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FINITE ELEMENTS APPROXIMATION FOR LINEAR ELLIPTIC EQUATIONS WITH L^1-DATA

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Abstract. In this paper we consider, in dimension $d \geq 2$, the $\mathbb{P}_1$ finite elements approximation of the linear elliptic equation which generalizes Laplace's equation. When the right-hand side belongs to $L^1(\Omega)$, we prove that the unique solution of the discrete problem converges in $L^1(\Omega)$ to the unique renormalized solution of the problem.

1. Introduction

In this paper we consider the $\mathbb{P}_1$ finite elements approximation of the boundary value problem

$$\begin{cases} \lambda u - div(A\nabla u + \Phi(u)) & = f \quad \text{ in } \Omega, \\ \\ \quad\quad u & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded set of $\mathbb{R}^d$, $d \geq 2$, A is a coercive matrix with coefficients in $L^\infty(\Omega)$, $\lambda > 0$, Φ is a linear function which belongs to $W^{1,\infty}_{loc}(\mathbb{R})^d$.
The fact that f belongs to $L^1(\Omega)$ is the outstanding feature of the present paper.
For this problem the standard $\mathbb{P}_1$ finite elements approximation, namely

$$\begin{cases} u_h \in V_h, \\ \forall v_h \in V_h, \ \displaystyle\int_\Omega A\nabla u_h \nabla v_h dx + \int_\Omega \Phi(u_h)\nabla v_h dx \\ \qquad\qquad\qquad \displaystyle +\lambda \int_\Omega u_h v_h dx = \int_\Omega f v_h dx, \end{cases} \tag{1.2}$$

where

$$V_h = \{v_h \in \mathcal{C}^0(\overline{\Omega}) : \forall\, T \in \mathcal{T}_h,\ v_{h|T} \in \mathbb{P}_1, v_{h|\partial\Omega} = 0\}, \tag{1.3}$$

has a unique solution (see Proposition 2.3 below).
More details on finite elements methods can be found in [3, 6, 7, 11, 12, 13, 16].
Actually, in order to correctly define the solution of (1.1), one has to consider a

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specific framework, the concept of renormalized solution. The definition of these solutions (see Section 2 below) has been introduced by P. Bénilan and L. Boccardo in [1]. These definitions allow one to prove that in this new sense problem (1.1) is well posed in the terminology of Hadamard, namely that the solution of (1.1) exists, is unique and depends continuously on the right-hand side f (see [14]).
Using the ideas which are the root of the definition of renormalized solution, we are able to prove in Section 3 (Theorem 3.1 below) that the unique solution u_h of (1.2) converges to the unique renormalized solution u of (1.1) in the following sense

$$\begin{cases} u_h \longrightarrow u \text{ strongly in } L^1(\Omega), \\ \Pi_h(T_k(u_h)) \longrightarrow T_k(u) \text{ strongly in } H_0^1(\Omega), \end{cases} \tag{1.4}$$

for every $k > 0$, where Π_h is the usual Lagrange interpolation operator in V_h and where T_k is the usual truncation at height k.
To prove (1.4), we assume that the family of triangulations is regular in the sense of P.G. Ciarlet [5], and that it satisfies an assumption which is close to the assumption which is usually made to ensure that the discrete maximum principle holds true. More precisely, denoting by φ_i the basis functions of V_h, we assume that the matrix with coefficients Q_{ij} and H_{ij} defined respectively by

$$Q_{ij} = \int_\Omega A\nabla\varphi_i \nabla\varphi_j dx$$

and

$$H_{ij} = \int_\Omega \varphi_i\varphi_j dx$$

is a diagonally dominant matrix (hypothesis (2.18)). This allows us to prove (Theorem 3.8) that the solution u_h of (1.2) satisfies

$$\alpha \int_\Omega |\nabla \Pi_h(T_k(u_h))|^2 dx + \lambda \int_\Omega |\Pi_h(T_k(u_h))|^2 dx \le k\|f\|_{L^1(\Omega)}$$

and

$$\frac{1}{n}\int_{\{n\le u_h\le 2n\}} |\nabla u_h|^2\, dx \le \frac{1}{\alpha}\int_{\{|u_h|\ge n\}} |f|\, dx,$$

for every $h > 0$ and every $k > 0$, $n > 1$. This is the main estimates of the present paper.
The assumption that Q and H are a diagonally dominant matrix is unfortunately a restriction on the coercive matrices A with $L^\infty(\Omega)$ coefficients and on the triangulations $\mathcal{T}_h$ of Ω.

2. Preliminaries

2.1. **Notations.** In the present paper, Ω denotes an open bounded subset of $\mathbb{R}^d$ with $d \ge 2$.
For a measurable set $\mathcal{B} \subset \Omega$, we denote by $|\mathcal{B}|$ the measure of $\mathcal{B}$, by $\mathcal{B}^c$ the complement $\Omega \backslash \mathcal{B}$ of $\mathcal{B}$, and by $\chi_{\mathcal{B}}$ the characteristic function of $\mathcal{B}$.
For $1 \le p < +\infty$, we denote by $W^{1,p}(\Omega)$ the standard Sobolev space

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega)^d\},$$

equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)^d}\right)^{\frac{1}{p}},$$

and by $W_0^{1,p}(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $C_c^\infty(\Omega)$, the space of those C^∞ functions whose support is contained in Ω. Since Ω is bounded, $W_0^{1,p}(\Omega)$ will be equipped with the equivalent norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)^d}.$$

We denote by $W^{-1,p'}(\Omega)$, with $p' = \frac{p}{p-1}$, the dual of $W_0^{1,p}(\Omega)$, and when $p = 2$, we denote as usual.

$$H^1(\Omega) = W^{1,2}(\Omega),\ H_0^1(\Omega) = W_0^{1,2}(\Omega) \text{ and } H^{-1}(\Omega) = W^{-1,2}(\Omega).$$

For every r with $1 < r < +\infty$, we denote by $L^{r,\infty}(\Omega)$ the Marcinkiewicz space whose norm is defined by

$$\|v\|_{L^{r,\infty}(\Omega)} = \sup_{\lambda > 0} \lambda |\{x \in \Omega : |v(x)| \geq \lambda\}|^{\frac{1}{r}}.$$

For every real number $k > 0$ we define the truncation $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

2.2. **Setting of the problems.** We consider a matrix A such that

$$A \in L^\infty(\Omega)^{d\times d}. \tag{2.1}$$

$$\text{a.e } x \in \Omega,\ \forall \xi \in \mathbb{R}^d,\ A(x)\xi\xi \geq \alpha|\xi|^2, \tag{2.2}$$

for some $\alpha > 0$, and

$$\begin{cases} \lambda > 0, \quad f \in L^1(\Omega), \\ \Phi : \mathbb{R}^d \to \mathbb{R}^d \text{ is a linear function which belongs to } W_{loc}^{1,\infty}(\mathbb{R})^d. \end{cases} \tag{2.3}$$

Let us recall the definition of the renormalized solution of the problem (1.1).

Definition 2.1. *A function u is a renormalized solution of* (1.1) *if u satisfies*

$$u \in L^1(\Omega), \tag{2.4}$$

$$\forall\ k > 0,\ T_k(u) \in H_0^1(\Omega), \tag{2.5}$$

$$\lim_{n\longrightarrow\infty} \frac{1}{n} \int_{\{n \leq u \leq 2n\}} |\nabla u|^2 dx = 0, \tag{2.6}$$

$$\begin{cases} \forall k > 0,\ \forall S \in C_c^1(\mathbb{R}) \textit{ with } \operatorname{supp} S \subset [-k,k],\ \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega \big(A\nabla T_k(u) \cdot \nabla v\big) S(u)\, dx + \int_\Omega S'(u)\big(A\nabla T_k(u) \cdot \nabla T_k(u)\big) v\, dx \\ \displaystyle\quad + \int_\Omega S(u)\big(\Phi(u) \cdot \nabla v\big)\, dx + \int_\Omega S'(u)\big(\Phi(u) \cdot \nabla T_k(u)\big) v\, dx \\ \displaystyle\quad + \lambda \int_\Omega T_k(u)\, S(u)\, v\, dx = \int_\Omega f\, S(u)\, v\, dx. \end{cases} \tag{2.7}$$

In (2.7) every term makes sense since $T_k(u)$ belongs to $H_0^1(\Omega)$. Equation (2.7) is the correct way to write the result which is obtained formally when using $vS(u)$ as test function in (1.1) and noting that $\nabla u = \nabla T_k(u) = 0$ in $\{|S| > k\}$.

When $\Phi \equiv 0$ and f belongs to $L^1(\Omega) \cap H^{-1}(\Omega)$, the usual weak solution of (1.1), namely

$$\begin{cases} u \in H_0^1(\Omega), \\ v \in H_0^1(\Omega), \displaystyle\int_\Omega A\nabla u \cdot \nabla v dx + \lambda \int_\Omega uv dx = \int_\Omega fv dx, \end{cases} \tag{2.8}$$

is also a renormalized solution of (1.1) and conversely (see [14], Remark 2.5).
The above definition of renormalized solution was introduced by DiPerna and Lions in [10] (see also [2, 8], [15]). Two others definitions of solutions, the entropy solution and the solution obtained as limit of approximations, were introduced as the same time respectively in [1] and [9]. The three definitions can be proved to be equivalent (see e.g. [12]).
The main interest of the definition of renormalized solution is the following existence, uniqueness and continuity theorem.

Theorem 2.2. *[see* [14]*, Theorem 1.1] Assume that A, Φ, λ and f satisfy* (2.1)-(2.3)*. Then there exists a unique renormalized solution u of* (1.1)*. Moreover, this unique solution depends continuously on the right-hand side f in the following sense:*
if f_1 and f_2 belong to $L^1(\Omega)$, and if u_1 and u_2 are the renormalized solutions of (1.1) *for the right-hand sides f_1 and f_2, then*

$$\lambda \|u_1 - u_2\|_{L^1(\Omega)} \le \|f_1 - f_2\|_{L^1(\Omega)}. \tag{2.9}$$

Now we consider a family of triangulations $\mathcal{T}_h$ satisfying for each $h > 0$, the following assumption:

$$\begin{cases} \text{the triangulation } \mathcal{T}_h \text{ is made of a finite number of closed } d\text{-simplices } T \\ \text{(namely triangles when } d = 2\text{, tetrahedra when } d = 3\text{, etc.) such that :} \\ (i)\ \Omega_h = \bigcup\{T : T \in \mathcal{T}_h\} \subset \overline{\Omega}, \\ (ii) \text{ for every compact set } K \text{ with } K \subset \Omega, \text{ there exists } h_0(K) > 0 \text{ such that,} \\ \quad \text{for every } h \text{ with } h < h_0(K), \quad K \subset \Omega_h, \\ (iii) \text{ For } T_1 \text{ and } T_2 \text{ of } \mathcal{T}_h \text{ with } T_1 \neq T_2, \text{ one has } |T_1 \cap T_2| = 0, \\ (iv) \text{ every face of every } T \text{ of } \mathcal{T}_h \text{ is either a subset of } \partial\Omega_h, \text{ or a face of another } T' \text{ of } \mathcal{T}_h. \end{cases} \tag{2.10}$$

Note that because of (iv) the triangulations are conforming. A particular case is where Ω is a polyhedron of $\mathbb{R}^d$, and where Ω_h coincides with Ω for every h.
The vertices of the d-simplexes T of $\mathcal{T}_h$ are denoted by a_i. There are interior and boundary vertices, namely vertices which belong to $\mathring{\Omega}_h$ and vertices which belong to $\partial\Omega_h$. We denote by I the set of indices corresponding to interior vertices and by B the set of indices corresponding to boundary vertices.

For every $T \in \mathcal{T}_h$, we denote by h_T the diameter of T and by ρ_T the diameter of the ball inscribed in T. We set

$$h = \sup_{T \in \mathcal{T}_h} h_T \tag{2.11}$$

and we consider this h as the parameter of the triangulation $\mathcal{T}_h$ and let it tend to zero. We also assume that the family of triangulations $\mathcal{T}_h$ is regular in the sense of P.G. Ciarlet [5] namely that there exists a constant σ such that

$$\forall h, \forall T \in \mathcal{T}_h,\ \frac{h_T}{\rho_T} \leq \sigma. \tag{2.12}$$

For every triangulation $\mathcal{T}_h$, we define the space V_h of those continuous functions which are affine on each d-simplex of $\mathcal{T}_h$ and which vanish on $\overline{\Omega} \setminus \mathring{\Omega}_h$,
namely

$$V_h = \{v_h \in \mathcal{C}^0(\overline{\Omega}) : v_h = 0 \text{ in } \overline{\Omega} \setminus \mathring{\Omega}_h,\ \forall T \in \mathcal{T}_h,\ v_{h|T} \in \mathbb{P}_1\}. \tag{2.13}$$

One has

$$V_h \subset H_0^1(\Omega).$$

For every (interior or boundary) vertex a_i of $\mathcal{T}_h$, i.e. for every $i \in I \cup B$, we define the function φ_i by:

$$\begin{cases} \varphi_i \in C^0(\Omega_h),\ \varphi_{i|T} \in \mathbb{P}_1 \text{ for every } T \in \mathcal{T}_h, \\ \varphi_i(a_i) = 1,\ \varphi_i(a_j) = 0 \text{ for every vertex } a_j\ of\ \mathcal{T}_h \text{ with } a_j \neq a_i. \end{cases} \tag{2.14}$$

One has

$$\sum_{i \in I \cup B} \varphi_i = 1 \text{ in } \Omega_h. \tag{2.15}$$

When a_i is an interior vertex, i.e. when $i \in I$, then the function φ_i belongs to $H_0^1(\mathring{\Omega}_h)$, and extending φ_i by zero to $\overline{\Omega} \setminus \mathring{\Omega}_h$, we obtain a function of V_h, still denoted by φ_i. The functions φ_i, $i \in I$, are a basis of the space V_h.
We define the interpolation operator Π_h by:

$$\begin{cases} \forall v \in C^0(\overline{\Omega}) \text{ with } v = 0 \text{ in } \overline{\Omega} \setminus \mathring{\Omega}_h, \\ \Pi_h(v) \in V_h,\ (\Pi_h(v))(a_i) = v(a_i) \text{ for every vertex } a_i \text{ of } \mathcal{T}_h, \end{cases}$$

or equivalently by

$$\Pi_h(v) = \sum_{i \in I} v(a_i)\varphi_i.$$

For all interior vertices a_i and a_j of $\mathcal{T}_h$, i.e. for every i and j of I, we define two real numbers Q_{ij} and H_{ij} respectively by

$$Q_{ij} = \int_\Omega A\nabla\varphi_i \cdot \nabla\varphi_j dx, \tag{2.16}$$

$$H_{ij} = \int_\Omega \varphi_i\varphi_j dx; \tag{2.17}$$

this defines an $I \times I$ matrix Q.
The $I \times I$ matrix H is a diagonally dominant matrix.

The main assumption of the present paper is that Q and H satisfies:

$$\begin{cases} & Q_{ii} - \sum_{j\in I \ j\neq i} |Q_{ij}| \geq 0; \\ \forall i \in I, & \\ & H_{ii} - \sum_{j\in I \ j\neq i} |H_{ij}| \geq 0. \end{cases} \tag{2.18}$$

In other words, Q and H are assumed to be a diagonally dominant matrix. This assumption is close to the usual assumption which ensures that the discrete maximum principle holds true.
We have the following.

Proposition 2.3. *For every triangulation $\mathcal{T}_h$, the problem*

$$\begin{cases} u_h \in V_h, \\ \int_\Omega A\nabla u_h \cdot \nabla v_h dx + \int_\Omega \Phi(u_h) \cdot \nabla v_h dx \\ +\lambda \int_\Omega u_h v_h dx = \int_\Omega f v_h dx, \quad \forall v_h \in V_h \end{cases} \tag{2.19}$$

has a unique solution u_h.

Proof. Note that the right-hand side of (2.19) makes sense since f belongs to $L^1(\Omega)$ and $v_h \in V_h \subset L^\infty(\Omega)$. We define the form $a_h : V_h \times V_h \mapsto \mathbb{R}$ as

$$a_h(u_h, v_h) = \int_\Omega A\nabla u_h \cdot \nabla v_h dx + \int_\Omega \Phi(u_h) \cdot \nabla v_h dx + \lambda \int_\Omega u_h v_h dx.$$

The form a_h is bilinear, symmetric and continuous.
It remains to prove the coerciveness of a_h. We have for $v_h \in V_h$,

$$a_h(v_h, v_h) \quad = \quad \int_\Omega A\nabla v_h \cdot \nabla v_h dx + \int_\Omega \Phi(v_h) \cdot \nabla v_h dx + \lambda \int_\Omega v_h v_h dx.$$

We claim that

$$\int_\Omega \Phi(v_h) \cdot \nabla v_h dx = 0.$$

Indeed, if we set

$$\Psi_i(t) = \int_0^t \Phi_i(s)\, ds \qquad \forall t \in \mathbb{R}$$

and

$$\Psi = (\Psi_1,\ \Psi_2, \cdots, \Psi_d),$$

we get

$$\Phi(u_h) \cdot \nabla u_h = \Psi'(u_h) \cdot \nabla u_h = \nabla \cdot \Psi(u_h),$$

which implies by the Divergence Theorem,

$$\int_\Omega \Phi(u_h) \cdot \nabla u_h\, dx = \int_\Omega \nabla \cdot \Psi(u_h)\, dx = \int_{\partial\Omega} \Psi(u_h) \cdot \mathbf{n} dS = 0. \tag{2.20}$$

Therefore,

$$\begin{aligned} a_h(v_h, v_h) &= \int_\Omega A\nabla v_h \cdot \nabla v_h dx + \lambda \int_\Omega v_h v_h dx \\ &\geq \int_\Omega A\nabla v_h \cdot \nabla v_h dx \\ &\geq \alpha \int_\Omega |\nabla v_h|^2 dx \\ &= \alpha \|\nabla v_h\|_2^2. \end{aligned}$$

i.e.

$$a_h(v_h, v_h) \geq \alpha \|\nabla v_h\|_2^2, \ \forall v_h \in V_h. \tag{2.21}$$

By Lax-Milgram Theorem, we conclude that problem (2.19) has a unique solution for $f = 0$. Since the bilinear form is defined in the finite dimensional space $V_h \subset L^\infty(\Omega)$, the result remains true for $f \in L^1(\Omega)$. □

3. Main Results

Our main result is the following.

Theorem 3.1. *Assume that A, f, Φ and λ satisfy* (2.1), (2.2), (2.3), (2.10) ,(2.11), (2.12), *and* (2.18). *Then the unique solution u_h of* (2.19) *satisfies for every $k > 0$*

$$\Pi_h(T_k(u_h)) \longrightarrow T_k(u) \text{ strongly in } H_0^1(\Omega), \tag{3.1}$$

$$u_h \longrightarrow u \text{ strongly in } L^1(\Omega), \tag{3.2}$$

when h tends to zero, where u is the unique renormalized solution of (1.1).

The proof of this Theorem will be made through Proposition 3.10 and Theorem 3.9 below. We begin by recalling various results which will be used in this proof.

3.1. A priori estimates and basic convergence.

The following result is a piecewise $\mathbb{P}_1$ variant of a result of L. Boccardo and Th. Gallouët [2], [3].

Theorem 3.2. *(see* [4]*)*
Assume that $v_h \in V_h$ satisfies

$$\forall k > 0, \int_\Omega |\nabla \Pi_h(T_k(v_h))|^2 dx \leq kM, \tag{3.3}$$

for some $M > 0$. Then, for every q avec $1 \leq q < \dfrac{d}{d-1}$

$$\|v_h\|_{W_0^{1,q}(\Omega)} \leq C_2(d, |\Omega|, q)M, \tag{3.4}$$

where the constant $C_2(d, |\Omega|, q)$ only depends on d, $|\Omega|$ and q.

The following lemmas show that when v_h satisfies (1.1), then $\Pi_h(T_k(v_h))$ and $T_k(v_h)$ are close in measure.

Lemma 3.3. *(see* [4]*, Lemma 2.4) Let $v_h \in V_h$. For every s and every k with $0 < s < k$, the set $B(k, s)$ defined by*

$$B(k,s) = \bigcup \{T \in \mathcal{T}_h : \ \exists (x,y) \in T \times T, \ |v_h(x)| \geq k, \ |v_h(y)| \leq s\} \tag{3.5}$$

satisfies

$$|B(k,s)| \leq \frac{h^2}{(k-s)^2} \int_\Omega |\nabla \Pi_h(T_k(v_h))|^2 dx. \tag{3.6}$$

Lemma 3.4. *(see* [4]*, Lemma 2.5) Let $v_h \in V_h$. For every s and every k with $0 < s < k$, one has*

$$T_s(\Pi_h(T_k(v_h))) = T_s(v_h) \quad \text{in } B(k,s)^c \tag{3.7}$$

and

$$\nabla T_s(\Pi_h(T_k(v_h))) = \nabla T_s(v_h) \text{ almost everywhere in } B(k,s)^c. \tag{3.8}$$

In view of (3.6), $|B(k,s)|$ tends to zero when h tends to zero if estimate (3.3) holds. The following result is therefore an immediate consequence of Lemmas 3.3 and 3.4.

Proposition 3.5. *(see* [4]*, Proposition 2.6) Assume that $v_h \in V_h$ satisfies* (1.1)*. Then for every s and every k, with $0 < s < k$, one has*

$$T_s(\Pi_h(T_k(v_h))) - T_s(v_h) \longrightarrow 0 \text{ in measure}, \tag{3.9}$$

$$\nabla T_s(\Pi_h(T_k(v_h))) - \nabla T_s(v_h) \longrightarrow 0 \text{ in measure}, \tag{3.10}$$

when h tends to zero.

The following proposition gives an analogue in V_h of the fact that in the continuous case, for every $v \in H_0^1(\Omega)$ and every $k > 0$, one has

$$A\nabla(v - T_k(v)) \cdot \nabla T_k(v) + \lambda(v - T_k(v))T_k(v) = 0 \text{ a. e. in } \Omega.$$

Proposition 3.6. *Under assumption* (2.18)*, one has for every $v_h \in V_h$ and every $k > 0$*

$$A_1 = \int_\Omega A\nabla(v_h - \Pi_h(T_k(v_h))) \cdot \nabla \Pi_h(T_k(v_h))dx \geq 0,$$

$$A_2 = \lambda \int_\Omega (v_h - \Pi_h(T_k(v_h)))(\Pi_h(T_k(v_h)))dx \geq 0. \tag{3.11}$$

Proof. The proof is carried out in several steps.

- We show firstly, that $A_1 \geq 0$

  Using Definition 2.16 of Q_{ij}, the fact that $v_h = \sum_{i \in I} v_h(a_i)\varphi_i$ and $\Pi_h(T_k(v_h)) = \sum_{i \in I} T_k(v_h)(a_i)\,\varphi_i$, we have

$$\int_\Omega A\nabla\Big(v_h - \Pi_h T_k(v_h)\Big) \cdot \nabla \Pi_h(T_k(v_h))dx = \sum_{i,j \in I} Q_{ij}\Big(v_h(a_i) - T_k(v_h(a_i)))T_k(v_h(a_j)\Big) = \sum_{i \in I} S_i,$$

where

$$S_i = Q_{ii}(v_h(a_i) - T_k(v_h(a_i)))T_k(v_h(a_i)) + \sum_{j \in I, j \neq i} Q_{ij}(v_h(a_i) - T_k(v_h(a_i)))T_k(v_h(a_j)).$$

Fix $i \in I$.

If $|v_h(a_i)| \leq k$, then $v_h(a_i) - T_k(v_h(a_i)) = 0$ and $S_i = 0$.

If $|v_h(a_i)| > k$, then $\Big(v_h(a_i)-T_k(v_h(a_i))\Big)T_k(v_h(a_i)) = |v_h(a_i)-T_k(v_h(a_i))|k$.
Since $|T_k(v_h(a_j))| \leq k$ for every j, one has:

$$\begin{aligned} S_i &\geq Q_{ii}|v_h(a_i) - T_k(v_h(a_i))|k - \sum_{j\in I, j\neq i} |Q_{ij}||v_h(a_i) - T_k(v_h(a_i))|k \\ &= |v_h(a_i) - T_k(v_h(a_i))|k(Q_{ii} - \sum_{j\in I, j\neq i} |Q_{ij}|) \geq 0, \end{aligned}$$

owing to hypothesis (2.18). This proves that for all $i \in I,\ S_i \geq 0$.

- Secondly, we prove that A_2 is positive.
  Using the definition (2.17) of H_{ij}, we have

$$\begin{aligned} A_2 &= \sum_{i,j\in I} H_{ij}(v_h(a_i) - T_k(v_h)))T_k(v_h(a_i)) \\ &= \sum_{i\in I} R_i, \end{aligned}$$

where

$$R_i = H_{ii}(v_h(a_i) - T_k(v_h(a_i)))T_k(v_h(a_i)) + \sum_{\substack{j\in I \\ i\neq j}} H_{ij}(v_h(a_i) - T_k(v_h(a_i)))T_k(v_h(a_j)).$$

Fix $i \in I$.
If $|v_h| \leq k$, then $v_h(a_i) - T_k(v_h(a_i)) = 0$ and $R_i = 0$.
If $|v_h(a_i)| > k$, then

$$\Big(v_h(a_i) - T_k(v_h(a_i))\Big)T_k(v_h(a_i)) = |v_h(a_i) - T_k(v_h(a_i))|k.$$

Since $|T_k(v_h(a_i))| \leq k$ for every j, one has

$$\begin{aligned} R_i &\geq H_{ii}|v_h(a_i - T_k(v_h(a_i)))|k - \sum_{\substack{j\in I \\ i\neq j}} |H_{ij}||v_h(a_i) - T_k(v_h(a_i))|k \\ &= |v_h(a_i) - T_k(v_h(a_i))|k(H_{ii} - \sum_{\substack{j\in I \\ i\neq j}} |H_{ij}|) \geq 0, \end{aligned}$$

owing to hypothesis (2.18). This proves that for all $i \in I, R_i \geq 0$.
The proof of Proposition 3.6 is then complete.

□

Proposition 3.7. *Under the assumptions of Theorem 3.2, the solution u_h of* (2.19) *satisfies for every $h > 0$ and every $k > 0$*

$$\begin{aligned} \int_\Omega A\nabla\Pi_h(T_k(u_h)) \cdot \nabla\Pi_h(T_k(u_h))dx + \lambda \int_\Omega \Big(\Pi_h(T_k(u_h))\Big)^2 dx \\ \leq \int_\Omega f\Pi_h(T_k(u_h))dx. \end{aligned} \tag{3.12}$$

Proof. As $T_k(u_h)$ is continuous with $T_k(0) = 0$, so the function $\Pi_h(T_k(u_h))$ belongs to V_h. Then, we can take it as a test function in (2.19) to obtain

$$\int_\Omega A\nabla u_h \cdot \nabla\Pi_h(T_k(u_h))dx + \int_\Omega \Phi(u_h) \cdot \nabla\Pi_h(T_k(u_h))dx$$

$$+\lambda \int_\Omega u_h\, \Pi_h(T_k(u_h))dx = \int_\Omega f\Pi_h(T_k(u_h))dx.$$

Recall that in Proposition 3.6 we proved that

$$\int_\Omega A\nabla\Pi_h(T_k(u_h)) \cdot \nabla\Pi_h(T_k(u_h))\, dx \le \int_\Omega A\nabla u_h \cdot \nabla\Pi_h(T_k(u_h))\, dx, \tag{3.13}$$

$$\lambda \int_\Omega |\Pi_h(T_k(u_h))|^2\, dx \le \lambda \int_\Omega \lambda u_h\, \Pi_h(T_k(u_h))\, dx. \tag{3.14}$$

On the other hand, we claim that

$$\int_\Omega \Phi(u_h) \cdot \nabla\Pi_h(T_k(u_h))\, dx = 0. \tag{3.15}$$

Indeed,

$$\begin{aligned} \nabla\Pi_h(T_k(u_h)) &= \Pi_h'(T_k(u_h))\nabla T_k(u_h) \\ &= \begin{cases} \Pi_h'(u_h)\nabla u_h & \text{if} \quad |u_h| < k \\ 0 & \text{if} \quad |u_h| \ge k. \end{cases} \end{aligned}$$

Therefore, if we set

$$(\Psi_{h,k})_i(t) = \int_0^t \Pi_h'(s)\Phi_i(s)\chi_{\{|s|<k\}}\, ds, \qquad \forall t \in \mathbb{R}$$

and

$$\Psi_{h,k} = \big((\Psi_{h,k})_1,\ (\Psi_{h,k})_2, \cdots, (\Psi_{h,k})_d\big),$$

we get

$$\begin{aligned} \Phi(u_h) \cdot \nabla\Pi_h(T_k(u_h)) &= \chi_{\{|s|<k\}}\Pi_h'(u_h)\Phi(u_h) \cdot \nabla u_h \\ &= (\Psi_{h,k})'(u_h) \cdot \nabla u_h \\ &= \nabla \cdot \Psi_{h,k}(u_h), \end{aligned}$$

which implies by the Divergence Theorem,

$$\int_\Omega \Phi(u_h) \cdot \nabla\Pi_h(T_k(u_h))\, dx = \int_\Omega \nabla \cdot \Psi_{h,k}(u_h)\, dx = \int_{\partial\Omega} \Psi_{h,k}(u_h) \cdot \mathbf{n} dS = 0.$$

From (3.13)-(3.15), we deduce (3.12). □

In the following theorem we prove a uniform estimate on the interpolation function Π_h and the truncated energy of u_h, which is crucial to pass to the limit in the approximate problem.

Theorem 3.8. *Assume that $u_h \in V_h$ is a solution of* (2.19)*, then*

$$\forall k > 0,\ \alpha\|\Pi_h(T_k(u_h))\|^2_{H^1_0(\Omega)} + \lambda\, \|\Pi_h(T_k(u_h))\|^2_{L^2(\Omega)} \le k\|f\|_{L^1(\Omega)}, \tag{3.16}$$

$$\frac{1}{n}\int_{\{n\le u_h\le 2n\}} |\nabla u_h|^2\, dx \le \frac{1}{\alpha}\int_{\{|u_h|\ge n\}} |f|\, dx. \tag{3.17}$$

Proof. The proof is done in two steps.

- Step 1: Proof of (3.16).
  The proof of (3.16) follows immediately from (3.12).

- Step 2: Proof of (3.17).

  Let's us introduce the function $h_n : \mathbb{R} \to \mathbb{R}$ defined, for any $n \geq 1$, by

$$h_n(s) = \begin{cases} -n & \text{if} \quad s \leq -2n, \\ s+n & \text{if} \quad -2n \leq s \leq -n, \\ 0 & \text{if} \quad -n \leq s \leq n, \\ s-n & \text{if} \quad n \leq s \leq 2n, \\ n & \text{if} \quad s \geq 2n. \end{cases} \tag{3.18}$$

  Note that $h_n(s) = T_{2n}(s) - T_n(s)$, so h_n is a Lipschitz-function with $h_n(0) = 0$ and then, $\Pi_h(h_n(u_h)) \in V_h$. We can take $\Pi_h(h_n(u_h))$ as a test function in (2.19) to obtain

$$\int_\Omega A\nabla u_h \cdot \nabla \Pi_h(h_n(u_h))\,dx + \int_\Omega \Phi(u_h) \cdot \nabla \Pi_h(h_n(u_h))\,dx + \lambda \int_\Omega u_h\, \Pi_h(h_n(u_h))\,dx$$
$$= \int_\Omega f\, \Pi_h(h_n(u_h))\,dx \leq n \int_{\{|u_h| \geq n\}} |f|\,dx. \tag{3.19}$$

  Observe that $\nabla h_n(u_h) = h_n'(u_h) \cdot \nabla u_h$ with

$$h_n'(s) = \begin{cases} 1 & \text{if} \quad n < |s| < 2n, \\ 0 & \text{if} \quad |s| < n \quad \text{or} \quad |s| > 2n. \end{cases}$$

  Therefore, we have

$$\int_\Omega A\nabla u_h \cdot \nabla \Pi_h(h_n(u_h))\,dx \geq \int_\Omega A\nabla \Pi_h(h_n(u_h)) \cdot \nabla \Pi_h(h_n(u_h))\,dx$$
$$\geq \alpha \int_\Omega |\nabla \Pi_h(h_n(u_h))|^2\,dx = \alpha \int_{\{n \leq |u_h| \leq 2n\}} |\nabla u_h|^2\,dx. \tag{3.20}$$

  On the other hand, as in the proofs of (3.14) and (3.15), we show that

$$\lambda \int_\Omega u_h\, \Pi_h(h_n(u_h))\,dx \geq \lambda \int_\Omega |\Pi_h(h_n(u_h))|^2\,dx \geq 0 \tag{3.21}$$

  and

$$\int_\Omega \Phi(u_h) \cdot \nabla \Pi_h(h_n(u_h))\,dx = 0. \tag{3.22}$$

  Combining (3.19)-(3.22), we obtain (3.17).

□

To pass to the limit as $h \to 0$ in (2.19), we need strong convergence of u_h and $\Pi_h(T_k(u_h))$.

3.2. **Strong convergence.**

Theorem 3.9. *Under the assumptions of Theorem 3.2, the solution u_h of* (2.19) *satisfies*

$$u_h \longrightarrow u \text{ strongly in } L^1(\Omega), \tag{3.23}$$

as h tends to zero, where u is the unique renormalized solution of (1.1).

Proof. Consider a sequence f^ε of functions such that

$$f^\varepsilon \in L^\infty(\Omega),\ f^\varepsilon \longrightarrow f \text{ strongly in } L^1(\Omega).$$

Such a sequence is easily obtained by taking for example $f^\varepsilon = T_{\frac{1}{\varepsilon}}(f)$. Let u_h^ε be the unique solution of (2.19) for the right-hand side f^ε. Then $u_h - u_h^\varepsilon$ satisfies

$$\begin{cases} u_h - u_h^\varepsilon \in V_h, \\ \forall v_h \in V_h, \displaystyle\int_\Omega A\nabla(u_h - u_h^\varepsilon)\nabla v_h dx + \int_\Omega (\Phi(u_h - u_h^\varepsilon)) \cdot \nabla v_h dx \\ \qquad +\lambda \displaystyle\int_\Omega (u_h - u_h^\varepsilon) v_h dx = \int_\Omega (f - f^\varepsilon) v_h dx. \end{cases}$$

Applying estimate (3.16) to this problem, we obtain for every $k > 0$, every $h > 0$ and every $\varepsilon > 0$

$$\alpha \int_\Omega |\nabla \Pi_h(T_k(u_h - u_h^\varepsilon))|^2 dx + \lambda \int_\Omega |\Pi_h(T_k(u_h - u_h^\varepsilon))|^2 dx \le k\|f - f^\varepsilon\|_{L^1(\Omega)},$$

which implies by Theorem 3.2 that for every q with $1 \le q < \dfrac{d}{d-1}$, every $h > 0$ and every $\varepsilon > 0$

$$\alpha\|u_h - u_h^\varepsilon\|_{W_0^{1,q}(\Omega)} \le C(d, |\Omega|, q)\|f - f^\varepsilon\|_{L^1(\Omega)}.$$

In particular, for $q = 1$, we deduce from the above inequality that

$$\|u_h - u_h^\varepsilon\|_{L^1(\Omega)} \le \frac{1}{\alpha} C(d, |\Omega|)\|f - f^\varepsilon\|_{L^1(\Omega)}. \tag{3.24}$$

On the other hand, since $f^\varepsilon \in L^\infty(\Omega) \subset L^2(\Omega)$ and since the family of triangulations $\mathcal{T}_h$ satisfies (2.10), (2.11) and (2.12), we have that for every fixed ε

$$u_h^\varepsilon \longrightarrow u^\varepsilon \text{ strongly in } H_0^1(\Omega), \tag{3.25}$$

as h tends to zero and where u^ε is the unique weak solution (see [14], Theorem 1.1) of

$$\begin{cases} u^\varepsilon \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ -div(A\nabla u^\varepsilon + \Phi^\varepsilon(u^\varepsilon)) + \lambda u^\varepsilon = f^\varepsilon \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{3.26}$$

Finally, the function u^ε, which is the unique weak solution of (3.26) and the unique renormalized solution (see [14], Theorem 1.1) in the sense of Definition 2.1 of the problem

$$\begin{cases} -div(A\nabla u^\varepsilon + \Phi^\varepsilon u^\varepsilon) + \lambda u^\varepsilon = f^\varepsilon \text{ in } \Omega, \\ u^\varepsilon = 0 \text{ on } \partial\Omega, \end{cases}$$

satisfy

$$\|u^\varepsilon - u\|_{L^1(\Omega)} \le \frac{1}{\lambda}\|f^\varepsilon - f\|_{L^1(\Omega)}, \tag{3.27}$$

Writing now

$$\|u_h - u\|_{L^1(\Omega)} \le \|u_h - u_h^\varepsilon\|_{L^1(\Omega)} + \|u_h^\varepsilon - u^\varepsilon\|_{L^1(\Omega)} + \|u^\varepsilon - u\|_{L^1(\Omega)},$$

and using (3.24), (3.25) and (3.27), we have proved that for every $\varepsilon > 0$

$$\limsup_{h \longrightarrow 0} \|u_h - u\|_{L^1(\Omega)} \le \Big(\frac{1}{\alpha} C(d, |\Omega|) + \frac{1}{\lambda}\Big)\|f^\varepsilon - f\|_{L^1(\Omega)}.$$

Taking the limit when ε tends to zero proves (3.23), and relation (3.2) of Theorem 3.2 is proved.

□

Now, we prove that $\Pi_h(T_k(u_h))$ converges strongly to $T_k(u)$ in $H_0^1(\Omega)$ in the following statement.

Proposition 3.10. *Under the assumptions of Theorem 3.2, the solution u_h of* (2.19) *satisfies for every $k > 0$*

$$\Pi_h(T_k(u_h)) \longrightarrow T_k(u) \text{ strongly in } H_0^1(\Omega) \quad \textit{as } h \to 0 \tag{3.28}$$

Proof. Fix $k > 0$. In view of estimate (3.16), we can extract a subsequence (which depends on k and is still denoted by u_h) such that for some $w_k \in H_0^1(\Omega)$

$$\Pi_h(T_k(u_h)) \rightharpoonup w_k \text{ weakly in } H_0^1(\Omega), \tag{3.29}$$

when h tends to zero. By estimate (3.16) and Proposition 3.5, u_h satisfies (2.14), namely

$$T_s(\Pi_h(T_k(u_h))) - T_s(u_h) \longrightarrow 0 \text{ in measure,}$$

when h tends to zero, for every s with $0 < s < k$. The convergence (3.29), the convergence (3.23), the Rellich-Kondrachov's compactness theorem and the continuity of the function T_s prove that

$$T_s(w_k) = T_s(u),$$

for every s with $0 < s < k$. Passing to the limit when s tends to k, we obtain $T_k(w_k) = T_k(u)$. But since $|\Pi_h(T_k(u_h)| \leq k$, the convergence (3.29) implies that $|w_h(x)| \leq k$, hence $T_k(w_k) = w_k$. This yields $w_k = T_k(u)$, and since the limit does not depend on the subsequence, we have proved that

$$\Pi_h(T_k(u_h)) \rightharpoonup T_k(u) \text{ weakly in } H_0^1(\Omega), \tag{3.30}$$

when h tends to zero without extracting a subsequence.
Let us now prove that this convergence is strong. Lebesgue's dominated convergence theorem combined with

$$|f\Pi_h(T_k(u_h))| \leq |f|k \in L^1(\Omega),$$

with the weak convergence (3.30) and with Rellich-Kondrachov's compactness theorem imply that

$$\int_\Omega f\Pi_h(T_k(u_h))dx \longrightarrow \int_\Omega fT_k(u)dx \qquad \text{as } h \to 0.$$

Therefore passing to the limit with respect to h in (3.12) yields

$$\limsup_{h\longrightarrow 0} \Big[\int_\Omega A\nabla\Pi_h(T_k(u_h))\nabla\Pi_h(T_k(u_h))dx + \int_\Omega \Phi(\Pi_h(T_k(u_h)))\nabla\Pi_h(T_k(u_h))dx + \lambda \int_\Omega (\Pi_h(T_k(u_h)))^2 dx\Big] \leq \int_\Omega fT_k(u)dx. \tag{3.31}$$

On the other hand, since u is the renormalized solution of (1.1), one has (see [14], Theorem 5.1)

$$\int_\Omega A\nabla T_k(u)\nabla T_k(u)dx + \lambda \int_\Omega uT_k(u)dx = \int_\Omega fT_k(u)dx. \tag{3.32}$$

From (3.31) and (3.32) we deduce that

$$\limsup_{h\longrightarrow 0}\Big[\int_\Omega A\nabla\Pi_h(T_k(u_h))\nabla\Pi_h(T_k(u_h))dx+\int_\Omega \Phi(\Pi_h(T_k(u_h)))\nabla\Pi_h(T_k(u_h))dx$$
$$+\lambda\int_\Omega(\Pi_h(T_k(u_h)))^2dx\Big]\leq\int_\Omega A\nabla T_k(u)\nabla T_k(u)dx+\lambda\int_\Omega uT_k(u)dx,$$

which combined with the weak convergence (3.30) implies the strong convergence (3.28), which proves relation (3.1) of Theorem 3.2. □

To achieve the proof of Theorem 3.2, it remains to prove that the limit u is a renormalized solution of problem (1.1).
We claim that u satisfies the decay (2.6) of the truncate energy, i. e.,

$$\lim_{n\longrightarrow\infty}\lim_{h\to 0}\frac{1}{n}\int_{\{n\leq u\leq 2n\}}|\nabla u_h|^2\,dx=0. \tag{3.33}$$

Indeed, from (3.17), we can write

$$\frac{1}{n}\int_\Omega|\nabla h_n(u_h)|^2\,dx=\frac{1}{n}\int_{\{n\leq u_h\leq 2n\}}|\nabla u_h|^2\,dx$$
$$\leq\frac{1}{\alpha}\int_{\{|u_h|\geq n\}}|f|\,dx\leq\frac{1}{\alpha}\int_{\{|u_h|\geq n\}}|f|\,\chi_{\{|u_h|\geq n\}}\,dx. \tag{3.34}$$

But

$$\limsup_{h\to 0}\chi_{\{|u_h|\geq n\}}\leq\chi_{\{|u|\geq n\}}\qquad\text{and}\qquad\limsup_{h\to 0}|f|\,\chi_{\{|u_h|\geq n\}}\leq|f|\,\chi_{\{|u|\geq n\}}$$

almost everywhere in Ω. Therefore, we use Fatou's Lemma to obtain

$$\limsup_{h\to 0}\int_{\{|u_h|\geq n\}}|f|\,dx\leq\int_{\{|u|\geq n\}}|f|\,dx. \tag{3.35}$$

On the other hand, as $h_n(u_h)$ is bounded in $H_0^1(\Omega)$, it's clear that

$$h_n(u_h)\rightharpoonup h_n(u)\text{ weakly in }H_0^1(\Omega)\text{ as }h\to 0. \tag{3.36}$$

Combining (3.34)-(3.36) and using the lower semi-continuity of the norm, we obtain

$$\frac{1}{n}\int_{\{n\leq u\leq 2n\}}|\nabla u_h|^2\,dx=\frac{1}{n}\int_\Omega|\nabla h_n(u)|^2\,dx\leq\frac{1}{n}\limsup_{h\to 0}\int_\Omega|\nabla h_n(u_h)|^2\,dx$$
$$\leq\frac{1}{\alpha}\limsup_{h\to 0}\int_{\{|u_h|\geq n\}}|f|\,dx\leq\frac{1}{\alpha}\int_{|u|\geq n}|f|\,dx. \tag{3.37}$$

Letting $n\to\infty$ in (3.37), we deduce (3.33).

To complete the proof of Theorem 3.9, it remains to prove that the limit of u_h satisfies (2.7). We use the same manage as the proof of Theorem 1.1 in [14] to obtain the desired result, that is

$$\begin{cases}\forall k>0,\ \forall S\in C_c^1(\mathbb{R})\ \textit{with}\ \operatorname{supp}S\subset[-k,k],\ \forall v\in H_0^1(\Omega)\cap L^\infty(\Omega),\\[2mm] \displaystyle\int_\Omega\big(A\nabla T_k(u)\cdot\nabla v\big)S(u)\,dx+\int_\Omega S'(u)\big(A\nabla T_k(u)\cdot\nabla T_k(u)\big)v\,dx\\ \displaystyle\qquad+\int_\Omega S(u)\big(\Phi(u)\cdot\nabla v\big)\,dx+\int_\Omega S'(u)\big(\Phi(u)\cdot\nabla T_k(u)\big)v\,dx\\ \displaystyle\qquad\qquad+\lambda\int_\Omega T_k(u)\,S(u)\,v\,dx=\int_\Omega f\,S(u)\,v\,dx.\end{cases} \tag{3.38}$$

Combining (3.23), (3.28), (3.33) and (3.38), we conclude that u is a renormalized solution of problem (1.1). This achives the proof of Theorem 3.2.

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