Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 16 Issue 3(2024), Pages 1-12
https://doi.org/10.54671/BMAA-2024-3-1

ON SOME TOPOLOGY GENERATED BY $\mathcal{I}$-DENSITY FUNCTION

INDRAJIT DEBNATH, AMAR KUMAR BANERJEE

Abstract. In this paper we study on $\mathcal{I}$-density function using the notion of $\mathcal{I}$-density, introduced by Banerjee and Debnath [2], where $\mathcal{I}$ is an ideal of subsets of the set of natural numbers. We explore certain properties of $\mathcal{I}$-density function and induce a topology using this function in the space of reals, namely $\mathcal{I}$-density topology, and we give a characterization of the Lebesgue measurable subsets of reals in terms of Borel sets in $\mathcal{I}$-density topology.

1. Introduction and preliminaries

In 1961, Casper Goffman and Daniel Waterman [10] introduced the notion of homogeneity of sets relative to metric density and Euclidean n-space was topologized by taking the homogeneous sets as open sets and this topology was referred to as d-topology or density topology. The idea of density functions and the corresponding density topology were studied in several spaces like the space of real numbers [21], Euclidean n-space [11], metric spaces [15] etc. In the recent past the notion of classical Lebesgue density point was generalized by many authors by weakening the assumptions on the sequences of intervals and as a result several notions like $\langle s \rangle$-density point by M. Filipczak and J. Hejduk [9], $\mathcal{J}$-density point by J. Hejduk and R. Wiertelak [13], $\mathcal{S}$-density point by F. Strobin and R. Wiertelak [23] were obtained. Significant generalizations on density topology was studied by Das and Banerjee in [1, 6], by W. Wilczynski in [24] and by W. Wojdowski in [26, 27]. Lately, Banerjee and Debnath have devised a new way to generalize classical density topology using ideals in [2].

We shall use the notation $\mathcal{L}$ for the σ-algebra of Lebesgue measurable sets and λ for the Lebesgue measure [12]. Throughout $\mathbb{R}$ stands for the set of all real numbers. The symbol $\mathcal{T}_U$ stands for the natural topology on $\mathbb{R}$. Wherever we write $\mathbb{R}$ it means that $\mathbb{R}$ is equipped with natural topology unless otherwise stated. The symmetric difference of two sets A and B is $(A \setminus B) \cup (B \setminus A)$ and it is denoted by $A \triangle B$. By "a sequence of closed intervals $\{Q_n\}_{n\in\mathbb{N}}$ about a point c" we mean $c \in \bigcap_{n\in\mathbb{N}} Q_n$.

For $H \in \mathcal{L}$ and a point $c \in \mathbb{R}$ we say the point c is a classical density point [25] of H if and only if $\lim_{t\to 0+} \frac{\lambda(H\cap[c-t,c+t])}{2t} = 1$. The set of all classical density point

1991 *Mathematics Subject Classification.* 26E99, 54C30, 40A35.

Key words and phrases. Density topology, ideal, $\mathcal{I}$-density topology.

Submitted January 03, 2024. Accepted August 15, 2024. Published September 11, 2024.
Communicated by Guest editor L. Kocinac.

of H is denoted by $\Phi(H)$. The collection $\mathcal{T}_d = \{H \in \mathcal{L} : H \subseteq \Phi(H)\}$ is a topology in the real line [25] and it is called the classical density topology. Lebesgue density theorem states that for any Lebesgue measurable set $H \subset \mathbb{R}$, $\lambda(H \triangle \Phi(H)) = 0$.

The convergence of sequences plays a significant role in the study of basic mathematical theory. The idea of statistical convergence of sequences was introduced in the middle of twentieth century by H. Fast [8]. For $J \subset \mathbb{N}$, a set of natural numbers and $n \in \mathbb{N}$ let $J_n = \{k \in J : k \leq n\}$. The natural density of a set J is defined by $d(J) = \lim_{n\to\infty} \frac{|J_n|}{n}$, provided the limit exists, where $|J_n|$ stands for the cardinality of the set J_n. A sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to α_0 if for each $\epsilon > 0$ the set $V(\epsilon) = \{k \in \mathbb{N} : |\alpha_k - \alpha_0| \geq \epsilon\}$ has natural density zero. Later on, in the year 2000, statistical convergence of real sequences were generalized to the idea of $\mathcal{I}$-convergence of real sequences by P. Kostyrko et al. [14] using the notion of ideal $\mathcal{I}$ of subsets of $\mathbb{N}$, the set of natural numbers.

A subcollection $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an ideal if $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and $A \in \mathcal{I}, B \subset A$ imply $B \in \mathcal{I}$. $\mathcal{I}$ is called nontrivial ideal if $\mathcal{I} \neq \{\emptyset\}$ and $\mathbb{N} \notin \mathcal{I}$. $\mathcal{I}$ is called admissible if it contains all the singletons. It is easy to verify that the family $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$ forms a non-trivial admissible ideal of subsets of $\mathbb{N}$. If $\mathcal{I}$ is a proper non-trivial ideal, then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \mathbb{N} \setminus M \in \mathcal{I}\}$ is a filter on $\mathbb{N}$ and it is called the filter associated with the ideal $\mathcal{I}$ of $\mathbb{N}$.

A sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent [14] to α_0 if the set $V(\epsilon) = \{k \in \mathbb{N} : |\alpha_k - \alpha_0| \geq \epsilon\}$ belongs to $\mathcal{I}$ for each $\epsilon > 0$. A sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-bounded if there is a real number $M > 0$ such that $\{k \in \mathbb{N} : |\alpha_k| > M\} \in \mathcal{I}$. Further many works were carried out in this direction by many authors [3, 4, 17]. For summability theory, sequence spaces and related topics the reader may refer to the textbooks [5, 19].

K. Demirci [7] introduced the notion of $\mathcal{I}$-limit superior and inferior of a real sequence and proved several basic properties.

Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ be a real sequence. Let, $B_\alpha = \{b \in \mathbb{R} : \{k : \alpha_k > b\} \notin \mathcal{I}\}$ and $A_\alpha = \{a \in \mathbb{R} : \{k : \alpha_k < a\} \notin \mathcal{I}\}$. Then the $\mathcal{I}$-limit superior of α is given by,

$$\mathcal{I} - \limsup \alpha = \begin{cases} \sup B_\alpha & \text{if } B_\alpha \neq \phi \\ -\infty & \text{if } B_\alpha = \phi \end{cases}$$

and the $\mathcal{I}$-limit inferior of α is given by,

$$\mathcal{I} - \liminf \alpha = \begin{cases} \inf A_\alpha & \text{if } A_\alpha \neq \phi \\ \infty & \text{if } A_\alpha = \phi \end{cases}$$

Further Lahiri and Das [16] carried out some more works in this direction. Throughout the paper the ideal $\mathcal{I}$ will always stand for a nontrivial admissible ideal of subsets of $\mathbb{N}$.

In this paper we try to give the notion of $\mathcal{I}$-density function with the help of $\mathcal{I}$-density introduced by Banerjee and Debnath [2] in the space of reals. In Section 3 we explore some properties of this function. Finally, in Section 4 we study that under certain conditions union of arbitrary collection of measurable sets can be measurable. We consider $\mathcal{T}^{\mathcal{I}}$ to be the collection of measurable subsets of $\mathbb{R}$ such that each point of the set is an $\mathcal{I}$-density point and we prove that the collection $\mathcal{T}^{\mathcal{I}}$ forms a topology on the set of reals. The mode of proofs are different from that given in [2]. At last we characterize the Lebesgue measurable sets in the usual topology on reals as the Borel sets in $\mathcal{T}^{\mathcal{I}}$.

2. $\mathcal{I}$-density

Definition 2.1. [2] *For a Lebesgue measurable set $H \in \mathcal{L}$, a point $c \in \mathbb{R}$ and $n \in \mathbb{N}$, the upper $\mathcal{I}$-density of H at the point c, denoted by $\mathcal{I} - d^{-}(c, H)$, and the lower $\mathcal{I}$-density of H at the point c, denoted by $\mathcal{I} - d_{-}(c, H)$, are defined as follows: Suppose $\{Q_n\}_{n\in\mathbb{N}}$ is a sequence of closed intervals about c such that*

$$\mathscr{S}(Q_n) = \left\{n \in \mathbb{N} : 0 < \lambda(Q_n) < \tfrac{1}{n}\right\} \in \mathcal{F}(\mathcal{I}).$$

For any such $\{Q_n\}_{n\in\mathbb{N}}$ we take

$$\alpha_n = \frac{\lambda(Q_n \cap H)}{\lambda(Q_n)} \text{ for all } n \in \mathbb{N}.$$

Then $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers. Now, if

$$B_{\alpha_k} = \{b \in \mathbb{R} : \{k : \alpha_k > b\} \notin \mathcal{I}\}$$

and

$$A_{\alpha_k} = \{a \in \mathbb{R} : \{k : \alpha_k < a\} \notin \mathcal{I}\}$$

we define,

$$\begin{aligned}\mathcal{I} - d^{-}(c, H) &= \sup\{\sup B_{\alpha_n} : \{Q_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I})\} \\ &= \sup\{\mathcal{I} - \limsup\ \alpha_n : \{Q_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I})\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{I} - d_{-}(c, H) &= \inf\{\inf A_{\alpha_n} : \{Q_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I})\} \\ &= \inf\{\mathcal{I} - \liminf\ \alpha_n : \{Q_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I})\}.\end{aligned}$$

In the above two expressions it is to be understood that $\{Q_n\}_{n\in\mathbb{N}}$'s are closed intervals about the point c. Now, if $\mathcal{I} - d_{-}(c, H) = \mathcal{I} - d^{-}(c, H)$, then we denote the common value by $\mathcal{I} - d(c, H)$, which we call $\mathcal{I}$-density of H at the point c.

A point $c_0 \in \mathbb{R}$ is an *$\mathcal{I}$-density point* of $H \in \mathcal{L}$ if $\mathcal{I} - d(c_0, H) = 1$.

If a point $c_0 \in \mathbb{R}$ is an $\mathcal{I}$-density point of the set $\mathbb{R} \setminus H$, then c_0 is an *$\mathcal{I}$-dispersion point* of H.

Note 2.2. *It was shown in* [2] *if $\mathcal{I} = \mathcal{I}_{fin}$, where $\mathcal{I}_{fin}$ is the class of all finite subsets of $\mathbb{N}$, then Definition 2.1 coincides with the definition of metric density which was introduced by Martin in* [18]*. Moreover, the notion of $\mathcal{I}$-density point is more general than the notion of classical density point as the collection of intervals about the point c considered in case of $\mathcal{I}$-density is larger than that considered in case of classical density which is illustrated in the following example.*

Example 2.3. *Let us consider the ideal $\mathcal{I}_d$ of subsets of $\mathbb{N}$, where $\mathcal{I}_d$ is the ideal containing all those subsets of $\mathbb{N}$ whose natural density is zero. Now, for any point $p \in \mathbb{R}$ consider the following collections of sequences of intervals:*

$$\mathscr{C}_{p(\mathcal{I}_{fin})} = \{\{K_n\}_{n\in\mathbb{N}} :$$
$$\{K_n\} \text{is a sequence of closed intervals about } p \text{ such that } \mathscr{S}(K_n) \in \mathcal{F}(\mathcal{I}_{fin})\} \text{ and}$$
$$\mathscr{C}_{p(\mathcal{I}_d)} = \{\{K_n\}_{n\in\mathbb{N}} :$$
$$\{K_n\} \text{is a sequence of closed intervals about } p \text{ such that } \mathscr{S}(K_n) \in \mathcal{F}(\mathcal{I}_d)\}.$$

We claim that $\mathscr{C}_{p(\mathcal{I}_{fin})} \subsetneqq \mathscr{C}_{p(\mathcal{I}_d)}$. Since any finite subset of $\mathbb{N}$ has natural density zero so $\mathcal{I}_{fin} \subset \mathcal{I}_d$.
Now, in particular, let us take the following sequence $\{Q_n\}_{n\in\mathbb{N}}$ of closed intervals about a point p.

$$Q_n = \begin{cases} \left[p - \frac{1}{2^n}, p + \frac{1}{2^n}\right] & \text{for } n \neq m^2 \text{ where } m \in \mathbb{N} \\ [p - \sqrt{n}, p + \sqrt{n}] & \text{for } n = m^2 \text{ where } m \in \mathbb{N}. \end{cases}$$

We observe that $\mathscr{S}(Q_n) = \left\{n \in \mathbb{N} : 0 < \lambda(Q_n) < \frac{1}{n}\right\} = \{n : n \neq m^2,$ *for some* $m \in \mathbb{N}\} \setminus \{2\}$ *so* $\mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I}_d)$. *But since* $\mathbb{N} \setminus \mathscr{S}(K_n) = \{n : n = m^2,$ *where* $m \in \mathbb{N}\} \cup \{2\}$ *is not a finite set, so it does not belong to* $\mathcal{I}_{fin}$. *As a result,* $\{Q_n\} \in \mathscr{C}_{p(\mathcal{I}_d)} \setminus \mathscr{C}_{p(\mathcal{I}_{fin})}$.

Let us take the set H *to be the open interval* $(-1, 1)$ *and the point* p *to be* 0. *Let* $\{Q_n\}_{n \in \mathbb{N}} \in \mathscr{C}_{0(\mathcal{I}_d)} \setminus \mathscr{C}_{0(\mathcal{I}_{fin})}$ *be taken as above. Now, if* $\alpha_n = \frac{\lambda(Q_n \cap H)}{\lambda(Q_n)}$, *then*

$$\alpha_n = \begin{cases} 1 & \text{if } n \neq m^2 \text{ where } m \in \mathbb{N} \\ \frac{1}{\sqrt{n}} & \text{if } n = m^2 \text{ where } m \in \mathbb{N}. \end{cases}$$

Now, let us calculate lim sup *and* lim inf *of the sequence* $\{\alpha_n\}$. *Thus,*

$$\limsup \alpha_n = \inf_n \sup_{k \geq n} \alpha_k = 1 \text{ and } \liminf \alpha_n = \sup_n \inf_{k \geq n} \alpha_k = 0.$$

Consequently, $\lim_n \alpha_n$ *does not exists. Next, we will show that* 0 *is* $\mathcal{I}_d$*-density point of the set* H.

Given any sequence of closed intervals $\{Q_n\}_{n \in \mathbb{N}}$ *about the point* 0 *such that* $\mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I}_d)$ *we have* $\{n : Q_n \subset H\} \in \mathcal{F}(\mathcal{I}_d)$. *For if* $\mathscr{S}(Q_n) = \{l_1 < l_2 < \cdots < l_n < \cdots\}$ *(say). Then there exists* $n_0 \in \mathbb{N}$ *such that for* $l_n > l_{n_0}$, $Q_{l_n} \subset H$. *Thus,* $\{n : Q_n \subset H\} \supset \mathscr{S}(Q_n) \setminus \{l_1, l_2, \cdots, l_{n_0}\}$. *Since* $\mathbb{N} \setminus \{l_1, l_2, \cdots, l_{n_0}\} \in \mathcal{F}(\mathcal{I}_d)$ *so*

$$\mathscr{S}(Q_n) \setminus \{l_1, l_2, \cdots, l_{n_0}\} = \mathscr{S}(Q_n) \cap (\mathbb{N} \setminus \{l_1, l_2, \cdots, l_{n_0}\}) \in \mathcal{F}(\mathcal{I}_d).$$

Now if, $Q_n \subset H$ *then* $\sigma_n = \frac{\lambda(Q_n \cap H)}{\lambda(Q_n)} = \frac{\lambda(Q_n)}{\lambda(Q_n)} = 1$. *Thus,* $\{n : \sigma_n = 1\} \supset \{n : Q_n \subset H\}$. *Therefore,* $\{n : \sigma_n = 1\} \in \mathcal{F}(\mathcal{I}_d)$. *Therefore,* $B_{\sigma_n} = (-\infty, 1)$ *and* $A_{\sigma_n} = (1, \infty)$ *and so,* $\mathcal{I}_d - \limsup \sigma_n = \sup B_{\sigma_n} = 1$ *and* $\mathcal{I}_d - \liminf \sigma_n = \inf A_{\sigma_n} = 1$. *This is true for all* $\{Q_n\}_{n \in \mathbb{N}} \in \mathscr{C}_{0(\mathcal{I}_d)}$. *Hence,*

$$\mathcal{I}_d - d^-(0, H) = \sup\{\sup B_{\sigma_n} : \{Q_n\}_{n \in \mathbb{N}} \text{ such that } \mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I}_d)\} = 1$$

and

$$\mathcal{I}_d - d_-(0, H) = \inf\{\inf A_{\sigma_n} : \{Q_n\}_{n \in \mathbb{N}} \text{ such that } \mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I}_d)\} = 1.$$

Hence $\mathcal{I}_d - d(0, H)$ *exists and equals to* 1. *So, 0 is an* $\mathcal{I}_d$*-density point of the set* H.

Here we are stating some important results which will be needed later in our discussion.

Theorem 2.4. [7] *For any real sequence* $\omega = \{\omega_n\}_{n \in \mathbb{N}}$, $\mathcal{I}-\liminf \omega \leq \mathcal{I}-\limsup \omega$.

Theorem 2.5. [2] *For any Lebesgue measurable set* $H \subset \mathbb{R}$ *and any point* $p \in \mathbb{R}$,

$$\mathcal{I} - d_-(p, H) \leq \mathcal{I} - d^-(p, H).$$

Theorem 2.6. [16] *If* $\omega = \{\omega_n\}_{n \in \mathbb{N}}$ *and* $\delta = \{\delta_n\}_{n \in \mathbb{N}}$ *are two* $\mathcal{I}$*-bounded real number sequences, then*

(i) $\mathcal{I} - \limsup(\omega + \delta) \leq \mathcal{I} - \limsup \omega + \mathcal{I} - \limsup \delta$
(ii) $\mathcal{I} - \liminf(\omega + \delta) \geq \mathcal{I} - \liminf \omega + \mathcal{I} - \liminf \delta$.

Proposition 2.7. [2] *Given an $\mathcal{I}$-bounded real sequence $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ and a real number c,*

(i) $\mathcal{I} - \liminf(c + \omega_n) = c + \mathcal{I} - \liminf \omega_n$
(ii) $\mathcal{I} - \limsup(c + \omega_n) = c + \mathcal{I} - \limsup \omega_n$.

Proposition 2.8. [2] *For any real sequence $\omega = \{\omega_n\}_{n\in\mathbb{N}}$,*

(i) $\mathcal{I} - \limsup(-\omega) = -(\mathcal{I} - \liminf \omega)$
(ii) $\mathcal{I} - \liminf(-\omega) = -(\mathcal{I} - \limsup \omega)$.

Lemma 2.9. [2] *For any disjoint Lebesgue measurable subsets G and H of $\mathbb{R}$ and any point $p \in \mathbb{R}$ if $\mathcal{I} - d(p, G)$ and $\mathcal{I} - d(p, H)$ exist, then $\mathcal{I} - d(p, G \cup H)$ exists and $\mathcal{I} - d(p, G \cup H) = \mathcal{I} - d(p, G) + \mathcal{I} - d(p, H)$.*

Lemma 2.10. [2] *If $\mathcal{I} - d(p, G)$ and $\mathcal{I} - d(p, H)$ exist and $G \subset H$. Then $\mathcal{I} - d(p, H \setminus G)$ exists and $\mathcal{I} - d(p, H \setminus G) = \mathcal{I} - d(p, H) - \mathcal{I} - d(p, G)$.*

Theorem 2.11. [2] *For any measurable set H, $\mathcal{I}$-density of H at a point p exists if and only if $\mathcal{I} - d^-(p, H) + \mathcal{I} - d^-(p, H^c) = 1$.*

Let $H \subset \mathbb{R}$ be a measurable set. Let us denote the set of points of $\mathbb{R}$ at which H has lower $\mathcal{I}$-density 1 by $\Theta_{\mathcal{I}}(H)$, i.e.

$$\Theta_{\mathcal{I}}(H) = \{p \in \mathbb{R} : \mathcal{I} - d_-(p, H) = 1\}.$$

Next we state the Lebesgue $\mathcal{I}$-density theorem which is as follows.

Theorem 2.12. [2] *For any measurable set $H \subset \mathbb{R}$,*

$$\lambda(H \triangle \Theta_{\mathcal{I}}(H)) = 0.$$

The statement of this theorem may also be read as almost all points of an arbitrary measurable set H are points of $\mathcal{I}$-density for H and moreover we can conclude that $\Theta_{\mathcal{I}}(H)$ is measurable.

3. $\mathcal{I}$-density function

The function $\Theta_{\mathcal{I}}(.) : \mathcal{L} \to 2^{\mathbb{R}}$ is called $\mathcal{I}$-density function since $\Theta_{\mathcal{I}}(.)$ takes measurable set as input and it returns the set of all points which have $\mathcal{I}$-density 1 in that measurable set. Now, we explore some properties of $\mathcal{I}$-density function.

Proposition 3.1. *If A and B are measurable sets and $\lambda(A \triangle B) = 0$, then $\Theta_{\mathcal{I}}(A) = \Theta_{\mathcal{I}}(B)$.*

Proof. Let $\{Q_n\}_{n\in\mathbb{N}}$ be any sequence of closed interval in $\mathbb{R}$. If $\lambda(A \triangle B) = 0$, then we claim that $\lambda(A \cap Q_n) = \lambda(B \cap Q_n)$ for each interval $Q_n \subset \mathbb{R}$. Now,

$$\begin{aligned} A &= A \cap (B \cup B^c) \\ &= (A \cap B) \cup (A \cap B^c) \\ &= (A \cap B) \cup (A \setminus B) \\ &\subset B \cup (A \triangle B). \end{aligned}$$

For any $Q_n \subset \mathbb{R}$ we have

$$\begin{aligned} \lambda(A \cap Q_n) &\leq \lambda((B \cup (A \triangle B)) \cap Q_n) \\ &\leq \lambda((A \triangle B)) \cap Q_n) + \lambda(B \cap Q_n) \\ &= \lambda(B \cap Q_n) \quad \text{since } \lambda((A \triangle B)) \cap Q_n) \leq \lambda(A \triangle B) = 0. \end{aligned}$$

Similarly, $\lambda(B\cap Q_n)\leq\lambda(A\cap Q_n)$ for any interval $Q_n\subset\mathbb{R}$. So we have $\lambda(A\cap Q_n)=\lambda(B\cap Q_n)$ for any interval $Q_n\subset\mathbb{R}$. Now, we will show $\Theta_{\mathcal{I}}(A)=\Theta_{\mathcal{I}}(B)$.

Let $x\in\Theta_{\mathcal{I}}(A)$. So, $\mathcal{I}-d_{-}(x,A)=1$. Now,

$$\begin{aligned}\mathcal{I}-d_{-}(x,A)&=\inf\{\mathcal{I}-\liminf\frac{\lambda(A\cap Q_n)}{\lambda(Q_n)}:\{Q_n\}_{n\in\mathbb{N}}\text{ such that }\mathscr{S}(Q_n)\in\mathcal{F}(\mathcal{I})\}\\&=\inf\{\mathcal{I}-\liminf\frac{\lambda(B\cap Q_n)}{\lambda(Q_n)}:\{Q_n\}_{n\in\mathbb{N}}\text{ such that }\mathscr{S}(Q_n)\in\mathcal{F}(\mathcal{I})\}\\&=\mathcal{I}-d_{-}(x,B).\end{aligned}$$

So, $\mathcal{I}-d_{-}(x,B)=1$ and hence $x\in\Theta_{\mathcal{I}}(B)$. So, $\Theta_{\mathcal{I}}(A)\subseteq\Theta_{\mathcal{I}}(B)$. Similarly, $\Theta_{\mathcal{I}}(B)\subseteq\Theta_{\mathcal{I}}(A)$. Thus, $\Theta_{\mathcal{I}}(A)=\Theta_{\mathcal{I}}(B)$. This completes the proof. □

Corollary 3.2. *Let $A\subseteq\mathbb{R}$ be measurable then $\Theta_{\mathcal{I}}(A)=\Theta_{\mathcal{I}}(\Theta_{\mathcal{I}}(A))$, i.e. $\mathcal{I}$-density function is idempotent.*

Proof. By Lebesgue $\mathcal{I}$-Density Theorem 2.12 $\lambda(A\triangle\Theta_{\mathcal{I}}(A))=0$. So by Proposition 3.1 we have $\Theta_{\mathcal{I}}(A)=\Theta_{\mathcal{I}}(\Theta_{\mathcal{I}}(A))$. □

Lemma 3.3. *Given a pair of Lebesgue measurable sets A and B such that $A\subseteq B$, $\Theta_{\mathcal{I}}(A)\subseteq\Theta_{\mathcal{I}}(B)$, i.e. $\mathcal{I}$-density function is monotonic.*

Proof. If $A\subseteq B$, $\lambda(A\cap Q_n)\leq\lambda(B\cap Q_n)$ for each interval $Q_n\subset\mathbb{R}$. So if $x\in\Theta_{\mathcal{I}}(A)$, then $\mathcal{I}-d_{-}(x,A)=1$. Hence,

$$\begin{aligned}\mathcal{I}-d_{-}(x,A)&=\inf\{\mathcal{I}-\liminf\frac{\lambda(A\cap Q_n)}{\lambda(Q_n)}:\{Q_n\}_{n\in\mathbb{N}}\text{ such that }\mathscr{S}(Q_n)\in\mathcal{F}(\mathcal{I})\}\\&\leq\inf\{\mathcal{I}-\liminf\frac{\lambda(B\cap Q_n)}{\lambda(Q_n)}:\{Q_n\}_{n\in\mathbb{N}}\text{ such that }\mathscr{S}(Q_n)\in\mathcal{F}(\mathcal{I})\}\\&=\mathcal{I}-d_{-}(x,B).\end{aligned}$$

Hence, $\mathcal{I}-d_{-}(x,B)\geq1$. So, $\mathcal{I}-d_{-}(x,B)=1$ and hence $x\in\Theta_{\mathcal{I}}(B)$. Consequently, $\Theta_{\mathcal{I}}(A)\subseteq\Theta_{\mathcal{I}}(B)$. □

Theorem 3.4. *For every pair of Lebesgue measurable sets $H,G\in\mathcal{L}$, $\Theta_{\mathcal{I}}(H\cap G)=\Theta_{\mathcal{I}}(H)\cap\Theta_{\mathcal{I}}(G)$.*

Proof. Since $H\cap G\subseteq H$ and $H\cap G\subseteq G$, so by Lemma 3.3, $\Theta_{\mathcal{I}}(H\cap G)\subseteq\Theta_{\mathcal{I}}(H)$ and $\Theta_{\mathcal{I}}(H\cap G)\subseteq\Theta_{\mathcal{I}}(G)$. Consequently, $\Theta_{\mathcal{I}}(H\cap G)\subseteq\Theta_{\mathcal{I}}(H)\cap\Theta_{\mathcal{I}}(G)$. Now we are to prove $\Theta_{\mathcal{I}}(H)\cap\Theta_{\mathcal{I}}(G)\subseteq\Theta_{\mathcal{I}}(H\cap G)$. Let $x\in\Theta_{\mathcal{I}}(H)\cap\Theta_{\mathcal{I}}(G)$. Thus $x\in\Theta_{\mathcal{I}}(H)$ and $x\in\Theta_{\mathcal{I}}(G)$. So, $\mathcal{I}-d_{-}(x,H)=1$ and $\mathcal{I}-d_{-}(x,G)=1$. We are to show $\mathcal{I}-d_{-}(x,H\cap G)=1$. It is sufficient to show $\mathcal{I}-d_{-}(x,H\cap G)\geq1$.

Let $\{Q_k\}_{k\in\mathbb{N}}$ be any sequence of closed intervals about the point x such that $\mathscr{S}(Q_k)\in\mathcal{F}(\mathcal{I})$. Then for all $k\in\mathscr{S}(Q_k)$, $\lambda(H\cap Q_k)+\lambda(G\cap Q_k)-\lambda(H\cap G\cap Q_k)\leq\lambda(Q_k)$.

So, for $k\in\mathscr{S}(Q_k)$ we have

$$\frac{\lambda(H\cap Q_k)}{\lambda(Q_k)}+\frac{\lambda(G\cap Q_k)}{\lambda(Q_k)}\leq1+\frac{\lambda((H\cap G)\cap Q_k)}{\lambda(Q_k)}.$$

Let us take $\alpha_k = \frac{\lambda(H\cap Q_k)}{\lambda(Q_k)}, \beta_k = \frac{\lambda(G\cap Q_k)}{\lambda(Q_k)}, \zeta_k = \frac{\lambda((H\cap G)\cap Q_k)}{\lambda(Q_k)}$. So, $\zeta_k \geq \alpha_k+\beta_k-1$. Thus,

$$\begin{aligned}\mathcal{I}-\liminf \zeta_n &\geq \mathcal{I}-\liminf(\alpha_n+\beta_n-1)\\ &= \mathcal{I}-\liminf(\alpha_n+\beta_n)-1 \text{ by Proposition 2.7}\\ &\geq \mathcal{I}-\liminf \alpha_n+\mathcal{I}-\liminf \beta_n-1 \text{ by Theorem 2.6.}\end{aligned}$$

Hence,

$$\begin{aligned}&\inf\{\mathcal{I}-\liminf \zeta_n : \{Q_n\} \text{ such that } \mathscr{S}(Q_n)\in\mathcal{F}(\mathcal{I})\}\\ &\geq \inf\{\mathcal{I}-\liminf \alpha_n+\mathcal{I}-\liminf \beta_n-1 : \{Q_n\} \text{ such that } \mathscr{S}(Q_n)\in\mathcal{F}(\mathcal{I})\}\\ &\geq \inf\{\mathcal{I}-\liminf \alpha_n : \{Q_n\} \text{ such that } \mathscr{S}(Q_n)\in\mathcal{F}(\mathcal{I})\}\\ &\quad+\inf\{\mathcal{I}-\liminf \beta_n : \{Q_n\} \text{ such that } \mathscr{S}(Q_n)\in\mathcal{F}(\mathcal{I})\}-1.\end{aligned}$$

So,

$$\begin{aligned}\mathcal{I}-d_-(x,H\cap G) &= \inf\{\mathcal{I}-\liminf \zeta_n : \{Q_n\} \text{ such that } \mathscr{S}(Q_n)\in\mathcal{F}(\mathcal{I})\}\\ &\geq \mathcal{I}-d_-(x,H)+\mathcal{I}-d_-(x,G)-1\\ &= 1+1-1=1.\end{aligned}$$

Therefore, $\mathcal{I}-d_-(x,H\cap G)=1$. So, $x\in\Theta_{\mathcal{I}}(H\cap G)$. Hence, $\Theta_{\mathcal{I}}(H)\cap\Theta_{\mathcal{I}}(G)\subseteq\Theta_{\mathcal{I}}(H\cap G)$. This completes the proof. □

Lemma 3.5. *Let $H,G\subseteq\mathbb{R}$ such that $\lambda(H\setminus G)=0$, then $\Theta_{\mathcal{I}}(H)\subseteq\Theta_{\mathcal{I}}(G)$.*

Proof. Let us assume $\lambda(H\setminus G)=0$ and $\Theta_{\mathcal{I}}(H)\nsubseteq\Theta_{\mathcal{I}}(G)$. Then there exists $x\in\Theta_{\mathcal{I}}(H)$ such that $x\notin\Theta_{\mathcal{I}}(G)$, i.e $\mathcal{I}-d_-(x,H)=1$ but $\mathcal{I}-d_-(x,G)<1$. Now we have the following two cases.

Case(i): If $\mathcal{I}-d_-(x,H\setminus G)>0$, then

$$\inf\left\{\mathcal{I}-\liminf\frac{\lambda((H\setminus G)\cap Q_n)}{\lambda(Q_n)} : \{Q_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(Q_n)\in\mathcal{F}(\mathcal{I})\right\}>0.$$

So, for some sequence of closed intervals about x, say $\{Q_n\}_{n\in\mathbb{N}}$ such that $\mathscr{S}(Q_n)\in\mathcal{F}(\mathcal{I})$ we have

$$\mathcal{I}-\liminf\frac{\lambda((H\setminus G)\cap Q_n)}{\lambda(Q_n)}>0.$$

Thus for some $n_0\in\mathbb{N}$ we have $\lambda((H\setminus G)\cap Q_{n_0})>0$. So there exists a measurable subset $W\subseteq H$ such that $W\cap G=\emptyset$ and $\lambda(W)>0$. So, $\lambda(H\setminus G)\geq\lambda(W)>0$. It contradicts the fact that $\lambda(H\setminus G)=0$.

Case(ii): If $\mathcal{I}-d_-(x,H\setminus G)=0$. Note that H can be written as a disjoint union of $(H\setminus G)$ and $(H\cap G)$. Since $(H\setminus G)$ and $(H\cap G)$ are measurable, so by Lemma 2.9

$$\mathcal{I}-d_-(x,H)=\mathcal{I}-d_-(x,H\setminus G)+\mathcal{I}-d_-(x,H\cap G)=\mathcal{I}-d_-(x,H\cap G).$$

Thus $\mathcal{I}-d_-(x,H\cap G)=1$. Now since $H\cap G\subset G$ so $\mathcal{I}-d_-(x,H\cap G)\leq\mathcal{I}-d_-(x,G)$ which implies $\mathcal{I}-d_-(x,G)\geq 1$ which is a contradiction since $x\notin\Theta_{\mathcal{I}}(G)$.

So our assumption was wrong. Hence the result follows. □

Lemma 3.6. *Let H be any subset of $\mathbb{R}$ such that $\lambda(H)=0$, then $\Theta_{\mathcal{I}}(H)=\emptyset$ and $\Theta_{\mathcal{I}}(\mathbb{R}\setminus H)=\mathbb{R}$.*

Proof. If $\lambda(H) = 0$, then at each point $x \in \mathbb{R}$ we have

$$\begin{aligned}\mathcal{I} - d^-(x, H) &= \sup\left\{\mathcal{I} - \limsup \frac{\lambda(H \cap Q_n)}{\lambda(Q_n)} : \{Q_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I})\right\}\\ &= 0.\end{aligned}$$

This implies, $\mathcal{I} - d_-(x, H) = 0$. Thus, $\Theta_{\mathcal{I}}(H)$ is an empty set.

Clearly, $\Theta_{\mathcal{I}}(\mathbb{R} \setminus H) \subseteq \mathbb{R}$. We need to show $\mathbb{R} \subseteq \Theta_{\mathcal{I}}(\mathbb{R} \setminus H)$. Let $x \in \mathbb{R}$ and let $\{Q_k\}_{k\in\mathbb{N}}$ be any sequence of closed intervals about x such that $\mathscr{S}(Q_k) \in \mathcal{F}(\mathcal{I})$. Then for $k \in \mathscr{S}(Q_k)$ we have

$$\lambda(\mathbb{R} \cap Q_k) = \lambda((\mathbb{R} \setminus H) \cap Q_k) + \lambda(H \cap Q_k) = \lambda((\mathbb{R} \setminus H) \cap Q_k).$$

Now,

$$\begin{aligned}\mathcal{I} - d_-(x, \mathbb{R} \setminus H) &= \inf\left\{\mathcal{I} - \liminf \frac{\lambda((\mathbb{R} \setminus H) \cap Q_n)}{\lambda(Q_n)} : \{Q_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I})\right\}\\ &= \inf\left\{\mathcal{I} - \liminf \frac{\lambda(\mathbb{R} \cap Q_n)}{\lambda(Q_n)} : \{Q_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I})\right\}\\ &= 1.\end{aligned}$$

Thus, $x \in \Theta_{\mathcal{I}}(\mathbb{R} \setminus H)$. So, $\mathbb{R} \subseteq \Theta_{\mathcal{I}}(\mathbb{R} \setminus H)$. This completes the proof. □

Theorem 3.7. *If A is a measurable subset of $\mathbb{R}$, then $\Theta_{\mathcal{I}}(A) \cap \Theta_{\mathcal{I}}(A^c) = \emptyset$.*

Proof. If possible let $\Theta_{\mathcal{I}}(A) \cap \Theta_{\mathcal{I}}(A^c) \neq \emptyset$. Then there exists a point $x \in \Theta_{\mathcal{I}}(A) \cap \Theta_{\mathcal{I}}(A^c)$. So, $\mathcal{I} - d_-(x, A) = 1$ and $\mathcal{I} - d_-(x, A^c) = 1$. Thus,

$$\inf\left\{\mathcal{I} - \liminf \frac{\lambda(A \cap Q_n)}{\lambda(Q_n)} : \{Q_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I})\right\} = 1$$

and

$$\inf\left\{\mathcal{I} - \liminf \frac{\lambda(A^c \cap Q_n)}{\lambda(Q_n)} : \{Q_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(Q_n) \in \mathcal{F}(\mathcal{I})\right\} = 1.$$

So for any fixed interval $\{Q_n\}_{n\in\mathbb{N}}$, $\mathcal{I}-\liminf \frac{\lambda(A\cap Q_n)}{\lambda(Q_n)} \geq 1$ and $\mathcal{I}-\liminf \frac{\lambda(A^c\cap Q_n)}{\lambda(Q_n)} \geq 1$. Therefore,

$$\mathcal{I} - \liminf \frac{\lambda(A \cap Q_n)}{\lambda(Q_n)} + \mathcal{I} - \liminf \frac{\lambda(A^c \cap Q_n)}{\lambda(Q_n)} \geq 2.$$

As a result,

$$\mathcal{I} - \liminf \left\{\frac{\lambda(A \cap Q_n)}{\lambda(Q_n)} + \frac{\lambda(A^c \cap Q_n)}{\lambda(Q_n)}\right\} = \mathcal{I} - \liminf \frac{\lambda(\mathbb{R} \cap Q_n)}{\lambda(Q_n)} = 1 \geq 2$$

which is a contradiction. So the result follows.

□

4. $\mathcal{I}$-density topology

First we see under certain conditions, union of arbitrary collection of measurable sets can be measurable.

Theorem 4.1. *If $\{A_\alpha\}_{\alpha\in\Lambda}$ is an arbitrary collection of measurable sets, where Λ is arbitrary indexing set, such that for all $\alpha \in \Lambda$, $A_\alpha \subseteq \Theta_{\mathcal{I}}(A_\alpha)$ and $\lambda(A_\alpha \setminus B) = 0$ for any measurable set B so that $B \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is measurable.*

Proof. Since for all $\alpha \in \Lambda$, $\lambda(A_\alpha \setminus B) = 0$ thus by Lemma 3.5, $\Theta_{\mathcal{I}}(A_\alpha) \subseteq \Theta_{\mathcal{I}}(B)$. So, $\bigcup_{\alpha\in\Lambda} \Theta_{\mathcal{I}}(A_\alpha) \subseteq \Theta_{\mathcal{I}}(B)$. Since, B is measurable so by Theorem 2.12, $\lambda(B \triangle \Theta_{\mathcal{I}}(B)) = 0$ which implies $\Theta_{\mathcal{I}}(B)$ is measurable. Thus,

$$B \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha \subseteq \bigcup_{\alpha\in\Lambda} \Theta_{\mathcal{I}}(A_\alpha) \subseteq \Theta_{\mathcal{I}}(B).$$

Since, B and $\Theta_{\mathcal{I}}(B)$ are both measurable and they differ by a null set, so $\bigcup_{\alpha\in\Lambda} A_\alpha$ is measurable. □

Next, we consider $\mathcal{T}^{\mathcal{I}}$ to be the collection of measurable subsets of $\mathbb{R}$ such that each point of the set is $\mathcal{I}$-density point. So,

$$\mathcal{T}^{\mathcal{I}} = \{A \in \mathcal{L} : A \subset \Theta_{\mathcal{I}}(A)\}.$$

Whether such a collection forms a topology is the next question. The difficulty lies in the fact that a topology must be closed under arbitrary unions and arbitrary union of measurable sets may not be necessarily measurable.

Theorem 4.2. *The Lebesgue measure λ on $\mathbb{R}$ satisfies the countable chain condition, i.e. any collection of measurable sets each with positive measure, such that the intersection of two distinct elements of that collection has measure zero, is countable.*

Proof. Let $\mathcal{A}$ be a collection of sets $\{A_\alpha\}_{\alpha\in\Lambda}$, where $A_\alpha \subseteq \mathbb{R}$ and Λ is arbitrary indexing set such that for each $\alpha \in \Lambda$, A_α is measurable, $\lambda(A_\alpha) > 0$ and $\lambda(A_\alpha \cap A_\beta) = 0$ whenever $\alpha \neq \beta$, then we need to show that $\mathcal{A}$ is countable. Let us assume that $\mathcal{A}$ is uncountable. Consider $\mathbb{R}$ as $\bigcup_{n\in\mathbb{Z}}[n, n+1]$ where $\mathbb{Z}$ is the set of integers. For any $\alpha \in \Lambda$, $A_\alpha = A_\alpha \cap \mathbb{R} = A_\alpha \cap (\bigcup_{k\in\mathbb{Z}}[k, k+1]) = \bigcup_{k\in\mathbb{Z}}(A_\alpha \cap [k, k+1])$. Therefore, $\lambda(A_\alpha) = \sum_{k\in\mathbb{Z}} \lambda(A_\alpha \cap [k, k+1])$. Since $\lambda(A_\alpha) > 0$ so there exists at least one k such that $\lambda(A_\alpha \cap [k, k+1]) > 0$. Thus for each A_α there exists some $i \in \mathbb{Z}$ such that $\lambda(A_\alpha \cap [i, i+1]) > 0$. Now, if each interval $[k, k+1]$ for $k \in \mathbb{Z}$ intersect with only countably many A_α's such that $\lambda(A_\alpha \cap [k, k+1]) > 0$ then the collection $\{A_\alpha\}_{\alpha\in\Lambda}$ will be countable since countable union of countably many elements is again countable; which is a contradiction. Therefore, there exists some $k_0 \in \mathbb{Z}$ such that $[k_0, k_0+1]$ intersect with uncountably many A_α in $\mathcal{A}$ such that $\lambda(A_\alpha \cap [k_0, k_0+1]) > 0$. Take, $\Lambda' = \{\alpha \in \Lambda : \lambda(A_\alpha \cap [k_0, k_0+1]) > 0\}$. Then clearly, Λ' is uncountable and $\Lambda' \subseteq \Lambda$. Now,

$$\Lambda' = \{\alpha \in \Lambda : \lambda(A_\alpha \cap [k_0, k_0+1]) > 0\} = \bigcup_{m\in\mathbb{N}} \left\{\alpha \in \Lambda : \lambda(A_\alpha \cap [k_0, k_0+1]) \geq \frac{1}{m}\right\}.$$

If each set in the above expression under union is countable then Λ' will be countable. So, there exists some m_0 such that $\left\{\alpha \in \Lambda : \lambda(A_\alpha \cap [k_0, k_0+1]) \geq \frac{1}{m_0}\right\}$ is uncountable.

Let $\Lambda'' = \left\{\alpha \in \Lambda : \lambda(A_\alpha \cap [k_0, k_0+1]) \geq \frac{1}{m_0}\right\}$. Then $\Lambda'' \subseteq \Lambda' \subseteq \Lambda$. Since we have assumed $\lambda(A_\alpha \cap A_\beta) = 0$ whenever $\alpha \neq \beta$, so

$$\lambda([k_0, k_0+1]) \geq \sum_{\alpha\in\Lambda'} \lambda(A_\alpha \cap [k_0, k_0+1]) \geq \sum_{\alpha\in\Lambda''} \lambda(A_\alpha \cap [k_0, k_0+1]) \geq \sum_{\alpha\in\Lambda''} \frac{1}{m_0} = \infty.$$

This is a contradiction. So, $\mathcal{A}$ must be a countable collection. This completes the proof.

□

Theorem 4.3. *The collection $\mathcal{T}^{\mathcal{I}}$ is a topology on $\mathbb{R}$.*

Proof. Clearly by Lemma 3.6, $\Theta_{\mathcal{I}}(\emptyset) = \emptyset$ and $\Theta_{\mathcal{I}}(\mathbb{R}) = \mathbb{R}$ and both $\emptyset$ and $\mathbb{R}$ are measurable. So, $\emptyset, \mathbb{R} \in \mathcal{T}^{\mathcal{I}}$. Now, we will show $\mathcal{T}^{\mathcal{I}}$ is closed under finite intersection. Let $\{A_{\alpha_1}, A_{\alpha_2}, \cdots, A_{\alpha_n}\}$ be any finite collection in $\mathcal{T}^{\mathcal{I}}$. So each A_{α_k} is measurable and $A_{\alpha_k} \subseteq \Theta_{\mathcal{I}}(A_{\alpha_k})$ for each k. Clearly, $\bigcap_{k=1}^{n} A_{\alpha_k}$ is measurable. By Theorem 3.4,

$$\bigcap_{k=1}^{n} A_{\alpha_k} \subseteq \bigcap_{k=1}^{n} \Theta_{\mathcal{I}}(A_{\alpha_k}) = \Theta_{\mathcal{I}}\left(\bigcap_{k=1}^{n} A_{\alpha_k}\right).$$

Therefore, $\bigcap_{k=1}^{n} A_{\alpha_k} \in \mathcal{T}^{\mathcal{I}}$.

Next, we need to show $\mathcal{T}^{\mathcal{I}}$ is closed under arbitrary unions. If $\{A_\alpha\}_{\alpha\in\Lambda}$ is an arbitrary collection of sets in $\mathcal{T}^{\mathcal{I}}$, where Λ is arbitrary indexing set, then $\bigcup_{\alpha\in\Lambda} A_\alpha \in \mathcal{T}^{\mathcal{I}}$, i.e. we are to show $\bigcup_{\alpha\in\Lambda} A_\alpha \subseteq \Theta_{\mathcal{I}}(\bigcup_{\alpha\in\Lambda} A_\alpha)$ and $\bigcup_{\alpha\in\Lambda} A_\alpha$ is measurable.

Since for each $\alpha \in \Lambda$, $A_\alpha \in \mathcal{T}^{\mathcal{I}}$ we have $A_\alpha \subseteq \Theta_{\mathcal{I}}(A_\alpha)$ and it follows that $\bigcup_{\alpha\in\Lambda} A_\alpha \subseteq \bigcup_{\alpha\in\Lambda} \Theta_{\mathcal{I}}(A_\alpha)$. Let $x \in \bigcup_{\alpha\in\Lambda} \Theta_{\mathcal{I}}(A_\alpha)$, then there exists $\beta \in \Lambda$ such that $x \in \Theta_{\mathcal{I}}(A_\beta)$. Note that $A_\beta \subset \bigcup_{\alpha\in\Lambda} A_\alpha$, so $\Theta_{\mathcal{I}}(A_\beta) \subseteq \Theta_{\mathcal{I}}(\bigcup_{\alpha\in\Lambda} A_\alpha)$. Thus $x \in \Theta_{\mathcal{I}}(\bigcup_{\alpha\in\Lambda} A_\alpha)$. So, $\bigcup_{\alpha\in\Lambda} \Theta_{\mathcal{I}}(A_\alpha) \subseteq \Theta_{\mathcal{I}}(\bigcup_{\alpha\in\Lambda} A_\alpha)$. Therefore, $\bigcup_{\alpha\in\Lambda} A_\alpha \subseteq \Theta_{\mathcal{I}}(\bigcup_{\alpha\in\Lambda} A_\alpha)$.

It remains to show that arbitrary union of members of $\mathcal{T}^{\mathcal{I}}$ is measurable. Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be an arbitrary collection of sets in $\mathcal{T}^{\mathcal{I}}$, where Λ is arbitrary indexing set. Since by Lemma 3.6, $\lambda(A) = 0$ implies $\Theta_{\mathcal{I}}(A) = \emptyset$, which in turn implies $A \notin \mathcal{T}^{\mathcal{I}}$ so clearly, $\lambda(A_\alpha) > 0$ for all α. We choose a sequence in Λ in the following way. By Well Ordering Principle, every set can be well ordered. So, we can linearly order the elements of Λ. Choose the first element of Λ to be α_0. Following the linear order on Λ compare each element $A_{\alpha'}$ with A_{α_0}. If $\lambda(A_{\alpha'} \setminus A_{\alpha_0}) > 0$, let $\alpha_1 = \alpha'$. If no such α' exists let us take the sequence to be $(\alpha_0, \alpha_0, \cdots)$. Once α_1 is chosen search through Λ starting after α_1 to find A_{α_2} such that $\lambda(A_{\alpha_2} \setminus (A_{\alpha_0} \cup A_{\alpha_1})) > 0$. If no such α_2 exists then take the sequence as $(\alpha_0, \alpha_1, \alpha_1, \cdots)$. Continuing for each $n \in \mathbb{N}$ at any step m, assuming α_{m-1} is already chosen, search through Λ starting after α_{m-1} to find A_{α_m} such that $\lambda(A_{\alpha_m} \setminus \bigcup_{n=0}^{m-1} A_{\alpha_n}) > 0$. If no A_{α_m} can be found let the sequence be $(\alpha_0, \alpha_1, \cdots, \alpha_{m-1}, \alpha_{m-1}, \cdots)$. Whether or not a unique α_n can be found for each n, by Theorem 4.1 the sequence may be atmost countably long. So we obtain a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ such that for any $\alpha \in \Lambda$ we have $\lambda(A_\alpha \setminus \bigcup_{n=0}^{\infty} A_{\alpha_n}) = 0$. Since $\{A_{\alpha_n}\}$ is a countable sequence of measurable sets so $\bigcup_{n=0}^{\infty} A_{\alpha_n}$ is measurable. Now, by Lemma 3.5 for any $\alpha \in \Lambda$,

$$\lambda(A_\alpha \setminus \bigcup_{n=0}^{\infty} A_{\alpha_n}) = 0 \implies \Theta_{\mathcal{I}}(A_\alpha) \subseteq \Theta_{\mathcal{I}}(\bigcup_{n=0}^{\infty} A_{\alpha_n}).$$

Hence, $\bigcup_{\alpha\in\Lambda} \Theta_{\mathcal{I}}(A_\alpha) \subseteq \Theta_{\mathcal{I}}\left(\bigcup_{n=0}^{\infty} A_{\alpha_n}\right)$. By Lebesgue $\mathcal{I}$-density Theorem 2.12,

$$\lambda\left(\bigcup_{n=0}^{\infty} A_{\alpha_n} \triangle \Theta_{\mathcal{I}}\left(\bigcup_{n=0}^{\infty} A_{\alpha_n}\right)\right) = 0.$$

So, $\Theta_{\mathcal{I}}\left(\bigcup_{n=0}^{\infty} A_{\alpha_n}\right)$ is measurable. Thus,

$$\bigcup_{n=0}^{\infty} A_{\alpha_n} \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha \subseteq \bigcup_{\alpha\in\Lambda} \Theta_{\mathcal{I}}(A_\alpha) \subseteq \Theta_{\mathcal{I}}\left(\bigcup_{n=0}^{\infty} A_{\alpha_n}\right).$$

Since, $\bigcup_{n=0}^{\infty} A_{\alpha_n}$ and $\Theta_{\mathcal{I}}\left(\bigcup_{n=0}^{\infty} A_{\alpha_n}\right)$ both are measurable and differ by a null set, so $\bigcup_{\alpha\in\Lambda} A_\alpha$ is measurable. Thus, $\mathcal{T}^{\mathcal{I}}$ is closed under arbitrary unions. This completes the proof.

□

We name the topology $\mathcal{T}^{\mathcal{I}}$ to be the $\mathcal{I}$-density topology on $\mathbb{R}$ and the pair $(\mathbb{R}, \mathcal{T}^{\mathcal{I}})$ is the corresponding topological space.

The σ-algebra that is generated by the open sets of any given topology is the collection of Borel sets of that topology. The collection of Borel sets can be characterized as the σ-algebra generated by the open sets. Thus, we can talk about Borel sets on any given topological space. The $\mathcal{I}$-density topology requires open sets to be Lebesgue measurable, but not all measurable sets are open in $(\mathbb{R}, \mathcal{T}^{\mathcal{I}})$. In [22], S. Scheinberg proved the existence of a topology on the space of reals in which the Borel sets are precisely the Lebesgue measurable sets. Likewise we give a characterization of Lebesgue measurable subsets of reals in the following theorem.

Theorem 4.4. *The Borel sets in the $\mathcal{I}$-density topology $\mathcal{T}^{\mathcal{I}}$ on the space of reals are precisely the Lebesgue measurable sets.*

Proof. Let B be a Borel set in $\mathcal{T}^{\mathcal{I}}$. So B is formed through the operation of countable union, countable intersection and relative complement of sets in $\mathcal{T}^{\mathcal{I}}$. Since each element of $\mathcal{T}^{\mathcal{I}}$ is measurable so B is measurable.

Conversely, let B be lebesgue measurable. Then by Theorem 2.12 we can write $B = C \cup D$, where $C = B \cap \Theta_{\mathcal{I}}(B)$ and D is measure zero set. Then clearly the set C is measurable since both B and $\Theta_{\mathcal{I}}(B)$ are measurable. Next, $\Theta_{\mathcal{I}}(C) = \Theta_{\mathcal{I}}(B) \cap \Theta_{\mathcal{I}}(\Theta_{\mathcal{I}}(B)) = \Theta_{\mathcal{I}}(B) \supseteq C$. Therefore, the set C is $\mathcal{T}^{\mathcal{I}}$-open. Again, $\lambda(D) = 0$, so D is measurable which implies $\mathbb{R} \setminus D$ is measurable and by Lemma 3.6, $\Theta_{\mathcal{I}}(\mathbb{R} \setminus D) = \mathbb{R} \supseteq \mathbb{R} \setminus D$. So, $\mathbb{R} \setminus D$ is $\mathcal{T}^{\mathcal{I}}$-open and consequently D is $\mathcal{T}^{\mathcal{I}}$-closed. Thus, B is union of $\mathcal{T}^{\mathcal{I}}$-open and $\mathcal{T}^{\mathcal{I}}$-closed set. So, B is a Borel set in $\mathcal{T}^{\mathcal{I}}$. Therefore, the result follows. □

Acknowledgments. The first author is thankful to The Council of Scientific and Industrial Research (CSIR), Government of India, for giving the award of Senior Research Fellowship (File no. 09/025(0277)/2019-EMR-I) during the tenure of preparation of this research paper. The authors also express their gratitude to the reviewer for his remarks and suggestions which have improved the quality of the paper.

References

[1] A. K. Banerjee, *Sparse set topology and the space of proximally continuous mappings*, South Asian Journal of Mathematics **6(2)** (2016) 58–63. http://www.sajm-online.com/uploads/sajm6-2-2.pdf

[2] A. K. Banerjee and I. Debnath, *On density topology using ideals in the space of reals*, Filomat **38(2)** (2024) 743–768. https://doi.org/10.2298/FIL2402743B

[3] A. K. Banerjee and A. Paul, *I-divergence and I^*-divergence in cone metric spaces*, Asian-Eur. J. Math. **13(08)** (2020) 2050139. https://doi.org/10.1142/S1793557120501399

[4] A. K. Banerjee and A. Paul, *On I and I^*-Cauchy condition in C^*-algebra valued metric spaces*, Korean J. Math. **29(3-4)** (2021) 621–629. https://doi.org/10.11568/kjm.2021.29.3.621

[5] F. Başar, *Summability theory and its applications*, 2nd ed., CRC Press/Taylor & Francis Group, Boca Raton London New York, 2022.

[6] P. Das and A. K. Banerjee, *On the sparse set topology*, Math. Slovaca **60(3)** (2010) 319–326. https://doi.org/10.2478/s12175-010-0014-x

[7] K. Demirci, *I-limit superior and limit inferior*, Math. Commun. **6** (2001) 165–172. https://hrcak.srce.hr/file/1589
[8] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951) 241–244. https://eudml.org/doc/209960
[9] M. Filipczak and J. Hejduk, *On topologies associated with the Lebesgue measure*, Tatra Mt. Math. Publ. **28** (2004) 187–197.
[10] C. Goffman and D. Waterman, *Approximately continuous transformations*, Proc. Amer. Math. Soc. **12(1)** (1961) 116–121. https://doi.org/10.2307/2034136
[11] C. Goffman, C. J. Neugebauer and T. Nishiura, *Density topology and approximate continuity*, Duke Math. J. **28** (1961) 497–503. DOI: 10.1215/S0012-7094-61-02847-2
[12] P. R. Halmos, *Measure Theory*, Springer-Verlag, New York, 1974.
[13] J. Hejduk and R. Wiertelak, *On the generalization of density topologies on the real line*, Math. Slovaca **64** (2014) 1267–1276. https://doi.org/10.2478/s12175-014-0274-y
[14] P. Kostyrko, T. alt and W. Wilczyski, *I-convergence*, Real Anal. Exchange **26(2)** (2000/2001) 669–686.
[15] B. K. Lahiri and P. Das, *Density topology in a metric space*, J. Indian Math. Soc.(N.S.) **65** (1998) 107–117.
[16] B. K. Lahiri and P. Das, *Further results on I-limit superior and I-limit inferior*, Math. Commun. **8** (2003) 151–156. https://hrcak.srce.hr/726
[17] B. K. Lahiri and P. Das, *I and $I^\star$-convergence in topological spaces*, Math. Bohem. **130(2)** (2005) 153–160. http://dml.cz/dmlcz/134133
[18] N. F. G. Martin, *A topology for certain measure spaces*, Trans. Amer. Math. Soc. **112** (1964) 1–18. https://doi.org/10.2307/1994005
[19] M. Mursaleen and F. Başar, *Sequence Spaces: Topics in Modern Summability Theory*, 1st ed., CRC Press, Taylor & Francis Group, Boca Raton London New York, 2020.
[20] J. C. Oxtoby, *Measure and Category*, Springer-Verlag, Berlin, 1987.
[21] F. Riesz., *Sur les points de densite au sens fort*, Fund. Math. **22** (1934) 221–225. https://eudml.org/doc/212687
[22] S. Scheinberg, *Topologies which generate a complete measure algebra*, Adv. Math. **7(3)** (1971) 231–239. https://core.ac.uk/download/pdf/82344361.pdf
[23] F. Strobin and R. Wiertelak, *On a generalization of density topologies on the real line*, Topology Appl. **199** (2016) 1–16. https://doi.org/10.1016/j.topol.2015.11.005
[24] W. Wilczyski, *A generalization of density topology*, Real Anal. Exchange **8(1)** (1982-83) 16–20.
[25] W. Wilczyski, *Density topologies, In: Handbook of Measure Theory*, North-Holland, Amsterdam, 2002, 675-702.
[26] W. Wojdowski, *A generalization of the density topology*, Real Anal. Exchange **32(2)** (2006/2007) 349–358.
[27] W. Wojdowski, *A further generalization of the τ_{A_d}-density topology*, J. Appl. Anal. **19(2)** (2013) 283–304. https://doi.org/10.1515/jaa-2013-0017

Indrajit Debnath
Department of Mathematics, The University of Burdwan, Burdwan-713104, West Bengal, India
E-mail address: ind31math@gmail.com

Amar Kumar Banerjee
Department of Mathematics, The University of Burdwan, Burdwan-713104, West Bengal, India
E-mail address: akbanerjee1971@gmail.com, akbanerjee@math.buruniv.ac.in