f_j|\, f\rangle|^2 \\
&\quad + \cdots + \sum_{j\in\sigma_k}|\langle U_k f_j|\, f\rangle|^2 - \sum_{j\in\sigma_m}|\langle (U_m - U_k)\, f_j|\, f\rangle|^2 \\
&\geqslant \sum_{j\in J}|\langle U_k f_j|\, f\rangle|^2 - \sum_{i\neq k}\sum_{j\in\sigma_i}|\langle (U_i - U_k)\, f_j|\, f\rangle|^2 \\
&= \sum_{j\in J}|\langle f_j|\, U_k^* f\rangle|^2 - \sum_{i\neq k}\sum_{j\in\sigma_i}|\langle (U_i - U_k)\, f_j|\, f\rangle|^2 \\
&\geqslant A\,\|U_k^* f\|^2 - \sum_{i\neq k}\sum_{j\in J}|\langle (U_i - U_k)\, f_j|\, f\rangle|^2 \\
&\geqslant \frac{A}{\|V\|^2}\|f\|^2 - B\sum_{i\neq k}\|U_i - U_k\|^2\|f\|^2 \\
&\geqslant \left(\frac{A}{\|V\|^2} - (m-1)B\max_{i\neq k}\|U_i - U_k\|^2\right)\|f\|^2. \qquad \square
\end{aligned}$$

Next, we present a characterization of weaving frames by synthesis operators of frames.

Theorem 3.8. *For $i \in [m]$, let $F_i = \{f_{ij}\}_{j\in\mathcal{J}}$ be a frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds A_i, B_i. Assume that for any $k \in [m]$, $\|T_{F_i} - T_{F_k}\| < \frac{A_k}{(m-1)(\sqrt{B_i}+\sqrt{B_k})}$, $i \in [m]$, $i \neq k$. Then the family of frames $\{F_i\}_{i\in[m]}$ is woven frame of $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. The family $\{f_{ij}\}_{j\in\sigma_i, i\in[m]}$ is a Bessel sequence with Bessel bound $\max_{i\in[m]} B_i$ by Proposition 2.1. Therefore, for any $f \in \mathbb{H}^R(\mathfrak{Q})$,

$$\left\|T_{F_i}^{\sigma_i} f\right\|^2 = \sum_{j\in\sigma_i}|\langle f_{ij}|\, f\rangle|^2 \leqslant \sum_{j\in\mathcal{J}}|\langle f_{ij}|\, f\rangle|^2 = \|T_{F_i} f\|^2 \leqslant \|T_{F_i}\|^2\|f\|^2$$

Thus, $\left\|T_{F_i}^{\sigma_i}\right\| \leqslant \|T_{F_i}\|$ and so

$$\begin{aligned}
\left\|T_{F_i}^{\sigma_i} T_{F_i}^{\sigma_i *} - T_{F_k}^{\sigma_i} T_{F_k}^{\sigma_i *}\right\| &= \left\|T_{F_i}^{\sigma_i} T_{F_i}^{\sigma_i *} - T_{F_i}^{\sigma_i} T_{F_k}^{\sigma_i *} + T_{F_i}^{\sigma_i} T_{F_k}^{\sigma_i *} - T_{F_k}^{\sigma_i} T_{F_k}^{\sigma_i *}\right\| \\
&\leq \left\|T_{F_i}^{\sigma_i}\left(T_{F_i}^{\sigma_i *} - T_{F_k}^{\sigma_i *}\right)\right\| + \left\|\left(T_{F_i}^{\sigma_i} - T_{F_k}^{\sigma_i}\right) T_{F_k}^{\sigma_i *}\right\| \\
&\leqslant \left\|T_{F_i}^{\sigma_i}\right\| \|T_{F_i} - T_{F_k}\| + \|T_{F_i} - T_{F_k}\| \|T_{F_k}\| \\
&\leqslant \left(\sqrt{B_i} + \sqrt{B_k}\right) \|T_{F_i} - T_{F_k}\|.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\sum_{i \in [m]} S_{F_i}^{\sigma_i} &= \sum_{i \in [m] \setminus \{k\}} S_{F_i}^{\sigma_i} + S_{F_k}^{\sigma_k} \\
&= \sum_{i \in [m] \setminus \{k\}} S_{F_i}^{\sigma_i} + \left(S_{F_k} - \sum_{i \in [m] \setminus \{k\}} S_{F_k}^{\sigma_i}\right) \\
&= S_{F_k} + \sum_{i \in [m] \setminus \{k\}} \left(S_{F_i}^{\sigma_i} - S_{F_k}^{\sigma_i}\right) \\
&\geqslant A_k I_{\mathbb{H}^R(\mathfrak{Q})} - \sum_{i \in [m] \setminus \{k\}} \left\|S_{F_i}^{\sigma_i} - S_{F_k}^{\sigma_i}\right\| I_{\mathbb{H}^R(\mathfrak{Q})} \\
&\geqslant \left(A_k - \sum_{i \in [m] \setminus \{k\}} \left(\sqrt{B_i} + \sqrt{B_k}\right) \|T_{F_i} - T_{F_k}\|\right) I_{\mathbb{H}^R(\mathfrak{Q})}.
\end{aligned}$$

Hence, the sequence $\{f_{ij}\}_{j \in \sigma_i, i \in [m]}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$, and the family of frames $\{F_i\}_{i \in [m]}$ is woven frame of $\mathbb{H}^R(\mathfrak{Q})$. $\square$

Now, recall that a right linear operator T on $\mathbb{H}^R(\mathfrak{Q})$ is said to be positive if $\langle T(u)|u\rangle \geq 0,\ u \in \mathbb{H}^R(\mathfrak{Q})$. Next, we provide conditions in terms of a positive right linear operator under which a family of frames is woven.

Theorem 3.9. *For $i \in [m]$, let $F_i = \{f_{ij}\}_{j \in \mathcal{J}}$ be a frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds A_i, B_i. For any $\sigma \subset \mathcal{J}$ and a fixed $k \in I$, let $P_i^\sigma(f) = \sum_{j \in \sigma} f_{ij} \langle f_{ij} | f\rangle - \sum_{j \in \sigma} f_{kj} \langle f_{kj} | f\rangle$ for $i \neq k$. If P_i^σ is a positive right linear operator, then the family of frames $\{F_i\}_{i \in [m]}$ form a woven frame.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathcal{J}$. Then, for every $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned}
A_k\|f\|^2 &\leqslant \sum_{j\in\mathcal{J}} |\langle f_{kj}|\, f\rangle|^2 = \sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle f_{kj}|\, f\rangle|^2 \\
&= \sum_{i\in[m]} \left\langle \sum_{j\in\sigma_i} f_{kj}\, \langle f_{kj}|\, f\rangle | f \right\rangle = \sum_{i\in[m]} \left\langle \sum_{j\in\sigma_i} f_{ij}\, \langle f_{ij}|f\rangle - P_i^{\sigma_i}(f)|f \right\rangle \\
&= \sum_{i\in[m]} \left\langle \sum_{j\in\sigma_i} f_{ij}\, \langle f_{ij}|\, f\rangle | f \right\rangle - \sum_{i=1}^{m} \langle P_i^{\sigma_i}(f)|f\rangle \\
&\leqslant \sum_{i\in[m]} \left\langle \sum_{j\in\sigma_i} f_{ij}\, \langle f_{ij}|\, f\rangle | f \right\rangle = \sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle f_{ij}|\, f\rangle|^2 \\
&\leqslant \sum_{i\in[m]}\sum_{j\in\mathcal{J}} |\langle f_{ij}|f\rangle|^2 \leqslant \left(\sum_{i\in[m]} B_i\right) \|f\|^2.
\end{aligned}$$

Hence,

$$A_k\|f\|^2 \leqslant \sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle f_{ij}|\, f\rangle|^2 \leqslant \sum_{i\in[m]} B_i\|f\|^2. \qquad \square$$

Finally, we consider the perturbation of a set of frames that is woven frame. We first need the following definition:

Definition 3.10. *Let $F = \{f_j\}_{j\in\mathcal{J}}$ and $G = \{g_j\}_{j\in\mathcal{J}}$ be sequences in right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$, let $0 < \mu, \lambda < 1$. If*

$$\sum_{j\in\mathcal{J}} |\langle f_j - g_j|\, f\rangle|^2 \leqslant \lambda \sum_{j\in\mathcal{J}} |\langle f_j|\, f\rangle|^2 + \mu\|f\|^2,\ f \in \mathbb{H}^R(\mathfrak{Q})$$

then we say that G is a (λ, μ)-perturbation of F.

In the following result, we obtain conditions under which a family of frames forms a woven frame for $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 3.11. *For $i \in [m]$, let $F_i = \{f_{ij}\}_{j\in\mathcal{J}}$ be a frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds A_i, B_i. For a fixed $k \in [m]$, let F_i be the (λ_i, μ_i)-perturbation of F_k, for all $i \in [m]/\{k\}$. If*

$$\sum_{i\neq k} \lambda_i < 1 \ \textit{and}\ A_k > \frac{\sum_{i\neq k}\mu_i}{1 - \sum_{i\neq k}\lambda_i},$$

then the family of frames $\{F_i\}_{i\in[m]}$ forms a woven frame.

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathcal{J}$. Now, observe that

$$\sum_{j\in\sigma_i} |\langle f_{kj} - f_{ij}|\, f\rangle|^2 \leqslant \sum_{j\in\mathcal{J}} |\langle f_{kj} - f_{ij}|\, f\rangle|^2 .$$

Also, for every $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned}
\left(\sum_{i\in[m]} B_i\right)\|f\|^2 &\geqslant \sum_{i\in[m]}\sum_{j\in\sigma_i} |\langle f_{ij}|\, f\rangle|^2 \\
&= \sum_{j\in\sigma_k} |\langle f_{kj}|\, f\rangle|^2 + \sum_{i\neq k}\sum_{j\in\sigma_i} |\langle f_{ij}|\, f\rangle|^2 \\
&= \sum_{j\in\sigma_k} |\langle f_{kj}|\, f\rangle|^2 + \sum_{i\neq k}\sum_{j\in\sigma_i} |\langle f_{ij}-f_{kj}+f_{kj}|\, f\rangle|^2 \\
&\geqslant \sum_{j\in\sigma_k} |\langle f_{kj}|\, f\rangle|^2 + \sum_{i\neq k}\sum_{j\in\sigma_i} |\langle f_{kj}|\, f\rangle|^2 - \sum_{i\neq k}\sum_{j\in\sigma_i} |\langle f_{ij}-f_{kj}|\, f\rangle|^2 \\
&= \sum_{i\in[m]}\sum_{j\in\sigma_k} |\langle f_{kj}|\, f\rangle|^2 - \sum_{i\neq k}\sum_{j\in\sigma_i} |\langle f_{ij}-f_{kj}|\, f\rangle|^2 \\
&\geqslant \sum_{j\in\mathcal{J}} |\langle f_{kj}|\, f\rangle|^2 - \sum_{i\neq k}\sum_{j\in\mathcal{J}} |\langle f_{ij}-f_{kj}|\, f\rangle|^2 \\
&\geqslant \sum_{j\in\mathcal{J}} |\langle f_{kj}|\, f\rangle|^2 - \sum_{i\neq k}\left[\lambda_i \sum_{j\in\mathcal{J}} |\langle f_{kj}|\, f\rangle|^2 + \mu_i\|f\|^2\right] \\
&\geqslant \left[\left(1-\sum_{i\neq k}\lambda_i\right)A_k - \sum_{i\neq k}\mu_i\right]\|f\|^2.
\end{aligned}$$

Hence, the family of frames $\{F_i\}_{i\in[m]}$ is a woven frame with bounds $\left(1-\sum_{i\neq k}\lambda_1\right)A_k - \sum_{i\neq k}\mu_i$ and $\sum_{i\in[m]} B_i$. $\square$

As an illustration of theorem 3.11, we give the following example.

Example 3.12. *Let $\mathbb{H}^R(\mathfrak{Q})$ be a three dimensional right quaternionic Hilbert space with an orthonormal basis $\{e_j\}_{j=1}^3$. Let $g_1 = e_1, g_2 = e_2, g_3 = e_1 + e_2$, and let $f_j = \frac{3}{2}g_j, h_j = \frac{1}{2}g_j$, for all $j = 1, 2, 3$. Then $F = \{f_j\}_{j=1}^3$, $G = \{g_j\}_{j=1}^3$ and $H = \{h_j\}_{j=1}^3$ are frames for $\mathbb{H}^R(\mathfrak{Q})$ with frame bounds $\left(\frac{9}{4}, \frac{27}{4}\right), (1, 3)$ and $\left(\frac{1}{4}, \frac{3}{4}\right)$, respectively.*

Choose $\lambda_1 = \frac{1}{9}, \mu_1 = \frac{1}{9}$ and $\lambda_2 = \frac{4}{9}, \mu_2 = \frac{2}{9}$. Then $\lambda_1 + \lambda_2 < 1$ and $A_1 > \frac{\mu_1+\mu_2}{1-(\lambda_1+\lambda_2)}$. For any $f \in \mathbb{H}^R(\mathfrak{Q})$, we compute

$$\begin{aligned}
\sum_{j=1}^3 |\langle f_j - g_j|\, f\rangle|^2 &= \frac{1}{4}\sum_{j=1}^3 |\langle g_j|\, f\rangle|^2 \\
&= \frac{1}{9}\sum_{j=1}^3 |\langle f_j|\, f\rangle|^2 \\
&\leqslant \lambda_1 \sum_{j=1}^3 |\langle f_j|\, f\rangle|^2 + \mu_1\|f\|^2
\end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{3}|\langle f_j-h_j|\, f\rangle|^2 &= \sum_{j=1}^{3}|\langle g_j|\, f\rangle|^2\\ &= \frac{4}{9}\sum_{j=1}^{3}|\langle f_j|\, f\rangle|^2\leqslant\lambda_2\sum_{j=1}^{3}|\langle f_j|\, f\rangle|^2+\mu_2\|f\|^2.\end{aligned}$$

Therefore, by Theorem 3.11, F, G and H are woven frames. In fact, for any partition $\{\sigma_1,\sigma_2,\sigma_3\}$ of $\mathcal{J}$,

$$\begin{aligned}\sum_{j\in\sigma_1}|\langle f_j|\, f\rangle|^2+\sum_{j\in\sigma_2}|\langle g_j|\, f\rangle|^2+\sum_{j\in\sigma_3}|\langle h_j|\, f\rangle|^2 &= \sum_{j\in\mathcal{J}}\left|\left\langle d_j^{\sigma_1}f_j+d_j^{\sigma_2}g_j+d_j^{\sigma_3}h_j\right|\, f\right\rangle\right|^2\\ &= \sum_{j\in\mathcal{J}}\left|\left\langle\left(\frac{3d_j^{\sigma_1}}{2}+d_j^{\sigma_2}+\frac{d_j^{\sigma_3}}{2}\right)(g_j)\right|\, f\right\rangle\right|^2,\end{aligned}$$

where $d_j^{\sigma_i}=1$ for $j\in\sigma_i (i=1,2,3)$ and 0 otherwise. Hence $\{f_j\}_{j\in\sigma_1}\cup\{g_j\}_{j\in\sigma_2}\cup\{h_j\}_{j\in\sigma_3}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$.

Definition 3.13. *Let $F=\{f_j\}_{j\in\mathcal{J}}$ and $G=\{g_j\}_{j\in\mathcal{J}}$ be sequences in $\mathbb{H}^R(\mathfrak{Q})$, let $0<\lambda<1$. Let $\{c_j\}_{j\in\mathcal{J}}$ be an arbitrary sequence of positive numbers such that $\sum\limits_{j\in\mathcal{J}}c_j^2<\infty$. If*

$$\left\|\sum_{j\in\mathcal{J}}c_j\left(f_j-g_j\right)\right\|_{\mathbb{H}^R(\mathfrak{Q})}\leqslant\lambda\left\|\{c_j\}_{j\in\mathcal{J}}\right\|_{l^2(\mathfrak{Q})}\tag{3.2}$$

then we say that G is a λ-perturbation of F.

Finally, in the following result, we give conditions under which $\lambda-$ perturbation of woven frame is woven.

Theorem 3.14. *Let $\left\{F_i=\{f_{ij}\}_{j\in\mathcal{J}}: i\in[m]\right\}$ be a family of woven frames with bounds A and B, and let $F_i'=\left\{f_{ij}'\right\}_{j\in\mathcal{J}}$ be λ_i-perturbation of F_i. If $\lambda_i<\frac{A}{2\sqrt{mB}}$, for all $i\in[m]$, then the family of frames $\{F_i'\}_{i\in[m]}$ forms a woven frame in Hilbert space $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Let T_{F_i} be the synthesis operator of F_i, by (3.2), it follows that

$$\left\|T_{F_i}-T_{F_i'}\right\|\leqslant\lambda_i,\ \forall\ \ i\in[m].$$

Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathcal{J}$. Then, for every $f\in\mathbb{H}^R(\mathfrak{Q})$,

$$A\|f\|^2\leqslant\sum_{i\in[m]}\left\|T_{F_i}^{\sigma_i *}(f)\right\|^2\leqslant B\|f\|^2.$$

Therefore, we obtain

$$\begin{aligned}
\sum_{i\in[m]} \left\|T^{\sigma_{i*}}_{F_i'}(f)\right\|^2 &= \sum_{i\in[m]} \left\|T^{\sigma_{i*}}_{F_i'}(f) - T^{\sigma_i *}_{F_i}(f) + T^{\sigma_i *}_{F_i}(f)\right\|^2 \\
&\leqslant 2\sum_{i\in[m]} \left\|T^{\sigma_{i*}}_{F_i'}(f) - T^{\sigma_i *}_{F_i}(f)\right\|^2 + 2\sum_{i\in[m]} \left\|T^{\sigma_i *}_{F_i}(f)\right\|^2 \\
&\leqslant 2\sum_{i\in[m]} \left\|\left(T^{\sigma_i}_{F_i'} - T^{\sigma_i}_{F_i}\right)^*\right\|^2 \|f\|^2 + 2\sum_{i\in[m]} \left\|T^{\sigma_i *}_{F_i}(f)\right\|^2 \\
&\leqslant 2\left(\sum_{i\in[m]} \lambda_i^2\right)\|f\|^2 + 2B\|f\|^2 = 2\left(B + \sum_{i\in[m]} \lambda_i^2\right)\|f\|^2.
\end{aligned}$$

On the other hand, by using the inequality

$$(a_1 + a_2 + \cdots + a_m) \leqslant \sqrt{m\left(a_1^2 + a_2^2 + \cdots + a_m^2\right)} \quad (a_i \geqslant 0)\,,$$

for any $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned}
&\sum_{i\in[m]} \left\|T^{\sigma_i *}_{F_i'}(f)\right\|^2 \\
&= \sum_{i\in[m]} \left\|T^{\sigma_i *}_{F_i'}(f) - T^{\sigma_{i*}}_{F_i}(f) + T^{\sigma_i *}_{F_i}(f)\right\|^2 \\
&\geqslant \sum_{i\in[m]} \left(\left\|T^{\sigma_i *}_{F_i}(f)\right\| - \left\|T^{\sigma_i *}_{F_i'}(f) - T^{\sigma_i *}_{F_i}(f)\right\|\right)^2 \\
&= \sum_{i\in[m]} \left(\left\|T^{\sigma_i *}_{F_i}(f)\right\|^2 + \left\|T^{\sigma_i *}_{F_i'}(f) - T^{\sigma_i *}_{F_i}(f)\right\|^2 - 2\left\|T^{\sigma_i *}_{F_i}(f)\right\| \left\|T^{\sigma_{i*} *}_{F_i'}(f) - T^{\sigma_i *}_{F_i}(f)\right\|\right) \\
&\geqslant \sum_{i\in[m]} \left\|T^{\sigma_i *}_{F_i}(f)\right\|^2 - 2\sum_{i\in[m]} \left\|T^{\sigma_i *}_{F_i}(f)\right\| \left\|T^{\sigma_i *}_{F_i'}(f) - T^{\sigma_i *}_{F_i}(f)\right\| \\
&\geqslant \sum_{i\in[m]} \left\|T^{\sigma_i *}_{F_i}(f)\right\|^2 - 2\max_{i\in[m]}\{\lambda_i\}\,\|f\| \sum_{i\in[m]} \left\|T^{\sigma_i *}_{F_i}(f)\right\| \\
&\geqslant A\|f\|^2 - 2\max_{i\in[m]}\{\lambda_i\}\,\|f\| \sqrt{m\sum_{i\in[m]} \left\|T^{\sigma_i *}_{F_i}(f)\right\|^2} \\
&\geqslant A\|f\|^2 - 2\sqrt{mB}\max_{i\in[m]}\{\lambda_i\}\,\|f\|^2. \qquad \square
\end{aligned}$$

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