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MULTIPLE SOLUTION FOR SOME p-KIRCHHOFF PROBLEMS WITH ψ-HILFER DERIVATIVE

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Abstract. In this paper, we will develop the variational framework for some Kirchhoff problems involving both the p-Laplace operator and the ψ-Hilfer derivative. Precisely, we use the Mountain pass theorem and Ekland's variational principle to prove the existence of multiple. Our main result generalizes the paper of César, Adv. Nonlinear Anal., **5(2)** (2016), 133-146..

1. Introduction

In the last decades, fractional calculus has attracted the attention of many researchers. Also, it appears in the application of many fields such as chemistry, electrodynamics of complex media, and physics see for example the monographs [7, 8, 12, 18, 20]. Consequently, several authors concentrated on the development of fractional operators like the Riemann Liouville derivative, Caputo derivative, and Hadamard derivative. Precisely, Hilfer [13] has developed a new fractional derivative, which is called Hilfer derivative, and generalizes both the Riemann-Liouville and the Caputo fractional derivatives. very recently, the authors in [22], have studied the question of the existence result of non-instantaneous impulsive stochastic differential equations involving Hilfer fractional derivative and driven by fractional Brownian motion.

Due to its importance and their various applications, many researchers are concentrated on the study of problems involving different fractional operators. For details and examples, one can see the papers [1, 2, 3, 9, 10, 11, 16, 19] and references therein.

We note that the first paper studying such a problem by using the variational approach is the paper of Jiao and Zhou [15]. After this, different methods are used in the study of such problems, we refer the readers to [3, 24, 5, 6, 11] and the references therein. More precisely, Torres in [23] uses the Mountain pass theorem

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to study the following problem

$$\begin{cases} -{}_sD_s^{\delta}\ {}_0D_s^{\delta}\varphi(s) = f(s,\varphi(s)),\ s\in(0,T) \\ \varphi(0)=\varphi(T)=0, \end{cases} \tag{1.1}$$

and obtain the existence of a nontrivial solution, where ${}_sD_s^{\delta}$ and ${}_0D_s^{\delta}$ are the right and left Riemann Liouville fractional derivatives.

In [24], Torres considered the following p-Laplacian Dirichlet problem

$$\begin{cases} -{}_sD_1^{\delta}\left(\Phi_p\left({}_0D_s^{\delta}\varphi(s)\right)\right) = f(s,\varphi(s)),\ s\in(0,T) \\ \varphi(0)=\varphi(T)=0, \end{cases} \tag{1.2}$$

where $0<\frac{1}{p}<\delta<1$, $f:[0,T]\times(-\infty,\infty)\to(-\infty,\infty)$ is a Carathéodory function and Φ_p is defined by

$$\Phi_p(x)=|x|^{p-2}x.$$

Under suitable assumptions on the nonlinearity f and using the direct variational method combined with the mountain pass theorem, the author proves that problem (1.2) has a nontrivial weak solution.

Given the large number of definitions of fractional operators introduced so far, several researchers are still looking for how to choose the best fractional derivative to discuss certain objectives. One of these operators is the fractional derivative with respect to another function. We cite, for example, the ψ-Riemann Liouville operators which are introduced in [16] (Chapter 2). Next, Almeida [4] introduced more general operators called ψ-Caputo operators, moreover, some other properties of Caputo and Hadamard derivatives are investigated. In 2018, Vanterler et al. [27] introduced a new and interesting fractional derivative called ψ-Hilfer derivative which is a generalization of other previous ones. In 2019 the authors extended in [25] other properties and applications for the ψ-fractional operators. In [26], the authors use the Nehari manifold method to prove the existence of a solution for some ψ-Hilfer p-Laplacian equation. Very recently, Vanterler et al. [28] investigated a problem involving the ψ-Hilfer operators using the variational method approach.

Motivated by the above-mentioned works, in this paper, we want to contribute to the development of this new area on differential equations involving fractional operators. Precisely, we will consider the following fractional boundary value problem involving the p-Laplace operator and the ψ-Hilfer fractional derivative

$$\begin{cases} K(\varphi(s))\,{}^H\mathcal{D}_s^{\mu,\delta,\psi}\left(\Phi_p\left({}^H\mathcal{D}_{0^+}^{\mu,\delta,\psi}\varphi(s)\right)\right) = \lambda g(s,\varphi(s))) + f(s,\varphi(s)), \quad s\in(0,T), \\ I_{0^+}^{\delta(\delta-1);\psi}(0) = I_s^{\delta(\delta-1);\psi}(T) = 0, \end{cases} \tag{1.3}$$

where $\lambda>0$, ${}^H\mathcal{D}_s^{\mu,\delta,\psi}$ and ${}^H\mathcal{D}_{0^+}^{\mu,\delta,\psi}$ are the right-sided and left sided ψ-Hilfer derivatives of order $0<\mu\in(\frac{1}{p},1]$, and of type $\delta\in[0,1]$, $I_{0^+}^{\delta(\delta-1);\psi}$ and $I_s^{\delta(\delta-1);\psi}$ are the left-sided and the right-sided ψ-Riemann-Liouville fractional integrals. The function $K:(-\infty,\infty)\to(-\infty,\infty)$ is defined by

$$K(\varphi(s)) = \left(a + b\int_0^s \left|{}^H\mathcal{D}_{0^+}^{\mu,\delta,\psi}\varphi(s)\right|^p ds\right)^{p-1}, \quad a,b\geq 1.$$

The functions $f,g:[0,T]\times(-\infty,\infty)\to(-\infty,\infty)$ are continuous, moreover, g is positively homogeneous of degree $q-1$, which means that for all $t>0$ we have

$$g(s,tu)=t^{q-1}g(s,u),\ \ (s,u)\in[0,T]\times(-\infty,\infty).$$

Hereafter, we put

$$F(x,s)=\int_0^s f(x,y)dy, \quad G(x,s)=\int_0^s g(x,y)dy,$$

and we assume that the nonlinearities satisfy the following hypotheses:
($\mathbf{H_1}$) There exists $\sigma > p$ such that

$$0<\sigma F(x,\varphi)\leq \varphi f(x,\varphi), \quad \forall\ (x,\varphi)\in[0,T]\times(-\infty,\infty), \tag{1.4}$$

Also, we assume that

$$|F(s,\varphi)|\leq C_0|\varphi|^{\sigma},\ C_0>0. \tag{1.5}$$

($\mathbf{H_2}$) $G:[0,T]\times(-\infty,\infty)\to(-\infty,\infty)$ is positively homogeneous of degree q, tha is

$$G(s,t\varphi)=t^qG(s,\varphi),\ \ \forall\ (t,s,\varphi)\in(0,\infty)\times[0,T]\times(-\infty,\infty).$$

We notes hypothesis ($\mathbf{H_2}$) leads to the so-called Euler identity:

$$\varphi g(s,\varphi)=qG(s,\varphi). \tag{1.6}$$

Moreover, for some positive constant C_1, we get

$$|G(s,\varphi)|\leq C_1|\varphi|^q. \tag{1.7}$$

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that hypotheses* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *are satisfied. If* $\frac{1}{p}<\delta<1$, *and* $1<p^2<\min(\sigma,q)$, *then there exists* $\lambda_0>0$, *such that for any* $\lambda\in(0,\lambda_0)$, *problem* (1.3) *admits two nontrivial weak solutions.*

We note that this result generalizes those of Torres stated in [23] for the cases $p=2$, $K\equiv 1$ and in [24] for the case $K\equiv 1$. This paper is organized as follows. In Section 2, some preliminaries and lemmas on fractional calculus are presented, moreover. In Section 3, we introduce the variational framework of problem (1.3)and we prove the main result of this paper (Theorem 1.1).

2. Preliminaries and variational setting

In this section, we present some preliminaries and background theory on the concept of ψ-Hilfer fractional derivative which will be used in the rest of this paper. First, let us start by introducing the definition of the fractional integral in the sense of Kilbas et al. [16] and Samko et al. [21]. Throughout this section, μ and δ denote positive real numbers, Γ denotes the Euler gamma function, and if $-\infty\leq a<b\leq\infty$, then $[a,b]$ denotes a finite or infinite interval in the real line. *psi* denotes an increasing positive function on $[a,b]$ with continuous derivative $\psi'(s)\neq 0$ on (a,b).
In the rest of this paper, φ denotes an integrable function defined on $[a,b]$, and ψ an increasing function in $C^1\big([a,b],(-\infty,\infty)\big)$ such that $\psi'(x)\neq 0$, for all $x\in[a,b]$. Also, for two given numbers x and y, we denote by $\psi_x(y)$ the following expression:

$$\psi_x(y)=\psi(x)-\psi(y).$$

Finally, we deonte by $d_{x,\psi}$ the following operator:

$$d_{x,\psi}=\frac{1}{\psi'(x)}\frac{d}{dx}.$$

Definition 2.1. ([16, 21]) The left and right fractional integrals of a function φ with respect to a function ψ are defined respectively as follows:

$$I_{a^+}^{\delta,\psi}\varphi(x) := \frac{1}{\Gamma(\delta)}\int_a^x \psi'(s)(\psi_x(s))^{\delta-1}\varphi(s)ds,$$

and

$$I_{b^-}^{\delta,\psi}\varphi(x) := \frac{1}{\Gamma(\delta)}\int_x^b \psi'(s)(\psi_s(x))^{\delta-1}\varphi(s)ds.$$

Definition 2.2. ([25, 27]) The ψ-Hilfer fractional derivatives left-sided and right-sided of order $\mu > 0$ and of type $0 \leq \delta \leq 1$ are defined respectively by:

$$^{H}\mathcal{D}_{a^+}^{\mu,\delta,\psi}\varphi(x) \quad := \quad I_{a^+}^{\delta(n-\mu),\psi}\left(d_{x,\psi}\right)^n I_{a^+}^{(1-\delta)(n-\mu),\psi}\varphi(x),$$

and

$$^{H}\mathcal{D}_{b^-}^{\mu,\delta,\psi}\varphi(x) \quad := \quad I_{b^-}^{\delta(n-\mu),\psi}\left(d_{x,\psi}\right)^n I_{b^-}^{(1-\delta)(n-\mu),\psi}\varphi(x),$$

where n is such that $n-1 < \mu \leq n$.

Remark. *The following statements hold.*

(i) From the ψ-Hilfer fractional derivatives, if δ tends to zero, then we obtain the ψ-Riemann-Liouville fractional derivatives which are defined by

$$\mathcal{D}_{a^+}^{\mu,\psi}\varphi(x) \quad = \quad \left(d_{x,\psi}\right)^n I_{a^+}^{n-\mu,\psi}\varphi(x),$$

and

$$\mathcal{D}_{b^-}^{\mu,\psi}\varphi(x) \quad = \quad \left(-d_{x,\psi}\right)^n I_{b^-}^{n-\mu,\psi}\varphi(x),$$

(ii) If $\delta \to 1$, then we obtain the ψ-Caputo fractional derivatives which are defined by

$$^{C}\mathcal{D}_{a^+}^{\mu,\psi}\varphi(x) \quad = \quad I_{a^+}^{n-\mu,\psi}\left(d_{x,\psi}\right)^n\varphi(x),$$

and

$$^{C}\mathcal{D}_{b^-}^{\mu,\delta,\psi}\varphi(x) \quad = \quad I_{b^-}^{n-\mu,\psi}\left(-d_{x,\psi}\right)^n\varphi(x),$$

(iii) There are direct relations between ψ-Hilfer fractional derivatives and ψ-Riemann-Liouville fractional derivatives, precisely we have

$$^{H}\mathcal{D}_{a^+}^{\mu,\delta,\psi}\varphi(x) \quad = \quad I_{a^+}^{\sigma-\mu,\psi}\mathcal{D}_{a^+}^{\sigma,\psi}\varphi(x),$$

and

$$^{H}\mathcal{D}_{b^-}^{\mu,\delta,\psi}\varphi(x) \quad = \quad I_{b^-}^{\sigma-\mu,\psi}\mathcal{D}_{b^-}^{\sigma,\psi}\varphi(x),$$

where $\sigma = \mu + \delta(n-\mu)$.

Now, by interchanging the order of integration by the Dirichlet formula in the particular case Fubini theorem, we can prove the following integration by parts for the ψ-Riemann-Liouville fractional integral:

$$\int_a^b I_{a^+}^{\delta,\psi}\varphi(x)v(x)\,dx = \int_a^b \varphi(x)\psi'(x)I_{b^-}^{\delta,\psi}\left(\frac{v(x)}{\psi'(x)}\right)dx.$$

For more details, we refer the readers to [26].
Also, we need the following result which is called fractional integration by parts for the ψ-Hilfer fractional derivatives:

Lemma 2.3 ([26]). *If φ is an absolutely countinous function on $[a,b]$ and s is a C^1 function on $[a,b]$ such that $s(a)=s(b)=0$. Then for $0<\mu\leq 1$ and $0\leq\delta\leq 1$, we have*

$$\int_a^b {}^H\mathcal{D}_{a^+}^{\mu,\delta,\psi}\varphi(x)s(x)\,dx = \int_a^b \varphi(x)\psi'(x)\ {}^H\mathcal{D}_{b^-}^{\mu,\delta,\psi}\left(\frac{s(x)}{\psi'(x)}\right)dx \tag{2.1}$$

For $1\leq r\leq\infty$, $L^r(a,b)$ denotes the set of all measurable function u on $[a,b]$, such that $\int_a^b|\varphi(s)|^r ds<\infty$.
Put

$$\|\varphi\|_{L^r(a,b)} = \left(\int_a^b |\varphi(s)|^r ds\right)^{\frac{1}{r}},$$

and

$$\|\varphi\|_\infty = ess\sup_{a\leq t\leq b}|\varphi(s)|.$$

Remark. *([17, 26]) If $0<\mu\leq 1$, $r\geq 1$ and $q=\frac{r}{r-1}$, then For each $\varphi\in L^r(a,b)$, we have:*

(i) *$I_{a^+}^{\mu,\psi}\varphi$ is bounded in $L^r(a,b)$, moreover we have*

$$\|I_{a^+}^{\mu,\psi}\varphi\|_{L^r(a,b)} \leq \frac{(\psi_b(a))^\mu}{\Gamma(\mu+1)}\|\varphi\|_{L^r(a,b)}.$$

(ii) *If $\frac{1}{r}<\mu<1$, then $I_{a^+}^{\mu,\psi}$ is Hölder continuous on $[a,b]$ with exponent $\mu-\frac{1}{r}$.*
(iii) *If $\frac{1}{r}<\mu<1$, then $\lim_{s\to a} I_{a^+}^{\mu,\psi}\varphi(s)=0$. That is $I_{a^+}^{\mu,\psi}\varphi$ can be continuously extended by zero in $t=a$. So, $I_{a^+}^{\mu,\psi}\varphi$ is continuous on $[a,b]$, moreover, we get*

$$\|I_{a^+}^{\mu,\psi}\varphi\|_\infty \leq \frac{(\psi_b(a))^{\mu-\frac{1}{r}}}{\Gamma(\mu)\Big((\mu-1)q+1\Big)^{\frac{1}{q}}}\|\varphi\|_{L^r(a,b)}.$$

The main tool in the proof of our main result is the following theorem.

Theorem 2.4. *(Mountain pass theorem) Let E be a real Banach space and $J\in C^1(E,(-\infty,\infty))$ satisfying the Palai Smale condition. Assume that that*

(i) *$J(0)=0$,*
(ii) *There is $\rho>0$ and $\sigma>0$ such that $J(z)\geq\sigma$ for all $z\in E$ with $\|z\|=\rho$.*
(iii) *There exists $z_1\in E$ with $\|z_1\|\geq\rho$ such that $J(z_1)<0$.*

Then J possesses a critical value at level $c\geq\sigma$. Moreover, c can be characterized as

$$c=\inf_{\gamma\in\Gamma}\max_{z\in[0,1]}J(\gamma(z)),$$

where $\Gamma=\{\gamma\in C([0,1],E):\gamma(0)=0,\gamma(1)=z_1\}$.

We note that The functional J satisfies the Palais-Smale condition if any Palais-Smale sequence has a strongly convergent subsequence. This means that if a sequence $\{\varphi_m\}$ in E is such that $J(\varphi_m)$ is bounded and $J'(\varphi_m)$ converges to 0 in the dual space E', then $\{\varphi_m\}$ has a convergent subsequence.

3. The proof of the main result

In this section, to apply the mountain pass theorem, we begin by introducing the fractional derivative space and some other interesting results. In order to formulate the variational setting to problem (1.3), we define the fractional derivative space $E_p^{\mu,\delta,\psi}$ by the closure of $C_0^\infty([0,T],(-\infty,\infty))$ with respect to the norm

$$\|\varphi\|_{E_p^{\mu,\delta,\psi}} = \left(\|\varphi\|^p_{L^p(0,T)} + \|{}_0\mathcal{D}_s^{\mu,\delta,\psi}u\|^p_{L^p(0,T)}\right)^{\frac{1}{p}}.$$

We note that, the space $E_p^{\mu,\psi}$ can be described as follows:

$$E_p^{\mu,\delta,\psi} = \left\{v \in L^p([0,T],(-\infty,\infty)) :\ \mathcal{D}_{0^+}^{\mu,\delta,\psi}v \in L^p([0,T],(-\infty,\infty)), I_{0^+}^{\delta(\delta-1);\psi}(0) = I_s^{\delta(\delta-1);\psi}(s) = 0\right\}.$$

Remark. *([17, 26]) If $0<\mu\leq 1$ and $0\leq\delta\leq 1$, then for all $\varphi\in E_p^{\mu,\delta,\psi}$, we have*

(i) *The space $E_p^{\mu,\delta,\psi}$ is a reflexive and separable Banach space.*

(ii) *If $1-\mu>\frac{1}{p}$ or $\mu>\frac{1}{p}$, then, we get*

$$\|\varphi\|_{L^p(0,T)} \leq \frac{(\psi_s(0))^\mu}{\Gamma(\mu+1)}\|\mathcal{D}_{0^+}^{\mu,\delta,\psi}\|_{L^p(0,T)}.$$

(ii) *If $\frac{1}{p}<\mu$, and $q=\frac{p}{p-1}$, then we have*

$$\|\varphi\|_\infty \leq \frac{(\psi_s(a))^{\mu-\frac{1}{r}}}{\Gamma(\mu)\Big((\mu-1)q+1\Big)^{\frac{1}{q}}}\|\mathcal{D}_{0^+}^{\mu,\delta,\psi}\|_{L^r(0,T)}.$$

We note tha, from Remark 3, one has

$$\|\varphi\|_\infty \leq \frac{(\psi_s(a))^{\mu-\frac{1}{r}}}{\Gamma(\mu)\Big((\mu-1)q+1\Big)^{\frac{1}{q}}}\|\varphi\|_{E_p^{\mu,\delta,\psi}}. \tag{3.1}$$

Definition 3.1. A function φ is said to be a weak solution of problem (1.3), if for every $v\in E_p^{\mu,\delta,\psi}$ we have :

$$\begin{aligned} K(\varphi(s))\int_0^s \Phi_p({}_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi(s))\,{}_0\mathcal{D}_s^{\mu,\delta,\psi}v(s)ds &= \lambda\int_0^s f(s,\varphi(s))v(s)ds \\ &\quad + \int_0^s g(s,\varphi(s))v(s)ds. \end{aligned}$$

Since we use the variational method, it is natural to define the associate functional $\mathcal{J}_\lambda : E_p^{\mu,\delta,\psi}\to(-\infty,\infty)$, which is defined by:

$$\mathcal{J}_\lambda(\varphi) = \frac{1}{bp^2}\left(a+b\|\varphi\|^p_{\mu,\psi}\right)^p - \lambda\int_0^s F(s,\varphi(s))ds - \int_0^s G(s,\varphi(s))ds - \frac{a^p}{bp^2}.$$

Since F and G are continuous, it is not difficult to show that $\mathcal{J}_\lambda\in C^1(E_p^{\mu,\delta,\psi},(-\infty,\infty))$, moreover for all $\varphi, v\in E_p^{\mu,\delta,\psi}$, we have

$$\begin{aligned} \langle \mathcal{J}_\lambda'(\varphi), v\rangle &= \left(a+b\|\varphi\|^p_{\mu,\psi}\right)^{p-1}\int_0^s \Phi_p({}_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi(s)){}_0\mathcal{D}_s^{\mu,\delta,\psi}v(s)ds \\ &- \lambda\int_0^s f(s,\varphi(s))v(s)ds - \int_0^s g(s,\varphi(s))v(s)ds \end{aligned} \tag{3.2}$$

So, critical points of $\mathcal{J}_\lambda$ are solutions of problem (1.3).

Lemma 3.2. *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold, and* $\min(\sigma, q) > p$. *Then there exist* $\rho > 0$ *and* $\sigma > 0$, *such that for all* $z \in E_p^{\mu,\delta,\psi}$ *we have*

$$\|z\| = \rho \implies \mathcal{J}_\lambda(z) \geq \sigma > 0.$$

Proof. Let $z \in E_p^{\mu,\delta,\psi}$, then, by equations (1.5), (1.7) and (3.1), we obtain

$$\begin{aligned}
\mathcal{J}_\lambda(z) &= \frac{1}{bp^2}\left(a + b\|z\|^p_{\mu,\psi}\right)^p - \lambda \int_0^s F(s, z(s))ds - \int_0^s G(s, z(s))ds - \frac{a^p}{bp^2} \\
&\geq \frac{1}{bp^2}\left(a + b\|z\|^p_{\mu,\psi}\right)^p - \lambda C_0 \int_0^s |z|^\sigma ds - C_1 \int_0^s |z|^q ds - \frac{a^p}{bp^2} \\
&\geq \frac{1}{bp^2}\left(a + b\|z\|^p_{\mu,\psi}\right)^p - \lambda C_0 T \left(\frac{(\psi_s(0))^{\mu - \frac{1}{p}}}{\Gamma(\mu)((\mu-1)q+1)^{\frac{1}{q}}}\right)^\sigma \|z\|^\sigma_{\mu,\psi} \\
&- C_1 T \left(\frac{(\psi_s(0))^{\mu - \frac{1}{p}}}{\Gamma(\mu)((\mu-1)q+1)^{\frac{1}{q}}}\right)^q \|z\|^q_{\mu,\psi} - \frac{a^p}{bp^2}. \qquad (3.3)
\end{aligned}$$

If $\|z\|_{\mu,\psi} = \rho > 0$, then, from (3.3) and the following elementary inequality

$$(x+y)^p \geq x^p + pyx^{p-1},$$

we get

$$\begin{aligned}
\mathcal{J}_\lambda(z) &\geq \frac{1}{bp^2}(a + b\rho^p)^p - \lambda C_0 T \left(\frac{(\psi_s(0))^{\mu - \frac{1}{p}}}{\Gamma(\mu)((\mu-1)q+1)^{\frac{1}{q}}}\right)^\sigma \rho^\sigma \\
&- C_1 T \left(\frac{(\psi_s(0))^{\mu - \frac{1}{p}}}{\Gamma(\mu)((\mu-1)q+1)^{\frac{1}{q}}}\right)^q \rho^q - \frac{a^p}{bp^2} \\
&\geq \frac{\rho^p a^{p-1}}{p} - \lambda C_0 T \left(\frac{(\psi_s(0))^{\mu - \frac{1}{p}}}{\Gamma(\mu)((\mu-1)q+1)^{\frac{1}{q}}}\right)^\sigma \rho^\sigma - C_1 T \left(\frac{(\psi_s(0))^{\mu - \frac{1}{p}}}{\Gamma(\mu)((\mu-1)q+1)^{\frac{1}{q}}}\right)^q \rho^q \\
&\geq \rho^p h(\rho).
\end{aligned}$$

where

$$h(t) = \frac{a^{p-1}}{p} - \lambda C_0 T \left(\frac{(\psi_s(0))^{\mu - \frac{1}{p}}}{\Gamma(\mu)((\mu-1)q+1)^{\frac{1}{q}}}\right)^\sigma t^{\sigma - p} - C_1 T \left(\frac{(\psi_s(0))^{\mu - \frac{1}{p}}}{\Gamma(\mu)((\mu-1)q+1)^{\frac{1}{q}}}\right)^q t^{q-p}.$$

Since $\min(\sigma, q) > p$, then we have

$$\lim_{\rho \to 0} h(\rho) = \frac{a^{p-1}}{p} > 0.$$

Hence, we can choose $\rho > 0$ small enough such that

$$\rho^p h(\rho) := \sigma > 0.$$

Which yields to $\mathcal{J}_\lambda(z) \geq \sigma > 0$. □

Lemma 3.3. *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. If* $\min(\sigma, q) > p^2$, *then, there exists* $z_1 \in E_p^{\mu,\delta,\psi}$ *with* $\|z_1\| \geq \rho$, *and* $\mathcal{J}_\lambda(z_1) < 0$.

Proof. Let $s > 0$ large enough such that $a < s^p$, then using hypothesis $(\mathbf{H_2})$ and equation (1.5), we obtain for all $\varphi \in E_p^{\mu,\delta,\psi}$

$$\begin{aligned}
\mathcal{J}_\lambda(s\varphi) &= \frac{1}{bp^2}\left(a+b\|s\varphi\|_{\mu,\psi}^p\right)^p - \lambda\int_0^s F(s,s\varphi(s))ds - \int_0^s G(s,s\varphi(s))ds - \frac{a^p}{bp^2} \\
&\leq \frac{s^{p^2}}{bp^2}\left(1+b\|\varphi\|_{\mu,\psi}^p\right)^p - s^q\int_0^s G(s,\varphi(s))ds \\
&\leq \frac{s^{p^2}}{bp^2}\left(1+b\|\varphi\|_{\mu,\psi}^p\right)^p - C_1 s^q\int_0^s |\varphi(s)|^q\,ds.
\end{aligned}$$

Since $q > p^2$, then we get $\lim\limits_{s\to\infty}\mathcal{J}_\lambda(s\varphi) = -\infty$. So, there exists $R > 0$ such that for $s > R$, we have $\mathcal{J}_\lambda(s\varphi) < 0$. If we fix $s_0 > \max(R, a^{\frac{1}{p}})$, then, for $z_1 = s_0\varphi$, we have $\|z_1\| \geq \rho$, and $\mathcal{J}_\lambda(z_1) < 0$. □

Lemma 3.4. *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Assume that* $\min(\sigma, q) > p^2$*, then* $\mathcal{J}_\lambda$ *satisfies the Palais-Smale condition*

Proof. Let $\{\varphi_k\} \in E_p^{\mu,\delta,\psi}$ be a Palai-Smale sequence. Then we can find $C_2 > 0$ and $C_3 > 0$, such that

$$|\mathcal{J}_\lambda(\varphi_k)| \leq C_2, \text{ and } |\langle \mathcal{J}_\lambda'(\varphi_k), \varphi_k\rangle| < C_3. \tag{3.4}$$

First of all, we claim that $\{u_k\}$ is bounded. indeed, If not, up to a subsequence, we can assume that

$$\|\varphi_k\| \to \infty, \text{ as } k \to \infty.$$

Put $\theta = \min(\sigma, q)$, then, by combining equation 3.4 with equations (1.4) and (1.6), we get

$$\begin{aligned}
C_2 &\geq \mathcal{J}_\lambda(\varphi_k) \\
&= \frac{1}{bp^2}\left(a+b\|\varphi_k\|_{\mu,\psi}^p\right)^p - \lambda\int_0^s F(s,\varphi_k(s))ds - \int_0^s G(s,\varphi_k(s))ds - \frac{a^p}{bp^2} \\
&= \frac{1}{bp^2}\left(a+b\|\varphi_k\|_{\mu,\psi}^p\right)^p - \frac{\lambda}{\sigma}\int_0^s f(s,\varphi_k(s))\varphi_k ds - \frac{1}{q}\int_0^s g(s,\varphi_k(s))\varphi_k ds - \frac{a^p}{bp^2} \\
&\geq \frac{1}{bp^2}\left(a+b\|\varphi_k\|_{\mu,\psi}^p\right)^p - \frac{\lambda}{\theta}\int_0^s f(s,\varphi_k(s))\varphi_k ds \\
&\quad -\frac{1}{\theta}\int_0^s g(s,\varphi_k(s))\varphi_k ds - \frac{a^p}{bp^2}.
\end{aligned} \tag{3.5}$$

Now, from (3.2), we have

$$\begin{aligned}
\langle \mathcal{J}_\lambda'(\varphi_k), \varphi_k\rangle &= \left(a+b\|\varphi_k\|_{\mu,\psi}^p\right)^{p-1}\|\varphi_k\|^p - \lambda\int_0^s f(s,\varphi_k(s))\varphi_k(s)ds \\
&- \int_0^s g(s,\varphi_k(s))\varphi_k(s)ds.
\end{aligned}$$

So from the definition of the sequence $\{\varphi_k\}$, we obtain

$$\begin{aligned}
C_3 &\geq -\langle \mathcal{J}_\lambda'(\varphi_k), \varphi_k\rangle \\
&= -\left(a+b\|\varphi_k\|_{\mu,\psi}^p\right)^{p-1}\|\varphi_k\|^p + \lambda\int_0^s f(s,\varphi_k(s))\varphi_k(s)ds \\
&\quad + \int_0^s g(s,\varphi_k(s))\varphi_k(s)ds.
\end{aligned} \tag{3.6}$$

Now, by combining (3.5) with (3.6), we get

$$\begin{aligned}
\theta C_2 + C_3 &\geq \theta \frac{1}{bp^2}\left(a+b\|\varphi_k\|^p_{\mu,\psi}\right)^p - \left(a+b\|\varphi_k\|^p_{\mu,\psi}\right)^{p-1}\|\varphi_k\|^p \\
&- (\theta-1)\frac{a^p}{bp^2} \\
&= \left(a+b\|\varphi_k\|^p_{\mu,\psi}\right)^{p-1}\left(\frac{\theta}{bp^2}(a+b\|\varphi_k\|^p_{\mu,\psi}) - \|\varphi_k\|^p_{\mu,\psi}\right) - (\theta-1)\frac{a^p}{bp^2} \\
&= \left(a+b\|\varphi_k\|^p_{\mu,\psi}\right)^{p-1}\left(\frac{a\theta}{bp^2} + (\frac{\theta}{p^2}-1)\|\varphi_k\|^p_{\mu,\psi})\right) - (\theta-1)\frac{a^p}{bp^2}.
\end{aligned}$$

Since $\theta > p^2$, then by letting k tend to infinity we obtain a contradiction. So $\{\varphi_k\}$ is bounded. Therefore, from Remark 3 (i), there exists $\varphi_* \in E_p^{\mu,\delta,\psi}$ such that, up to a subsequence, we have

$$\begin{cases} \varphi_k \rightharpoonup \varphi_*, \text{ weakly in } E_p^{\mu,\delta,\psi}, \\ \varphi_k \to \varphi_*, \text{ in } C([0,T],(-\infty,\infty)). \end{cases}$$

From (3.2), we get
$\langle \mathcal{J}'_\lambda(\varphi_k) - \mathcal{J}'_\lambda(\varphi_*), \varphi_k - \varphi_* \rangle$

$$\begin{aligned}
&= \left(a+b\|\varphi_k\|^p_{\mu,\psi}\right)^{p-1}\int_0^s \Phi_p({}_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi_k(s))_0\mathcal{D}_s^{\mu,\delta,\psi}(\varphi_k(s)-\varphi_*(s))ds \\
&- \left(a+b\|\varphi_*\|^p_{\mu,\psi}\right)^{p-1}\int_0^s \Phi_p({}_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi_*(s))_0\mathcal{D}_s^{\mu,\delta,\psi}(\varphi_k-\varphi_*(s))ds \\
&-\lambda\int_0^s \Big(f(s,\varphi_k(s)-f(s,\varphi_*(s)\Big)(\varphi_k(s)-\varphi_*(s))ds \\
&-\lambda\int_0^s \Big(g(s,\varphi_k(s)-g(s,\varphi_*(s)\Big)(\varphi_k(s)-\varphi_*(s))ds \\
&= \left(a+b\|\varphi_k\|^p_{\mu,\psi}\right)^{p-1}\left(\|\varphi_k\|^p_{\mu,\psi} - \int_0^s \Phi_p({}_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi_k(s))_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi_*(s)ds\right) \\
&+ \left(a+b\|\varphi_*\|^p_{\mu,\psi}\right)^{p-1}\left(\|\varphi_k\|^p_{\mu,\psi} - \int_0^s \Phi_p({}_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi_*(s))_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi_k(s)ds\right) \\
&-\lambda\int_0^s \Big(f(s,\varphi_k(s)-f(s,\varphi_*(s)\Big)(\varphi_k(s)-\varphi_*(s))ds \\
&-\lambda\int_0^s \Big(g(s,\varphi_k(s)-g(s,\varphi_*(s)\Big)(\varphi_k(s)-\varphi_*(s))ds.
\end{aligned}$$

By the Hölder inequality, we obtain

$$\int_0^s \Phi_p({}_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi_k(s))_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi_*(s)ds \leq \|\varphi_k\|^{p-1}_{\mu,\psi}\|\varphi_*\|^p_{\mu,\psi},$$

$$\int_0^s \Phi_p({}_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi_*(s))_0\mathcal{D}_s^{\mu,\delta,\psi}\varphi_k(s)ds \leq \|\varphi_*\|^{p-1}_{\mu,\psi}\|\varphi_k\|^p_{\mu,\psi}.$$

So,

$$\begin{aligned}
\langle \mathcal{J}'_\lambda(\varphi_k) - \mathcal{J}'_\lambda(\varphi_*), \varphi_k - \varphi_* \rangle \quad &\geq \quad \Big(a + b\|\varphi_k\|^p_{\mu,\psi}\Big)^{p-1} \Big(\|\varphi_k\|^p_{\mu,\psi} - \|\varphi_k\|^{p-1}_{\mu,\psi}\|\varphi_*\|_{\mu,\psi}\Big) \\
& \quad\quad \Big(a + b\|\varphi_*\|^p_{\mu,\psi}\Big)^{p-1} \Big(\|\varphi_k\|^p_{\mu,\psi} - \|\varphi_*\|^{p-1}_{\mu,\psi}\|\varphi_k\|_{\mu,\psi}\Big) \\
& - \quad \lambda \int_0^s \Big(f(s,\varphi_k(s) - f(s,\varphi_*(s)\Big)(\varphi_k(s) - \varphi_*(s))ds \\
& - \quad \lambda \int_0^s \Big(g(s,\varphi_k(s) - g(s,\varphi_*(s)\Big)(\varphi_k(s) - \varphi_*(s))ds \\
& \geq \quad \Big(a + b\|\varphi_k\|^p_{\mu,\psi}\Big)^{p-1} \|\varphi_k\|^{p-1}_{\mu,\psi} (\|\varphi_k\|_{\mu,\psi} - \|\varphi_*\|_{\mu,\psi}) \\
& + \quad \Big(a + b\|\varphi_*\|^p_{\mu,\psi}\Big)^{p-1} \|\varphi_*\|^{p-1}_{\mu,\psi} (\|\varphi_k\|_{\mu,\psi} - \|\varphi_k\|_{\mu,\psi}) \\
& - \quad \lambda \int_0^s \Big(f(s,\varphi_k(s) - f(s,\varphi_*(s)\Big)(\varphi_k(s) - \varphi_*(s))ds \\
& - \quad \lambda \int_0^s \Big(g(s,\varphi_k(s) - g(s,\varphi_*(s)\Big)(\varphi_k(s) - \varphi_*(s))ds
\end{aligned} \tag{3.7}$$

Since $\varphi_k \to \varphi_*$, in $C([0,T],(-\infty,\infty))$, and $|f(s,\varphi_k(s) - f(s,\varphi_*(s)|, |g(s,\varphi_k(s) - g(s,\varphi_*(s)|$ are bounded, then as $k \to \infty$, we have

$$\int_0^s \Big(f(s,\varphi_k(s) - f(s,\varphi_*(s)\Big)(\varphi_k(s) - \varphi_*(s))ds \to 0, \tag{3.8}$$

and

$$\int_0^s \Big(g(s,\varphi_k(s) - g(s,\varphi_*(s)\Big)(\varphi_k(s) - \varphi_*(s))ds \to 0. \tag{3.9}$$

On the other hand, since $\varphi_k \rightharpoonup \varphi_*$, weakly in $E_p^{\mu,\delta,\psi}$, and $\mathcal{J}'_\lambda(\varphi_k) \to 0$, then we get

$$\langle \mathcal{J}'_\lambda(\varphi_k) - \mathcal{J}'_\lambda(\varphi_*), \varphi_k - \varphi_* \rangle \to 0, \text{ as } k \to \infty. \tag{3.10}$$

By combining equations (3.8), (3.9), (3.10) with equation (3.7), we get

$$\|\varphi_k\|_{\mu,\psi} \to \|\varphi_*\|_{\mu,\psi}, \text{ as } k \to \infty.$$

Finally, by the uniform convexity of $E_p^{\mu,\delta,\psi}$, the weakly convergence of φ_k to φ_* in $E_p^{\mu,\delta,\psi}$ and the KadecKlee property (see [14]), we obtain that

$$\varphi_k \to \varphi_* \text{ strongly in } E_p^{\mu,\delta,\psi}.$$

This ends the proof of Lemma 3.4. □

Now, we are ready to prove the main result of this paper.

Proof of Theorem 1.1 We begin this proof by notting that $\mathcal{J}_\lambda(0) = 0$. So the condition (i) of Theorem 2.4 is satisfied.

Next, by Lemma 3.2, There exist $\rho > 0$ and $\sigma > 0$ such that for all $z \in E_p^{\mu,\delta,\psi}$, we have

$$\|z\| = \rho \implies \mathcal{J}_\lambda(z) \geq \sigma > 0. \tag{3.11}$$

Therefore, the condition (ii) of Theorem 2.4 is also fulfilled. On the other hand, from Lemma 3.3, there exists $z_1 \in E_p^{\mu,\delta,\psi}$ satisfying

$$\|z_1\| \geq \rho, \text{ and } \mathcal{J}_\lambda(z_1) < 0. \tag{3.12}$$

By combining equations (3.11), (3.12), Lemma 3.4 with Theorem 2.4, we deduce the existence of a critical point φ_λ of $\mathcal{J}_\lambda$, which is a weak solution for problem (1.3). Moreover, by (3.11) we get

$$\mathcal{J}_\lambda(\varphi_\lambda) > 0. \tag{3.13}$$

Hence φ_λ is nontrivial.

Now, we will the existence of a second solution for problem (1.3). By Lemma 3.2, we get

$$\inf_{\varphi\in\partial B(0,\rho)} \mathcal{J}_\lambda(\varphi) > 0, \text{ and } \quad -\infty < \underline{c} = \inf_{u\in\overline{B(0,\rho)}} \mathcal{J}_\lambda(\varphi) < 0,$$

where

$$B(0,\rho) = \{\varphi \in E_p^{\delta,\psi} : \|\varphi\|_{\delta,\psi} < \rho\} \text{ and } \overline{B(0,\rho)} = \{\varphi \in E_p^{\delta,\psi} : \|\varphi\|_{\delta,\psi} \le \rho\}.$$

Let n be an integer which is large enough such that

$$0 < \frac{1}{n} < \inf_{u\in\partial B(0,\rho)} \mathcal{J}_\lambda(\varphi) - \inf_{u\in B(0,\rho)} \mathcal{J}_\lambda(\varphi). \tag{3.14}$$

So, if we consider $\mathcal{J}_\lambda : \overline{B(0,\rho)} \to (-\infty,\infty)$, then by the Ekeland's variational principle there exists $\varphi_n \in \overline{B(0,\rho)}$, such that

$$\begin{cases} \underline{c} \le \mathcal{J}_\lambda(\varphi_n) \le \underline{c} + \frac{1}{n} \\ \\ \mathcal{J}_\lambda(\varphi_n) < \mathcal{J}_\lambda(\varphi) + \frac{1}{n}||\varphi - \varphi_n||, \ \varphi \neq \varphi_n. \end{cases} \tag{3.15}$$

Therefore, by combining equation (3.14) with equation (3.15), we obtain

$$\begin{aligned} \mathcal{J}_\lambda(\varphi_n) \quad &\le \quad \inf_{\varphi\in\overline{B(0,\rho)}} \mathcal{J}_\lambda(\varphi) + \frac{1}{n} \\ &\le \quad \inf_{\varphi\in B(0,\rho)} \mathcal{J}_\lambda(\varphi) + \frac{1}{n} \\ &< \quad \inf_{\varphi\in\partial B(0,\rho)} \mathcal{J}_\lambda(\varphi). \end{aligned}$$

So, we deduce that $\varphi_n \in B(0,\rho)$.

Now, we define the functional $\Psi : \overline{B(0,\rho)} \to (-\infty,\infty)$, as follows

$$\Psi(\varphi) = \mathcal{J}_\lambda(\varphi) + \frac{1}{n}\|\varphi - \varphi_n\|.$$

It is clear that φ_n is a minimum of Ψ. Therefore, for $s \in (0,1)$ small enough and for any $v \in B(0,1) \cap B(0,\rho)$, we have

$$\frac{\Psi(\varphi_n + sv) - \Psi(\varphi_n)}{s} \ge 0,$$

which means that

$$\frac{\mathcal{J}_\lambda(\varphi_n + sv) - \mathcal{J}_\lambda(\varphi_n)}{s} + \frac{1}{n}\,\|v\| \ge 0.$$

By letting s tend to zero in the last inequality, we get

$$\mathcal{J}'_\lambda(\varphi_n)(v) + \frac{1}{n}\|v\| \ge 0.$$

This implies that

$$\|\mathcal{J}'_\lambda(\varphi_n)\| \le \frac{1}{n}.$$

From the above information, we have

$$\mathcal{J}_\lambda(\varphi_n) \to \underline{c} < 0, \text{ and } \mathcal{J}'_\lambda(\varphi_n) \to 0.$$

Since $\{\varphi_n\} \subset B(0,\rho)$, then, $\{\varphi_n\}$, is bounded in $E_p^{\delta,\psi}$. So, from Lemma 3.4, up to a subsequence, there exists $\psi_\lambda \in E_p^{\delta,\psi}$, such that, $\varphi_n \to \psi_\lambda$ strongly in $E_p^{\delta,\psi}$.
Since $\mathcal{J}_\lambda \in C^1(E_p^{\delta,\psi}, (-\infty,\infty))$, then

$$\mathcal{J}'_\lambda(\varphi_n) \to \mathcal{J}'_\lambda(\psi_\lambda), \text{ as } n \to \infty$$

Hence, we conclude that

$$\mathcal{J}'_\lambda(\psi_\lambda) = 0, \ \|\psi_\lambda\| < \rho, \text{ and } \mathcal{J}_\lambda(\psi_\lambda) < 0. \tag{3.16}$$

This implies that ψ_λ is a nontrivial solution for problem (1.3). Finally, by combining equation (3.13) with equation (3.16), we deduce that φ_λ and ψ_λ are two distinct nontrivial weak solutions from problem (1.3).

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