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PARACOMPACTNESS IN A BISPACE

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Abstract. The idea of pairwise paracompactness was studied by many authors in a bitopological space. Here we study the same in the setting of more general structure of a bispace using the thoughts of the same given by Bose et al. [2].

1. Introduction

The idea of paracompactness given by Dieudonn$e^{'}$ in the year 1944 came out as a generalization of the notion of compactness. It has many implication in field of differential geometry and it plays important roll in metrization theory. The concept of the Alexandroff space [1] (i.e., a σ-space or simply a space) was introduced by A. D. Alexandroff in the year 1940 as a generalization of a topological space where the union of open sets were taken to be open for only countable collection of open sets instead of arbitrary collection. Another kind of generalization of a topological space is the idea of a bitopological space introduced by J.C. Kelly in [14]. Using these ideas Lahiri and Das [17] introduced the idea of a bispace as a generalization of a σ-space. Many works on topological properties were carried out by many authors ([21], [22], [25] etc.) in the setting of a bitopological space. Datta [11] studied the idea of paracompactness in a bitopological space and tried to get analogous results of topological properties given by Michael [19] in respect of paracompactness. In 1986 Raghavan and Reilly [23] gave the idea of paracompactness in a bitopological space in another way. Later in 2008 M. K. Bose et al. [2] studied the same in a bitopological space as a generalization of pairwise compactness. Here we have studied pairwise paracompactness using the thoughts given by Bose et al. [2] in a bispace and discussed some its results in the setting of a bispace, which was firstly introduced by Lahiri and Das [17] as a generalization of the notion of bitopological spaces in 2001.

2. Preliminaries

Definition 2.1. [1] *A set X is called an Alexandroff space or σ- space or simply space if it is chosen a system $\mathcal{F}$ of subsets of X, satisfying the following axioms*

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(i) The intersection of countable number sets in $\mathcal{F}$ is a set belonging to $\mathcal{F}$.
(ii) The union of finite number of sets from $\mathcal{F}$ is a set belonging to $\mathcal{F}$.
(iii) The empty set and X is a set belonging to $\mathcal{F}$.

Sets of $\mathcal{F}$ are called closed sets. There complementary sets are called open.It is clear that instead of closed sets in the definition of a space, one may put open sets with subject to the conditions of countable summability, finite intersectability and the condition that X and the void set should be open.
The collection of such open will sometimes be denoted by $\mathcal{P}$ and the space by $(X, \mathcal{P})$. It is noted that $\mathcal{P}$ is not a topology in general as can be seen by taking $X = \mathbb{R}$, the set of real numbers and τ as the collection of all F_σ sets in $\mathbb{R}$.

Definition 2.2. [1] *To every set M we correlate its closure $\overline{M}$= the intersection of all closed sets containing M.*

Generally the closure of a set in a σ-space is not a closed set. We denote the closure of a set M in a space $(X, \mathcal{P})$ by $\mathcal{P}$-cl(M) or simply $\overline{M}$ when there is no confusion about $\mathcal{P}$. The idea of limit points, derived set, interior of a set etc. in a space are similar as in the case of a topological space which have been thoroughly discussed in [16].

Definition 2.3. [3] *Let $(X, \mathcal{P})$ be a space. A family of open sets B is said to form a base (open) for $\mathcal{P}$ if and only if every open set can be expressed as countable union of members of B.*

Theorem 2.1. [3] *A collection of subsets B of a set X forms an open base of a suitable space structure $\mathcal{P}$ of X if and only if*
1) the empty set $\emptyset$ belongs to B
2) X is the countable union of some sets belonging to B.
3) intersection of any two sets belonging to B is expressible as countable union of some sets belonging to B.

Definition 2.4. [17] *Let X be a non-empty set. If $\mathcal{P}$ and $\mathcal{Q}$ be two collection of subsets of X such that $(X, \mathcal{P})$ and $(X, \mathcal{Q})$ are two spaces, then X is called a bispace.*

Definition 2.5. [17] *A bispace $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise T_1 if for any two distinct points x, y of X, there exist $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.*

Definition 2.6. [17] *A bispace $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise Hausdorff if for any two distinct points x, y of X, there exist $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.*

Definition 2.7. [17] *In a bispace $(X, \mathcal{P}, \mathcal{Q})$, $\mathcal{P}$ is said to be regular with respect to $\mathcal{Q}$ if for any $x \in X$ and a $\mathcal{P}$-closed set F not containing x, there exist $U \in \mathcal{P}$, $V \in \mathcal{Q}$ such that $x \in U$, $F \subset V$, $U \cap V = \emptyset$. $(X, \mathcal{P}, \mathcal{Q})$ is said to be pairwise regular if $\mathcal{P}$ and $\mathcal{Q}$ are regular with respect to each other.*

Definition 2.8. [17] *A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be pairwise normal if for any $\mathcal{P}$-closed set F_1 and $\mathcal{Q}$-closed set F_2 satisfying $F_1 \cap F_2 = \emptyset$, there exist $G_1 \in \mathcal{P}$, $G_2 \in \mathcal{Q}$ such that $F_1 \subset G_2$, $F_2 \subset G_1$, $G_1 \cap G_2 = \emptyset$.*

3. Pairwise paracompactness

We called a space (or a set) is bicompact [17] if every open cover of it has a finite subcover. Also similarly as [17] a cover B of $(X, \mathcal{P}, \mathcal{Q})$ is said to be pairwise open if $B \subset \mathcal{P} \cup \mathcal{Q}$ and B contains at least one nonempty member from each of $\mathcal{P}$ and $\mathcal{Q}$. Bourbaki and many authors defined the term paracompactness in a topological space including the requirement that the space is Hausdorff. Also in a bitopological space some authors follow this idea. But in our discussion we shall follow the convention as adopted in Munkresh[20] to define the following terminologies as in the case of a topological space.

Definition 3.1. cf.[20] *In a space X a collection of subsets $\mathcal{A}$ is said to be locally finite in X if every point has a neighborhood that intersects only a finitely many elements of $\mathcal{A}$.*

Similarly a collection of subsets $\mathcal{B}$ in a space X is said to be countably locally finite in X if $\mathcal{B}$ can be expressed as a countable union of locally finite collection.

Definition 3.2. cf.[20] *Let $\mathcal{A}$ and $\mathcal{B}$ be two covers of a space X. Then $\mathcal{B}$, is said to be a refinement of $\mathcal{A}$ if for $B \in \mathcal{B}$ there exists a $A \in \mathcal{A}$ containing B.*

We call $\mathcal{B}$ is an open refinement of $\mathcal{A}$ if the elements of $\mathcal{B}$ are open and similarly we call $\mathcal{B}$ is an closed refinement if the elements of $\mathcal{B}$ are closed.

Definition 3.3. cf.[20] *A space X is said to be paracompact if every open covering $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X.*

As in the case of a topological space [11, 2] we define the following terminologies. Let $\mathcal{A}$ and $\mathcal{B}$ be two pairwise open covers of a bispace $(X, \mathcal{P}, \mathcal{Q})$. Then $\mathcal{B}$ is said to be a parallel refinement [11] of $\mathcal{A}$ if for any $\mathcal{P}$-open set(respectively $\mathcal{Q}$-open set) B in $\mathcal{B}$ there exists a $\mathcal{P}$-open set(respectively $\mathcal{Q}$-open set) A in $\mathcal{A}$ containing B. Let $\mathcal{U}$ be a pairwise open cover in a bispace $(X, \mathcal{P}_1, \mathcal{P}_2)$. If x belongs to X and M be a subset of X, then by "M is $\mathcal{P}_{\mathcal{U}x}$-open" we mean M is $\mathcal{P}_1$-open(respectively $\mathcal{P}_2$-open set) if x belongs to a $\mathcal{P}_1$-open set(respectively $\mathcal{P}_2$-open set) in $\mathcal{U}$.

Definition 3.4. cf. [2] *Let $\mathcal{A}$ and $\mathcal{B}$ be two pairwise open covers of a bispace $(X, \mathcal{P}_1, \mathcal{P}_2)$. Then $\mathcal{B}$ is said to be a locally finite refinement of $\mathcal{A}$ if for each x belonging to X, there exists a $\mathcal{P}_{\mathcal{A}x}$-open open neighborhood of x intersecting only a finite number of sets of $\mathcal{B}$.*

Definition 3.5. cf. [2] *A bispace $(X, \mathcal{P}_1, \mathcal{P}_2)$ is said to be pairwise paracompact if every pairwise open cover of X has a locally finite parallel refinement.*

To study the notion of paracompactness in a bispace the idea of pairwise regular and strongly pairwise regular spaces play significant roll as discussed below.

As in the case of a bitopological space a bispace $(X, \mathcal{P}_1, \mathcal{P}_2)$ is said to be strongly pairwise regular[2] if $(X, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise regular and both the spaces $(X, \mathcal{P}_1)$ and $(X, \mathcal{P}_2)$ are regular.

Now we present two examples, the first one is of a strongly regular bispace and the second one is of a pairwise regular bispace without being a strongly pairwise regular bispace.

Example 3.1. Let $X = \mathbb{R}$ and (x, y) be an open interval in X. We consider the collection τ_1 with sets A in $\mathbb{R}$ such that either $(x, y) \subset \mathbb{R} \setminus A$ or $A \cap (x, y)$ can be expressed as some union of open subintervals of (x, y) and τ_2 be the collection of all countable subsets in (x, y). Also if τ be the collection of all countable union of members of $\tau_1 \cup \tau_2$ then clearly (X, τ) is a σ-space but not a topological space. Also consider the bispace (X, τ, σ), where σ is the usual topology on X.

We first show that (X, τ) is regular. Let $p \in X$ and P be any τ-closed set not containing p. Then $A = \{p\}$ is a τ-open set containing p. Also $A = \{p\}$ is closed in (X, τ) because if $p \notin (x, y)$ then $A^c \cap (x, y) = (x, y)$ and if $p \in (x, y)$ then $A^c \cap (x, y) = (x, p) \cup (p, y)$ and hence A^c is a τ-open set containing P.

Now we show that the bispace (X, τ, σ) is pairwise regular. Let $p \in X$ and M be a τ-closed set not containing p. Then $A = \{p\}$ is a τ-open set containing p and also as every singleton set is closed in (X, σ), A^c is a σ-open set containing M.

Now let $p \in X$ and P be a σ-closed set not containing p. Now consider the case when $P \cap (x, y) = \emptyset$ then P is a τ-open set containing P and P^c is a σ-open set containing p.

Now we consider the case when $P \cap (x, y) \neq \emptyset$. Since $p \notin P$, P^c is a σ-open set containing p and hence there exists an open interval I containing p be such that $p \in I \subset P^c$ and $p \in \overline{I} \subset P^c$, where $\overline{I}$ denotes the closer of I with respect to σ. If I intersects (x, y) then let $I_1 = (x, y) \setminus \overline{I}$. Clearly I_1 is non empty because $P \cap (x, y) \neq \emptyset$. Also $\overline{I} \subset P^c$ and hence $(x, y) \setminus P^c \subset (x, y) \setminus \overline{I}$ and its follows that $P \cap (x, y) \subset I_1$. So clearly $P \cup I_1$ is a τ-open set containing P and I is a σ-open set containing p and which are disjoint. Again if I does not intersect (x, y) then $P \cup (x, y)$ is a τ-open set containing P and I itself a σ-open set containing p and which are disjoint. Therefore the bispace (X, τ, σ) is strongly pairwise regular.

Example 3.2. Let $X = \mathbb{R}$ and (X, τ_1, τ_2) be a bispace where (X, τ_1) is cocountable topological space and $\tau_2 = \{X, \emptyset\} \cup \{\text{countable subsets of real numbers}\}$. Clearly τ_2 is not a topology and hence (X, τ_1, τ_2) is not a bitopological space. We show that (X, τ_1, τ_2) is a pairwise regular bispace but not a strongly pairwise regular bispace. Let $p \in X$ and A be a τ_1-closed set not containing p. Then clearly A itself a τ_2-open set containing A and A^c is a τ_1-open set containing p and clearly they are disjoint.

Similarly if B is a τ_2-closed set such that $p \notin B$, then B being a complement of a countable set is τ_1-open set containing B. Also B^c being countable is τ_2-open set containing p.

Now let $p \in X$ and P be a closed set in (X, τ_2) such that $p \notin P$. Then P must be a complement of a countable set in $\mathbb{R}$ and hence it must be a uncountable set. So clearly the only open set containing P is $\mathbb{R}$ itself. Therefore (X, τ_2) is not regular and hence (X, τ_1, τ_2) can not be strongly pairwise regular.

Remark 3.1. *In a bitopological space, pairwise Hausdorffness and pairwise paracompactness together imply pairwise normality but similar result holds in a bispace if an additional condition $C(1)$ holds.*

Theorem 3.1. *Let $(X, \mathcal{P}, \mathcal{Q})$ be a bispace, which is pairwise Hausdorff and pairwise paracompact and satisfies the condition $C(1)$ as stated below then it is pairwise normal.*

C(1) : If $A \subset X$ is expressible as an arbitrary union of $\mathcal{P}$-open sets and $A \subset B$, B is an arbitrary intersection of $\mathcal{Q}$-closed sets, then there exists a $\mathcal{P}$-open set K, such that $A \subset K \subset B$, the role of $\mathcal{P}$ and $\mathcal{Q}$ can be interchangeable.

Proof. We first show that X is pairwise regular. So let us suppose F be a $\mathcal{P}$-closed set not containing $x \in X$. Since X is pairwise Hausdorff for $\xi \in F$, there exists a $U_\xi \in \mathcal{P}$ and $V_\xi \in \mathcal{Q}$, such that $x \in U_\xi$ and $\xi \in V_\xi$ and $U_\xi \cap V_\xi = \emptyset$. Then the collection $\{V_\xi : \xi \in F\} \cup (X \setminus F)$ forms a pairwise open cover of X. Therefore it has a locally finite parallel refinement $\mathcal{W}$. Let $H = \cup\{W \in \mathcal{W} : W \cap F \neq \emptyset\}$. Now $x \in X \setminus F$ and $X \setminus F$ is $\mathcal{P}$-open set and hence there exists a $\mathcal{P}$-open neighborhood D of x intersecting only a finite number of members $W_1, W_2, \dots, W_n$ of $\mathcal{W}$. Now if $W_i \cap F = \emptyset$ for all $n = 1, 2, \dots, n$, then $H \cap D = \emptyset$. Therefore by C(1) we must have a $\mathcal{Q}$-open set K such that $F \subset H \subset K \subset D^c$. Hence we have a $\mathcal{Q}$-open set K containing F and $\mathcal{P}$-open set D containing x with $D \cap K = \emptyset$. If there exists a finite number of elements $W_{p_1}, W_{p_2}, \dots, W_{p_k}$ from the collection $\{W_1, W_2, \dots, W_n\}$ such that $W_{p_i} \cap F \neq \emptyset$, then we consider $V_{\xi_{p_i}}$ such that $W_{p_i} \subset V_{\xi_{p_i}}$, $\xi_{p_i} \in F$ and $i = 1, 2, \dots, k$, since $\mathcal{W}$ is a locally finite parallel refinement of $\{V_\xi : \xi \in F\} \cup (X \setminus F)$. Now, if $U_{\xi_{p_i}}$'s are the corresponding member of $V_{\xi_{p_i}}$, then $x \in D \cap (\bigcap_{i=1}^{n} U_{\xi_{p_i}}) = G(say) \in \mathcal{P}$. Since $\mathcal{W}$ is a cover of X it covers also D and since D intersects only finite number of members $W_1, W_2, \dots, W_n$, these n sets covers D. Now since the members $W_{p_1}, W_{p_2}, \dots, W_{p_k}$ be such that $W_{p_i} \cap F \neq \emptyset$, we have $D \cap F \subset \bigcup_{i=1}^{k} W_{p_i}$. Now let $W_{p_i} \subset V_{\xi_{p_i}}$ for some $\xi_{p_i} \in F$ and consider $U_{\xi_{p_i}}$ corresponding to $V_{\xi_{p_i}}$ be such that $U_{\xi_{p_i}} \cap V_{\xi_{p_i}} = \emptyset$. Now we claim that $G \cap F = \emptyset$. If not let $y \in G \cap F = [D \cap (\bigcap_{i=1}^{n} U_{\xi_{p_i}})] \cap F = [D \cap F] \cap (\bigcap_{i=1}^{n} U_{\xi_{p_i}})$. Then $y \in D \cap F$ and hence there exists W_{p_i} for some $i = 1, 2, \dots, k$ such that $y \in W_{p_i} \subset V_{\xi_{p_i}}$. Also $y \in (\bigcap_{i=1}^{n} U_{\xi_{p_i}}) \subset U_{\xi_{p_i}}$ and hence $y \in U_{\xi_{p_i}} \cap V_{\xi_{p_i}}$, which is a contradiction. So $G \cap F = \emptyset$. Now we have a $\mathcal{P}$-open neighborhood G of x intersecting only a finite number of members $W_{r_1}, W_{r_2} \dots W_{r_k}$ of $\mathcal{W}$ where $W_{r_i} \cap F = \emptyset$. So by similar argument there exists a $\mathcal{Q}$-open set K such that $F \subset H \subset K \subset G^c$. Thus we have a $\mathcal{Q}$-open set K containing F and a $\mathcal{P}$-open set G containing x such that $G \cap K = \emptyset$.

Next let A be a $\mathcal{Q}$-closed set and B be a $\mathcal{P}$-closed set and $A \cap B = \emptyset$. Then for every $x \in B$ and $\mathcal{Q}$-closed set A there exists $\mathcal{P}$-open set U_x containing A and $\mathcal{Q}$-open set V_x containing x with $U_x \cap V_x = \emptyset$. Now the collection $\mathcal{U} = (X \setminus B) \cup \{V_x : x \in B\}$ forms a pairwise open cover of X. Hence there exists a locally finite parallel refinement $\mathcal{M}$ of $\mathcal{U}$. Clearly $B \subset Q$ where $Q = \cup\{M \in \mathcal{M} : M \cap B \neq \emptyset\}$. Now for $x \in X \setminus B$, a $\mathcal{P}$-open set there exists a $\mathcal{P}$-open neighborhood of x intersecting only a finite number of elements of $\mathcal{M}$. Since $A \subset X \setminus B$, so for $x \in A$ there exists a $\mathcal{P}$-open neighborhood D_x of x intersecting only a finite number of elements $M_{x_1}, M_{x_2}, \dots, M_{x_n}$ of $\mathcal{M}$ with $M_{x_i} \cap B \neq \emptyset$ for some $i = 1, 2, \dots, n$. Suppose if $M_{x_i} \subset V_{x_i}$, $i = 1, 2, \dots, n$ and let $P_x = D_x \cap (\bigcap_{i=1}^{n} U_{x_i})$ where $U_{x_i} \cap V_{x_i} = \emptyset$. If $M_{x_i} \cap B = \emptyset$ for all $i = 1, 2, \dots, n$, then we consider $P_x = D_x$. Now if $P = \bigcup\{P_x : x \in A\}$ then $A \subset P$ and $P \subset Q^c$.

Now by the given condition C(1) there exists a $\mathcal{P}$-open set R be such that $A \subset P \subset R \subset Q^c$. Again by the same argument there exists a $\mathcal{Q}$-open set S be such that $B \subset Q \subset S \subset R^c$. Hence there exists a $\mathcal{P}$-open set R containing A and $\mathcal{Q}$-open set S containing B with $R \cap S = \emptyset$. □

Theorem 3.2. *If the bispace $(X, \mathcal{P}_1, \mathcal{P}_2)$ is strongly pairwise regular and satisfies the condition $C(2)$ given below, then the following statements are equivalent:*
(i) X is pairwise paracompact.
(ii) Each pairwise open cover $\mathcal{C}$ of X has a countably locally finite parallel refinement.

(iii) Each pairwise open cover $\mathcal{C}$ of X has a locally finite refinement.
(iv) Each pairwise open cover $\mathcal{C}$ of X has a locally finite refinement $\mathcal{B}$ such that if $B \subset C$ where $B \in \mathcal{B}$ and $C \in \mathcal{C}$, then $\mathcal{P}_1$-$cl(B) \cup \mathcal{P}_2$-$cl(B) \subset C$.

C(2) : If $M \subset X$ and $\mathcal{B}$ is a subfamily of $\mathcal{P}_1 \cup \mathcal{P}_2$ such that $\mathcal{P}_i$-$cl(B) \cap M = \emptyset$, for all $B \in \mathcal{B}$, then there exists a $\mathcal{P}_i$- open set S such that $M \subset S \subset [\bigcup_{B\in\mathcal{B}} \mathcal{P}_i\text{-}cl(B)]^c$.

Proof. $(i) \Rightarrow (ii)$

Let $\mathcal{C}$ be a pairwise open cover of X. Let $\mathcal{U}$ be a locally finite parallel refinement of $\mathcal{C}$. Then the collection $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$, where $\mathcal{V}_n = \mathcal{U}$ for all $n \in \mathbb{N}$, becomes the countably locally finite parallel refinement of $\mathcal{C}$.

$(ii) \Rightarrow (iii)$

We consider a pairwise open cover $\mathcal{C}$ of X. Let $\mathcal{V}$ be a parallel refinement of $\mathcal{C}$, such that $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$, where for each n and for each x there exists a $\mathcal{P}_{\mathcal{C}x}$-open neighborhood of x intersecting only a finite number of members of $\mathcal{V}_n$. For each $n \in \mathbb{N}$, let us agree to write $\mathcal{V}_n$ as $\mathcal{V}_n = \{\mathcal{V}_{n\alpha} : \alpha \in \wedge_n\}$ and we consider $M_n = \bigcup_{\alpha\in\wedge_n} \mathcal{V}_{n\alpha}$, $n \in \mathbb{N}$. Clearly the collection $\{M_n\}_{n\in\mathbb{N}}$ is a cover of X. Let $N_n = M_n - \bigcup_{k<n} M_k$. Clearly for $x \in X$ if $x \in M_n$, where n is the least positive integer then $x \in N_n$ and hence $\{N_n : n \in \mathbb{N}\}$ covers X. Also $N_n \subset M_n$ for every n, so $\{N_n : n \in \mathbb{N}\}$ is a refinement of $\{M_n : n \in \mathbb{N}\}$. The family $\{N_n : n \in \mathbb{N}\}$ is locally finite because for $x \in X$ there exists a $\mathcal{V}_{n\alpha} \in \mathcal{V}$ which can intersects only some or all of $N_1, N_2, \ldots, N_n$. Now the collection $\{\mathcal{V}_{n\alpha} \cap N_n : \alpha \in \wedge_n, n \in \mathbb{N}\}$ covers X as if $x \in \mathcal{V}_{p\alpha}$ for the least positive integer p then $x \in N_p$ and hence $x \in \mathcal{V}_{p\alpha} \cap N_p$. So clearly $\{\mathcal{V}_{n\alpha} \cap N_n : \alpha \in \wedge_n, n \in \mathbb{N}\}$ is a refinement of $\mathcal{V}$ and hence of $\mathcal{C}$. Also for $x \in X$ there exists a $\mathcal{P}_{\mathcal{C}x}$-open neighborhood $\mathcal{V}_{k\alpha}$ intersecting only a finite number of members of $\{N_n : n \in \mathbb{N}\}$ and hence it intersects only a finite number of members of $\{\mathcal{V}_{n\alpha} \cap N_n : \alpha \in \wedge_n, n \in \mathbb{N}\}$.

$(iii) \Rightarrow (iv)$

Let $\mathcal{C}$ be a pairwise open cover of X. Let $x \in X$ and suppose that $x \in C_x$ for some $C_x \in \mathcal{C}$. Without any loss of generality let $C_x \in \mathcal{P}_1$. Then $x \notin C_x^c$ and hence by using the condition of strongly pairwise regularity of X there exists a $\mathcal{P}_1$-open set D_1 containing x and a $\mathcal{P}_1$-open set $D_1^{'}$ containing C_x^c with $D_1 \cap D_1^{'} = \emptyset$. Now $(D_1^{'})^c \subset C_x$ and hence $(D_1^{'})^c$ is a $\mathcal{P}_1$-closed set such that $x \in (D_1^{'})^c \subset C_x$. Therefore $\mathcal{P}_1$-cl$(D_1) \subset C_x$ as $D_1 \subset (D_1^{'})^c \subset C_x$. Again $x \notin D_1^c$, a $\mathcal{P}_1$-closed set and hence by pairwise regularity of X there exists a $\mathcal{P}_1$-open set D_2 containing x and a $\mathcal{P}_2$-open set $D_2^{'}$ containing D_1^c with $D_2 \cap D_2^{'} = \emptyset$. Now $D_2 \subset (D_2^{'})^c$ and $D_2 \subset (D_2^{'})^c \subset D_1 \subset C_x$. Hence $\mathcal{P}_2$-cl$(D_2) \subset C_x$ and also $D_2 \subset D_1$. Therefore $\mathcal{P}_1$-cl$(D_2) \subset \mathcal{P}_1$-cl$(D_1)$ and hence $\mathcal{P}_1$-cl$(D_2) \cup \mathcal{P}_2$-cl$(D_2) \subset C_x$. Similarly if $C_x \in \mathcal{P}_2$ then there exists a $\mathcal{P}_2$-open set D_2 containing x such that $\mathcal{P}_1$-cl$(D_2) \cup \mathcal{P}_2$-cl$(D_2) \subset C_x$. Let us denote D_2 by a general notation G_x and then we can write $\mathcal{P}_1$-cl$(G_x) \cup \mathcal{P}_2$-cl$(G_x) \subset C_x$. Then, since $\mathcal{C}$ be a pairwise open cover $\{G_x : x \in X, C_x \in \mathcal{C}\}$ is a pairwise open cover of X which refines of $\mathcal{C}$. Therefore by (iii) there exists a locally finite refinement $\mathcal{B}$ of $\{G_x : x \in X\}$ and hence of $\mathcal{C}$. If $B \in \mathcal{B}$ then for some G_x we have $B \subset G_x \subset C_x$ and so $\mathcal{P}_1$-cl$(B) \cup \mathcal{P}_2$-cl$(B) \subset \mathcal{P}_1$-cl$(G_x) \cup \mathcal{P}_2$-cl$(G_x) \subset C_x$.

$(iv) \Rightarrow (i)$

Let $\mathcal{C}$ be a pairwise open cover of X and without any loss of generality we assume that there does not exist any element of $\mathcal{C}$ which is both $\mathcal{P}_1$-open and $\mathcal{P}_2$-open. So there exists a locally finite refinement $\mathcal{A}$ of $\mathcal{C}$. For $x \in X$ we must have a $C \in \mathcal{C}$

containing x. Let us suppose C is $\mathcal{P}_i$-open. Let W_x be a $\mathcal{P}_i$-open neighborhood of x intersecting only a finite number of elements of $\mathcal{A}$. So the collection $\mathcal{W} = \{W_x : x \in X\}$ is a pairwise open cover of X and let $E = \{E_\lambda : \lambda \in \wedge\}$ be a locally finite refinement of $\mathcal{W}$ such that if $E_\lambda \subset W_x$ then $\mathcal{P}_1\text{-cl}(E_\lambda) \cup \mathcal{P}_2\text{-cl}(E_\lambda) \subset W_x$. Now for $A \in \mathcal{A}$ we consider $C_A \in \mathcal{C}$ such that $A \subset C_A$. Then if C_A is $\mathcal{P}_i$-open, then we consider the set $F_A = \cup\{\mathcal{P}_i\text{-}cl(E_\lambda) : E_\lambda \in E, \mathcal{P}_i\text{-}cl(E_\lambda) \cap A = \emptyset\}$. Let $G_A = X \setminus F_A$, then by the given condition C(2) there exists a $\mathcal{P}_i$-open set S_A such that $A \subset S_A \subset G_A$. We write $H_A = S_A \cap C_A$ and since $A \subset H_A$, the collection $\{H_A : A \in \mathcal{A}\}$ covers X. Also $H_A \subset C_A$ and H_A is $\mathcal{P}_i$-open. Thus $\{H_A : A \in \mathcal{A}\}$ is a parallel refinement of $\mathcal{C}$. Now we show that $\{H_A : A \in \mathcal{A}\}$ is a locally finite refinement of $\mathcal{C}$.

We show that if M is a $\mathcal{P}_{\mathcal{W}x}$-open set containing x then it is also a $\mathcal{P}_{\mathcal{C}x}$-open set containing x. Let M be a $\mathcal{P}_{\mathcal{W}x}$-open set containing x and M is $\mathcal{P}_i$-open set then x must be contained in a $\mathcal{P}_i$-open set W_x in $\mathcal{W}$. So there exists a $\mathcal{P}_i$-open set C in $\mathcal{C}$ containing x. This shows that M is also a $\mathcal{P}_{\mathcal{C}x}$-open set containing x.

Now let $x \in X$ and J_x be a $\mathcal{P}_{\mathcal{W}x}$-open neighborhood of x intersecting only a finite numbers of members $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_n}$ of E. Hence J_x is also a $\mathcal{P}_{\mathcal{C}x}$-open neighborhood of x intersecting only a finite numbers of members $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_n}$ of E. Clearly J_x can be covered by these members of E. Now each E_{λ_i} is contained in some W_x with $\mathcal{P}_1\text{-cl}(E_{\lambda_i}) \cup \mathcal{P}_2\text{-cl}(E_{\lambda_i}) \subset W_x$. Also W_x can intersects only a finite number of members of $\mathcal{A}$. Hence each $\mathcal{P}_1\text{-cl}(E_{\lambda_i})$ or $\mathcal{P}_2\text{-cl}(E_{\lambda_i})$ can intersect only a finite number of sets in $\mathcal{A}$. So each $\mathcal{P}_1\text{-cl}(E_{\lambda_i})$ or $\mathcal{P}_2\text{-cl}(E_{\lambda_i})$ can intersect only a finite number of sets in $\{G_A : A \in \mathcal{A}\}$. Therefore J_x can intersect only a finite number of sets of $\{G_A : A \in \mathcal{A}\}$. Now $\{H_A : A \in \mathcal{A}\}$ covers X and $H_A \subset G_A$, hence J_x can intersect only a finite number of sets in $\{H_A : A \in \mathcal{A}\}$. Also $H_A \subset C_A$ and hence clearly $\{H_A : A \in \mathcal{A}\}$ refines $\mathcal{C}$. Therefore $\{H_A : A \in \mathcal{A}\}$ is a locally finite parallel refinement of $\mathcal{C}$. □

Theorem 3.3. *Let $\mathcal{A}$ be a locally finite collection in a σ-space X. Then the collection $\mathcal{B} = \{\overline{A}\}_{A \in \mathcal{A}}$ is also locally finite.*

Proof. Let $x \in X$ and U be a neighborhood of x intersecting only a finite number of members of $\mathcal{A}$. Now if for $A \in \mathcal{A}$, $A \cap U = \emptyset$ then $A \subset U^c$ and hence $A \subset \overline{A} \subset U^c$. Therefore $\overline{A} \subset U^c$ so $\overline{A} \cap U = \emptyset$. Therefore U can intersect only a finite number of members of $\mathcal{B}$. □

Theorem 3.4. *In a space any sub collection of a locally finite collection of sets is locally finite.*

Proof. Let $\mathcal{A}$ be a locally finite collection of sets in a space X and $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda, \text{an indexing set}\}$ be a sub collection of $\mathcal{A}$. If $x \in X$ then there exists a neighborhood U of X intersecting only a finite number of sets in $\mathcal{A}$. Hence U can not intersect infinite number of sets in $\mathcal{B}$. If U does not intersect any member of $\mathcal{B}$, then consider $B_p \in \mathcal{B}$ such that $M = B_p \setminus \bigcup_{\alpha \in \Lambda}^{\alpha \neq p} B_\alpha \neq \emptyset$. Then $M \cup U$ is a neighborhood of x intersecting only B_p of $\mathcal{B}$. Hence $\mathcal{B}$ is locally finite. □

It has been discussed in [10] that in a regular topological space X the following four conditions are equivalent:

(i) The space X is paracompact.

(ii) If $\mathcal{U}$ is a open cover of X then it has an open refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} V_n$, where V_n is a locally finite collection in X for each n.

(iii)For every open cover of the space X there exists its locally finite refinement.
(iv)For every open cover of the space X there exists its closed locally finite refinement.

In a σ-space it is not true because closure of a set may not be closed. But a similar kind of result has been discussed below.

Theorem 3.5. *In a regular space X for the following four conditions we have $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$:*
(i) *The space X is paracompact.*
(ii) *If $\mathcal{U}$ is a open cover of X then it has an open refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} V_n$, where V_n is a locally finite collection in X for each n.*
(iii)*For every open cover of the space X there exists its locally finite refinement.*
(iv)*For every open cover $\mathcal{A}$ of the space X there exists its locally finite refinement $S = \{S_\alpha : \alpha \in \Lambda\}$ such that $\{\overline{S_\alpha} : S_\alpha \in S\}$ is also its locally finite refinement, Λ being an indexing set.*

Proof. $(i) \Rightarrow (ii)$
The proof is straightforward.
$(ii) \Rightarrow (iii)$
Let $\mathcal{A}$ be an open cover of X. Then by (ii) there exists an open refinement $\mathcal{B} = \bigcup_{n=1}^{\infty} B_n$ where B_n is a locally finite collection in X for each n. Let $B_n = \{B_{n\alpha} : \alpha \in \Lambda_n\}$ and $C_n = \bigcup_{\alpha \in \Lambda_n} B_{n\alpha}$, Λ_n being an indexing set. Now clearly the collection $\{C_n\}$ covers X. Let us consider $D_n = C_n \setminus \bigcup_{k<n} C_k$. For $x \in X$, suppose that k be the least natural number for which $x \in B_{k\alpha}$, then $B_{k\alpha}$ can intersect at most k members $D_1, D_2, \ldots, D_k$ of $\{D_n : n \in \mathbb{N}\}$. Hence $\{D_n : n \in \mathbb{N}\}$ is a locally finite refinement of $\{C_n : n \in \mathbb{N}\}$. Now we show that $M = \{D_n \cap B_{n\alpha} : n \in \mathbb{N}, \alpha \in \Lambda_n\}$ is a locally finite refinement of $\mathcal{B}$. For $n \in \mathbb{N}$ we have $\bigcup_{\alpha \in \Lambda_n}(D_n \cap B_{n\alpha}) = D_n \cap (\bigcup_{\alpha \in \Lambda_n} B_{n\alpha}) = D_n \cap C_n = D_n$ as $D_n \subset C_n$. Also D_n covers X and hence $\bigcup_{n \in \mathbb{N}} \bigcup_{\alpha \in \Lambda_n}(D_n \cap B_{n\alpha}) = X$. Let $x \in X$ then there exists an neighborhood U of x intersecting only a finite number members $D_{i_1}, D_{i_2}, \ldots, D_{i_n}(say)$ of $\{D_n : n \in \mathbb{N}\}$. Also there exists an open set U_{i_n} intersecting only a finite number of members of B_{i_n}. Now $U \cap (\bigcap_{k=1}^{n} U_{i_k})$ is an neighborhood of x intersecting only a finite numbers of M as M covers X. Also $D_n \cap B_{n\alpha} \subset B_{n\alpha}$ and hence $M = \{D_n \cap B_{n\alpha} : n \in \mathbb{N}, \alpha \in \Lambda_n\}$ is a locally finite refinement of $\mathcal{B}$. And also since $D_n \cap B_{n\alpha} \subset B_{n\alpha} \subset A$ for some $A \in \mathcal{A}$, $M = \{D_n \cap B_{n\alpha} : n \in \mathbb{N}, \alpha \in \Lambda_n\}$ is a locally finite refinement of $\mathcal{A}$.
$(iii) \Rightarrow (iv)$
Let $\mathcal{U}$ be an open cover of X. Now for $x \in X$ we have a $U_x \in \mathcal{U}$ such that $x \in U_x$. So $x \notin (U_x)^c$ and hence by regularity of X, there exist disjoint open sets P_x and Q_x containing x and $(U_x)^c$ respectively. Hence $x \in P_x \subset (Q_x)^c \subset U_x$ and clearly $x \in \overline{P_x} \subset U_x$. Now $P = \{P_x : x \in X\}$ is an open cover of X and by (iii) it has a locally finite refinement $S = \{S_\alpha : \alpha \in \Lambda, \text{an indexing set}\}$(say). Also the collection $\{\overline{S_\alpha} : S_\alpha \in S\}$ is locally finite by previous lemma. Now for $\alpha \in \Lambda$, $S_\alpha \subset P_x \subset U_x$ for some $P_x \in P$ and hence $\overline{S_\alpha} \subset \overline{P_x} \subset U_x$ for some $U_x \in \mathcal{U}$. Therefore S is a locally finite refinement of $\mathcal{U}$ such that $\{\overline{S_\alpha} : S_\alpha \in S\}$ is also a locally finite refinement of $\mathcal{U}$.

□

We have discussed some results associated with paracompactness in a σ-space because our motivation was to establish the statement "If $(X, \mathcal{P}_1, \mathcal{P}_2)$ is a pairwise paracompact bispace with $(X, \mathcal{P}_2)$ regular, then every $\mathcal{P}_1$-F_σ proper subset is $\mathcal{P}_2$ paracompact". This has been discussed in a bitopological space [2]. But we failed due to the fact that arbitrary union of open sets in a σ-space may not be open.

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