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ON BOUNDS OF TOEPLITZ DETERMINANTS FOR A SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. In the present article, our aim is to investigate the problem of obtaining upper bounds for $|T_2(2)|$, $|T_2(3)|$, $|T_3(2)|$ and $|T_3(1)|$, which are special cases of the symmetric Toeplitz determinant for functions belonging to the $M(\lambda, n)$ subclass.

1. INTRODUCTION

Let A denote the family of normalized analytic functions in the open unit disk $\Delta = \{z \in C : |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \Delta) \tag{1.1}$$

and S be the subclass of A consisting of all univalent functions in Δ.

Let f be analytic in Δ and be given by (1.1). Then a function f is starlike and convex, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \ Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0.$$

We denote the class of starlike functions by S^* and convex functions by C, respectively.

For $f \in A$, $n \in \mathrm{N} = \{0, 1, 2, , 3, ...\}$, the operator $D^n f$ is defined by $D^n : A \to A$ [13]

$$D^0 f(z) = f(z)$$
$$D^{n+1} f(z) = z\left[D^n f(z)\right]', \ (z \in \Delta).$$

If $f \in A$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \ (z \in \Delta).$$

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Let $n \in \mathrm{N} = \{0, 1, 2, , 3, ...\}$ and $\lambda \geq 0$. We let D_λ^n denote the operator defined by [10]

$$D_\lambda^n : A \to A,$$
$$D_\lambda^0 f(z) = f(z),$$
$$D_\lambda^1 f(z) = (1-\lambda) D_\lambda^0 f(z) + \lambda z \left(D_\lambda^0 f(z)\right)' = (1-\lambda) f(z) + \lambda z f'(z),$$
$$\cdots$$
$$D_\lambda^{n+1} f(z) = (1-\lambda) D_\lambda^n f(z) + \lambda z \left(D_\lambda^n f(z)\right)'.$$

We observe that D_λ^n is a linear operator and for

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

we have [14]

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda (k-1)]^n a_k z^k \tag{1.2}$$

Hankel determinants play important role in several branches of mathematics such as quantum mechanics,image processing, statistics and probability, queueing networks, signal processing and time series analysis to mention a few [18].

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke ([2, 3]) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{1.3}$$

and define the symmetric Toeplitz determinant $T_q(n)$ as follows:

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}. \tag{1.4}$$

In particular,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}, \; H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}, \; H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

and

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \; T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix},$$
$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, \; T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

We note that $H_2(1)$ is the well-known Fekete-Szegö functional [11].

In recent years a lot of papers has been devoted to the estimation of determinants built with using coefficients of functions in the class A or its subclasses ([6, 7, 9, 17, 19, 23, 24]). In the univalent function theory, an extensive focus has been given to estimate the bounds of Hankel matrices. Hankel determinants play a vital role in different branches and have many applications. The closer relation from the Hankel determinants are the Toeplitz determinants. A Toeplitz determinant can be thought of as an 'upside-down' Hankel determinant, in that Hankel determinant have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. A good summary of the applications of Toeplitz determinant to a wide range of areas of pure and applied mathematics can also be found in [18].

In 2017, Ramachandran et al. [5] studied the problem of obtaining upper bounds for some special types of Toeplitz determinants obtained from the coefficients of functions belonging to a subclass of analytic functions denoted by M_α. In 2018, Radhika et al. [25] studied the Toeplitz matrices whose elements are the coefficients of Bazilevic functions and obtained upper bounds for the first four determinants of these Toeplitz matrices. In 2019,Arif et al. [20] studied the Hankel determinant of order three for familier subsets of analytic functions related with sine function. In 2021, Ayinla et al. [22] defined the new subclass of analytic functions denote by $R_n(\alpha,\beta)$ and obtained upper bounds of $T_2(2)$, $T_2(3)$, $T_3(2)$, $T_3(1)$ Toeplitz determinants for functions belonging to this class.

In this study, we will consider the subclass of analytic functions defined as follows:

Definition 1.1. *Let $\lambda \geq 0$ and suppose that $f(z)$ is defined by (1.1) if*

$$Re\left\{\frac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)}\right\} > 0, \quad (z \in \Delta).$$

We let the class of these functions be defined by $M(\lambda, n)$.

2. A SET OF LEMMAS

Let P denote the family of all functions p which are analytic in Δ with $Rep(z) > 0$ and has the following series representation

$$p(z) = 1 + p_1 z + p_2 z^2 + \ p_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} p_n z^n \ \ (z \in \Delta). \tag{2.1}$$

Here $p(z)$ is called the Caratheodory function [1].

Lemma 2.1. *Let $p(z) \in P$. Then $|p_n| \leq 2,\ n = 1, 2, ...$ [21]*

Lemma 2.2. *([15, 16, 20]) The power series for $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Let the function $p(z) \in P$ be given by (2.1), then*

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right) \tag{2.2}$$

for some $x,\ |x| \leq 1$,and

$$4p_3 = p_1^3 + 2p_1\left(4 - p_1^2\right)x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)\eta\,(2.3) \tag{2.3}$$

for some complex value $\eta,\ |\eta| \leq 1$.

Lemma 2.3. *([20, 22]) Let $p(z) \in P$ and has the form (2.1), then*

$$\left|p_2 - \frac{p_1^2}{2}\right| \leq 2 - \frac{|p_1|^2}{2} \tag{2.4}$$

$$\left|p_{n+2k} - \mu p_n p_k^2\right| \leq 2\left(1 + 2\mu\right) for \mu \in \mathrm{R}, \tag{2.5}$$

$$\left|p_{n+k} - \eta p_n p_k\right| < 2, \quad for 0 \leq \eta \leq 1, \tag{2.6}$$

$$|p_m p_n - p_k p_l| \leq 4 \qquad form + n = k + l, \tag{2.7}$$

and for complex number λ, *we have*

$$\left|p_2 - \lambda p_1^2\right| \leq \max\{2, 2|\lambda - 1|\}. \tag{2.8}$$

For the results in (2.4)-(2.7) see [4], *see* [12] *for the inequality (2.8).*

Lemma 2.4. [20] *Let* $p(z) \in P$*and has the form (2.1), then*

$$\left|Jp_1^3 - Kp_1p_2 + Lp_3\right| \leq 2|J| + 2|K - 2J| + 2|J - K + L|. \tag{2.9}$$

Proof. Consider the left hand side of (2.8) and rearranging the terms, we have

$$\begin{aligned}
\left|Jp_1^3 - Kp_1p_2 + Lp_3\right| &\leq \left|J\left(p_1^3 - 2p_1p_2 + p_3\right) - (K - 2J)(p_1p_2 - p_3) + (J - K + L)p_3\right| \\
&\leq |J|\left|p_1^3 - 2p_1p_2 + p_3\right| + |K - 2J||p_1p_2 - p_3| + |p_3||J - K + L| \\
&\leq 2|J| + 2|K - 2J| + 2|J - K + L|
\end{aligned}$$

where we have used Lemma 2.1, (2.6) and the result $\left|p_1^3 - 2p_1p_2 + p_3\right| \leq 2$ due to [16]. □

3. MAIN RESULTS

Theorem 3.1. *If the function* $f(z) \in M(\lambda, n)$ *and of the form (1.1), then*

$$|a_2| \leq \frac{2}{(1+\lambda)^n}, \quad |a_3| \leq \frac{3}{(1+2\lambda)^n}, \quad |a_4| \leq \frac{4}{(1+3\lambda)^n}.$$

Proof. Let $f \in M(\lambda, n)$. Then, there exists a $p \in P$ such that

$$z\left(D_\lambda^n f(z)\right)' = \left(D_\lambda^n f(z)\right) p(z).$$

From this last equation ,we write

$$\begin{aligned}
& z + 2(1+\lambda)^n a_2 z^2 + 3(1+2\lambda)^n a_3 z^3 + 4(1+3\lambda)^n a_4 z^4 + ... \\
= \;& z + [p_1 + (1+\lambda)^n a_2] z^2 + [p_2 + (1+\lambda)^n a_2 p_1 + (1+2\lambda)^n a_3] z^3 \\
& + [p_3 + (1+\lambda)^n a_2 p_2 + (1+2\lambda)^n a_3 p_1 + (1+3\lambda)^n a_4] z^4 + ...
\end{aligned}$$

Thus,we obtain

$$2(1+\lambda)^n a_2 = p_1 + (1+\lambda)^n a_2 \Rightarrow a_2 = \frac{p_1}{(1+\lambda)^n}, \tag{3.1}$$

$$3(1+2\lambda)^n a_3 = p_2 + (1+\lambda)^n a_2 p_1 + (1+2\lambda)^n a_3 \Rightarrow a_3 = \frac{p_2 + p_1^2}{2(1+2\lambda)^n}, \tag{3.2}$$

and

$$4(1+3\lambda)^n a_4 = p_3 + (1+\lambda)^n a_2 p_2 + (1+2\lambda)^n a_3 p_1 + (1+3\lambda)^n a_4 \Rightarrow$$

$$a_4 = \frac{2p_3 + 3p_1p_2 + p_1^3}{6(1+3\lambda)^n}. \tag{3.3}$$

Applying relation $|p_1| \leq 2$ in (3.1), we obtain

$$|a_2| = \frac{|p_1|}{(1+\lambda)^n} \leq \frac{2}{(1+\lambda)^n}. \tag{3.4}$$

Applying relation (2.4) in (3.2), we obtain

$$\begin{aligned}
|a_3| &= \frac{1}{2(1+2\lambda)^n}\left|p_2+p_1^2\right| = \frac{1}{2(1+2\lambda)^n}\left|p_2-\frac{1}{2}p_1^2+\frac{3}{2}p_1^2\right| \\
&\leq \frac{1}{2(1+2\lambda)^n}\left\{\left|p_2-\frac{1}{2}p_1^2\right|+\left|\frac{3}{2}p_1^2\right|\right\} \leq \frac{1}{2(1+2\lambda)^n}\left\{\left(2-\frac{1}{2}p_1^2\right)+\frac{3}{2}p_1^2\right\} \\
&= \frac{1}{2(1+2\lambda)^n}\left\{2+p_1^2\right\}.
\end{aligned}$$

Let $\Phi(p_1) = \frac{1}{2(1+2\lambda)^n}\left\{2+p_1^2\right\}$ with $p_1 \in [0,2]$. A simple computation leads to

$$\Phi'(p_1) = \frac{1}{2(1+2\lambda)^n}\{2p_1\} = \frac{p_1}{(1+2\lambda)^n} \Rightarrow p_1 = 0.$$

with a simple calculation.

$\Phi(p_1)$ has a maximum value attained at $p_1 = 2$,also which is

$$|a_3| \leq \frac{1}{2(1+2\lambda)^n}\left\{2+2^2\right\} = \frac{3}{(1+2\lambda)^n}. \tag{3.5}$$

Applying relation (2.9) in (3.3), we obtain

$$\begin{aligned}
a_4 &= \frac{1}{6(1+3\lambda)^n}\left|p_1^3+3p_1p_2+2p_3\right| \\
&\leq \frac{1}{6(1+3\lambda)^n}\left\{2|1|+2|-3-2|+2|1+3+2|\right\} \\
&= \frac{4}{(1+3\lambda)^n}.
\end{aligned}$$

□

Theorem 3.2. *Let $\lambda \geq 0$, and if the function $f(z)$ be of the form (1.1) belongs to the class$M(\lambda,n)$, then we have the sharp bound*

$$|T_2(2)| = \left|a_3^2-a_2^2\right| \leq \left|\frac{9}{(1+2\lambda)^{2n}}-\frac{4}{(1+\lambda)^{2n}}\right|. \tag{3.6}$$

Proof. In view of (3.1) and (3.2), a simple computation leads to

$$a_3^2-a_2^2 = \frac{p_2^2}{4(1+2\lambda)^{2n}}+\frac{p_2p_1^2}{2(1+2\lambda)^{2n}}+\frac{p_1^4}{4(1+2\lambda)^{2n}}-\frac{p_1^2}{(1+\lambda)^{2n}}. \tag{3.7}$$

Note that, by Lemma 2.2, we may write $2p_2 = p^2+x\left(4-p^2\right)$ where without loss of generality we let $0 \leq p_1 = p \leq 2$. Substituting this into the above equation, we obtain the following quadratic equation in terms of x :

$$a_3^2-a_2^2 = \frac{\left(4-p^2\right)^2}{16(1+2\lambda)^{2n}}x^2+\frac{3p^2\left(4-p^2\right)}{8(1+2\lambda)^{2n}}x+\frac{9p^4(1+\lambda)^{2n}-16p^2(1+2\lambda)^{2n}}{16(1+2\lambda)^{2n}(1+\lambda)^{2n}}.$$

Applying the triangle inequality, we obtain

$$\left|a_3^2-a_2^2\right| \leq \frac{\left(4-p^2\right)^2}{16(1+2\lambda)^{2n}}+\frac{3p^2\left(4-p^2\right)}{8(1+2\lambda)^{2n}}+\frac{9p^4(1+\lambda)^{2n}+16p^2(1+2\lambda)^{2n}}{16(1+2\lambda)^{2n}(1+\lambda)^{2n}} = \Psi(p)$$

Differentiating $\Psi(p)$ with respect to p, we write

$$\psi'(p)=\frac{p\left(p^2(1+\lambda)^{2n}+2(1+\lambda)^{2n}+2(1+2\lambda)^{2n}\right)}{(1+2\lambda)^{2n}(1+\lambda)^{2n}}$$

Setting $\Psi'(p)=0$ yields either $p=0$ or $p^2=-\frac{2\left[(1+\lambda)^{2n}+(1+2\lambda)^{2n}\right]}{(1+\lambda)^{2n}}$.

Since $\Psi'(p)>0$ on $0\leq p\leq 2$ and so $\Psi(p)\leq\psi(2)$. For $p=2$ we have $a_2=\frac{2}{(1+\lambda)^n}$ and $a_3=\frac{3}{(1+2\lambda)^n}$ which yields

$$\left|a_3^2-a_2^2\right|\leq\left|\frac{9}{(1+2\lambda)^{2n}}-\frac{4}{(1+\lambda)^{2n}}\right|.$$

□

Remark. *For $n=0$, as a special case of Theorem 3.2 we get the sharp bound as $|T_2(2)|=\left|a_3^2-a_2^2\right|\leq 5$. This result agree with bound obtained for the class of starlike function S^* by Thomas and Halim* [8].

Theorem 3.3. *Let$\lambda\geq 0$, and if the function $f(z)$ be of the form (1.1) belongs to the class$M(\lambda,n)$, then*

$$|T_2(3)|=\left|a_4^2-a_3^2\right|\leq\max\left\{\frac{1}{(1+2\lambda)^{2n}},\left|\frac{16}{(1+3\lambda)^{2n}}-\frac{9}{(1+2\lambda)^{2n}}\right|\right\}.$$

Proof. In view of (3.3) and (3.2) and applying Lemma 2.2, denoting $F=4-p_1^2$ and $G=\left(1-|x|^2\right)\eta$, where $0\leq p_1\leq 2$ and $|\eta|<1$, we get

$$\begin{aligned}a_4^2-a_3^2 &= \left(\frac{2p_3+3p_1p_2+p_1^3}{6(1+3\lambda)^n}\right)^2-\left(\frac{p_2+p_1^2}{2(1+2\lambda)^n}\right)^2\\ &= -\frac{9p_1^4}{16(1+2\lambda)^{2n}}+\frac{p_1^6}{4(1+3\lambda)^{2n}}-\frac{3p_1^2xF}{8(1+2\lambda)^{2n}}+\frac{5p_1^4xF}{12(1+3\lambda)^{2n}}-\frac{p_1^4x^2F}{12(1+3\lambda)^{2n}}\\ &\quad -\frac{x^2F^2}{16(1+2\lambda)^{2n}}+\frac{25p_1^2x^2F^2}{144(1+3\lambda)^{2n}}-\frac{5p_1^2x^3F^2}{72(1+3\lambda)^{2n}}+\frac{p_1^2x^4F^2}{144(1+3\lambda)^{2n}}+\frac{p_1^3FG}{6(1+3\lambda)^{2n}}\\ &\quad +\frac{5p_1xF^2G}{36(1+3\lambda)^{2n}}-\frac{p_1x^2F^2G}{36(1+3\lambda)^{2n}}+\frac{F^2G^2}{36(1+3\lambda)^{2n}}.\end{aligned}$$

As in the proof of Theorem 3.2, without loss of generality, we can write letting$p_1 = p$, where $0 \leq p_1 \leq 2$. Then an application of triangle inequality gives,

$$\begin{aligned}
\left|a_4^2 - a_3^2\right| &\leq \frac{\left(p^2-4p+4\right)\left(4-p^2\right)^2}{144\left(1+3\lambda\right)^{2n}}|x|^4 + \frac{\left(5p^2-10p\right)\left(4-p^2\right)^2}{72\left(1+3\lambda\right)^{2n}}|x|^3 \\
&\quad + \frac{\begin{array}{c}\left(12p^4-24p^3\right)\left(4-p^2\right)\left(1+2\lambda\right)^{2n} \\ +\left[9\left(1+3\lambda\right)^{2n}+25\left(1+2\lambda\right)^{2n}p^2+4\left(1+2\lambda\right)^{2n}p-8\left(1+2\lambda\right)^{2n}\right]\left(4-p^2\right)^2\end{array}}{144\left(1+3\lambda\right)^{2n}\left(1+2\lambda\right)^{2n}}|x|^2 \\
&\quad + \frac{27\left(1+3\lambda\right)^{2n}\left(4-p^2\right)p^2+30\left(1+2\lambda\right)^{2n}\left(4-p^2\right)p^4+10\left(1+2\lambda\right)^{2n}\left(4-p^2\right)^2p}{72\left(1+3\lambda\right)^{2n}\left(1+2\lambda\right)^{2n}}|x| \\
&\quad + \frac{\left(4-p^2\right)^2+6p^3\left(4-p^2\right)}{36\left(1+3\lambda\right)^{2n}} + \left|\frac{p^6}{4\left(1+3\lambda\right)^{2n}} - \frac{9p^4}{16\left(1+2\lambda\right)^{2n}}\right| \\
&= \varphi\left(p,|x|\right).
\end{aligned}$$

Now to find the maximum value of φ over the region D, differentiatingφ with respect to $|x|$, we get

$$\begin{aligned}
\frac{\partial\psi}{\partial|x|} &= \frac{\left(p^2-4p+4\right)\left(4-p^2\right)^2}{36\left(1+3\lambda\right)^{2n}}|x|^3 + \frac{\left(5p^2-10p\right)\left(4-p^2\right)^2}{24\left(1+3\lambda\right)^{2n}}|x|^2 + \left\{\frac{\left(12p^4-24p^3\right)\left(4-p^2\right)}{72\left(1+3\lambda\right)^{2n}}\right. \\
&\quad \left. + \frac{\left[9\left(1+3\lambda\right)^{2n}+25\left(1+2\lambda\right)^{2n}p^2+4\left(1+2\lambda\right)^{2n}p-8\left(1+2\lambda\right)^{2n}\right]\left(4-p^2\right)^2}{72\left(1+3\lambda\right)^{2n}\left(1+2\lambda\right)^{2n}}\right\}|x| \\
&\quad + \frac{27\left(1+3\lambda\right)^{2n}\left(4-p^2\right)p^2+30\left(1+2\lambda\right)^{2n}\left(4-p^2\right)p^4+10\left(1+2\lambda\right)^{2n}\left(4-p^2\right)^2p}{72\left(1+3\lambda\right)^{2n}\left(1+2\lambda\right)^{2n}}.
\end{aligned}$$

We need to find the maximum value of $\psi\left(p,|x|\right)$ on$[0,2]\times[0,1]$. First, assume that there is a maximum at an interior point $\psi\left(p_0,|x_0|\right)$ of $[0,2]\times[0,1]$. Differentiating $\psi\left(p,|x|\right)$ with respect to $|x|$ and equating it to zero implies that $p = p_0 = 2$, which is a contradiction. Thus for the maximum of $\psi\left(p,|x|\right)$, we need only to consider the end points of $[0,2]\times[0,1]$.For $p = 0$ we have

$$\begin{aligned}
\psi\left(0,|x|\right) &= \frac{4}{9\left(1+3\lambda\right)^{2n}}|x|^4 + \frac{\left[9\left(1+3\lambda\right)^{2n}-8\left(1+2\lambda\right)^{2n}\right]}{9\left(1+3\lambda\right)^{2n}\left(1+2\lambda\right)^{2n}}|x|^2 + \frac{4}{9\left(1+3\lambda\right)^{2n}} \\
&\leq \frac{1}{\left(1+2\lambda\right)^{2n}}.
\end{aligned}$$

For $p = 2$ we obtain

$$\psi\left(2,|x|\right) = \left|\frac{16}{\left(1+3\lambda\right)^{2n}} - \frac{9}{\left(1+2\lambda\right)^{2n}}\right|$$

For $|x| = 0$ we have

$$\psi\left(p,0\right) = \frac{\left(4-p^2\right)^2+6p^3\left(4-p^2\right)}{36\left(1+3\lambda\right)^{2n}} + \left|\frac{p^6}{4\left(1+3\lambda\right)^{2n}} - \frac{9p^4}{16\left(1+2\lambda\right)^{2n}}\right|$$

which has the maximum value $\left|\frac{16}{(1+3\lambda)^{2n}} - \frac{9}{(1+2\lambda)^{2n}}\right|$ on $[0,2]$.

For $|x|=1$we obtain

$$\begin{aligned}\psi(p,1) &= \frac{\left(p^2-4p+4\right)\left(4-p^2\right)^2}{144\left(1+3\lambda\right)^{2n}}+\frac{\left(5p^2-10p\right)\left(4-p^2\right)^2}{72\left(1+3\lambda\right)^{2n}}+\frac{\left(12p^4-24p^3\right)\left(4-p^2\right)}{144\left(1+3\lambda\right)^{2n}}\\ &+\frac{\left[9\left(1+3\lambda\right)^{2n}+25\left(1+2\lambda\right)^{2n}p^2+4\left(1+2\lambda\right)^{2n}p-8\left(1+2\lambda\right)^{2n}\right]\left(4-p^2\right)^2}{144\left(1+3\lambda\right)^{2n}\left(1+2\lambda\right)^{2n}}\\ &+\frac{27\left(1+3\lambda\right)^{2n}\left(4-p^2\right)p^2+30\left(1+2\lambda\right)^{2n}\left(4-p^2\right)p^4+10\left(1+2\lambda\right)^{2n}\left(4-p^2\right)^2p}{144\left(1+3\lambda\right)^{2n}\left(1+2\lambda\right)^{2n}}\\ &+\frac{\left(4-p^2\right)^2+6p^3\left(4-p^2\right)}{36\left(1+3\lambda\right)^{2n}}+\left|\frac{p^6}{4\left(1+3\lambda\right)^{2n}}-\frac{9p^4}{16\left(1+2\lambda\right)^{2n}}\right|\end{aligned}$$

which has the maximum value $\left|\frac{16}{(1+3\lambda)^{2n}}-\frac{9}{(1+2\lambda)^{2n}}\right|$ for $p=2$ and $\frac{1}{(1+2\lambda)^{2n}}$ for $p=0$. □

Remark. *For $n=0$,as a special case of Theorem 3.3 we get the sharp bound as $|T_2(3)|=\left|a_4^2-a_3^2\right|\le 7$. This result agree with bound obtained for the class of starlike function S^* by Thomas and Halim* [8].

Theorem 3.4. *Let$\lambda\ge 0$, and if the function $f(z)$ be of the form (1.1) belongs to the class$M(\lambda,n)$, then*

$$|T_3(2)|\le\delta_1.\delta_2,$$

where

$$\delta_1=\max\left\{\frac{2}{(1+2\lambda)^{2n}},\left|\frac{4}{(1+\lambda)^{2n}}-\frac{18}{(1+2\lambda)^{2n}}+\frac{8}{(1+\lambda)^n(1+3\lambda)^n}\right|\right\},$$

and

$$\delta_2=\max\frac{2}{(1+\lambda)^n}\left\{\frac{1}{3}\left(\frac{1+\lambda}{1+3\lambda}\right)^n,\left|1-2\left(\frac{1+\lambda}{1+3\lambda}\right)^n\right|\right\}.$$

Proof. With a simple calculation, we can write

$$|T_3(2)|=\begin{vmatrix}a_2&a_3&a_4\\a_3&a_2&a_3\\a_4&a_3&a_2\end{vmatrix}=\left|(a_2-a_4)\left(a_2^2-2a_3^2+a_2a_4\right)\right|$$

Now, let's first calculation to expression $|a_2-a_4|$. In view of (3.1) and (3.3), we obtain

$$a_2-a_4=\frac{p_1}{(1+\lambda)^n}-\frac{p_3}{3(1+3\lambda)^n}-\frac{p_1p_2}{2(1+3\lambda)^n}-\frac{p_1^3}{6(1+3\lambda)^n}.$$

Using Lemma 2.2 to express p_2 and p_3 in terms ofp_1,we obtain ,with $F=4-p_1^2$ and$G=\left(1-|x|^2\right)\eta$, we have

$$a_2-a_4=\frac{p_1}{(1+\lambda)^n}-\frac{1}{(1+3\lambda)^n}\left\{\frac{p_1^3}{2}+\frac{5p_1xF}{12}-\frac{p_1x^2F}{12}+\frac{FG}{6}\right\}.$$

In this last expression, if $0\le p_1=p\le 2$ is taken and the triangle inequality is used, we write

$$|a_2-a_4|\le\left|\frac{p}{(1+\lambda)^n}-\frac{p^3}{2(1+3\lambda)^n}\right|+\frac{5p(4-p^2)}{12(1+3\lambda)^n}|x|+\frac{(4-p^2)p}{12(1+3\lambda)^n}|x|^2+\frac{(4-p^2)\left(1-|x|^2\right)}{6(1+3\lambda)^n}\Rightarrow$$

$$\begin{aligned}|a_2-a_4| &\le \frac{(p-2)(4-p^2)}{12\left(1+3\lambda\right)^n}|x|^2+\frac{5p(4-p^2)}{12\left(1+3\lambda\right)^n}|x|+\frac{4-p^2}{6\left(1+3\lambda\right)^n}+\left|\frac{p}{\left(1+\lambda\right)^n}-\frac{p^3}{2\left(1+3\lambda\right)^n}\right|\\ &= \Phi\left(p,|x|\right).\end{aligned}$$

Differentiating $\Phi\left(p,|x|\right)$ with respect to $|x|$,we get

$$\frac{\partial\Phi}{\partial|x|}=\frac{5p\left(4-p^2\right)}{12\left(1+3\lambda\right)^n}+\frac{p|x|\left(4-p^2\right)}{6\left(1+3\lambda\right)^n}-\frac{\left(4-p^2\right)|x|}{3\left(1+3\lambda\right)^n}.$$

We need to find the maximum value of $\Phi\left(p,|x|\right)$ on$[0,2]\times[0,1]$. First, assume that there is a maximum at an interior point $\Phi\left(p_0,|x_0|\right)$ of $[0,2]\times[0,1]$. Differentiating $\Phi\left(p,|x|\right)$ with respect to $|x|$ and equating it to zero implies that$p=p_0=2$, which is a contradiction. Thus for the maximum of $\Phi\left(p,|x|\right)$, we need only to consider the end points of $[0,2]\times[0,1]$. For $p=0$ we obtain

$$\Phi\left(0,|x|\right)=-\frac{2}{3\left(1+3\lambda\right)^n}|x|^2+\frac{2}{3\left(1+3\lambda\right)^n}\le\frac{2}{3\left(1+3\lambda\right)^n}$$

For $p=2$ we have

$$\Phi\left(2,|x|\right)=\left|\frac{2}{\left(1+\lambda\right)^n}-\frac{4}{\left(1+3\lambda\right)^n}\right|=\frac{2}{\left(1+\lambda\right)^n}\left|1-2\left(\frac{1+\lambda}{1+3\lambda}\right)^n\right|.$$

For $|x|=0$we obtain

$$\Phi\left(p,0\right)=\frac{4-p^2}{6\left(1+3\lambda\right)^n}+\left|\frac{p}{\left(1+\lambda\right)^n}-\frac{p^3}{2\left(1+3\lambda\right)^n}\right|$$

which has maximum value

$$\Phi\left(p,0\right)=\left|\frac{2}{\left(1+\lambda\right)^n}-\frac{4}{\left(1+3\lambda\right)^n}\right|=\frac{2}{\left(1+\lambda\right)^n}\left|1-2\left(\frac{1+\lambda}{1+3\lambda}\right)^n\right|$$

attained at the point $p=2$..

For $|x|=1$ we obtain

$$\Phi\left(p,1\right)=\frac{(p-2)(4-p^2)}{12\left(1+3\lambda\right)^n}+\frac{5p(4-p^2)}{12\left(1+3\lambda\right)^n}+\frac{4-p^2}{6\left(1+3\lambda\right)^n}+\left|\frac{p}{\left(1+\lambda\right)^n}-\frac{p^3}{2\left(1+3\lambda\right)^n}\right|$$

Which has maximum value $\Phi\left(p,1\right)=0$ at $p=0$, and

$$\Phi\left(p,1\right)=\left|\frac{2}{\left(1+\lambda\right)^n}-\frac{4}{\left(1+3\lambda\right)^n}\right|=\frac{2}{\left(1+\lambda\right)^n}\left|1-2\left(\frac{1+\lambda}{1+3\lambda}\right)^n\right|$$

at $p=2$. Hence

$$|a_2-a_4|\le\max\left\{\frac{2}{3\left(1+3\lambda\right)^n},\frac{2}{\left(1+\lambda\right)^n}\left|1-2\left(\frac{1+\lambda}{1+3\lambda}\right)^n\right|\right\}\Rightarrow$$

$$|a_2-a_4|\le\max\frac{2}{\left(1+\lambda\right)^n}\left\{\frac{1}{3}\left(\frac{1+\lambda}{1+3\lambda}\right)^n,\left|1-2\left(\frac{1+\lambda}{1+3\lambda}\right)^n\right|\right\}.$$

In view of (3.1), (3.2) and (3.3), we write

$$a_2^2-2a_3^2+a_2a_4=\left(\frac{p_1}{\left(1+\lambda\right)^n}\right)^2-2\left(\frac{p_2+p_1^2}{2\left(1+2\lambda\right)^n}\right)^2+\left(\frac{p_1}{\left(1+\lambda\right)^n}\right)\left(\frac{2p_3+3p_1p_2+p_1^3}{6\left(1+3\lambda\right)^n}\right).$$

Using Lemma 2.2 to express p_2 and p_3 in terms of p_1, we obtain ,with $F = 4 - p_1^2$ and$G = \left(1 - |x|^2\right)\eta$, we have

$$\begin{aligned}
a_2^2 - 2a_3^2 + a_2a_4 &= \frac{p_1^2}{(1+\lambda)^{2n}} - \frac{p_1^4}{8(1+2\lambda)^{2n}} - \frac{p_1^2xF}{4(1+2\lambda)^{2n}} - \frac{x^2F^2}{8(1+2\lambda)^{2n}} - \frac{p_1^4}{2(1+2\lambda)^{2n}} \\
&\quad - \frac{p_1^2xF}{2(1+2\lambda)^{2n}} - \frac{p_1^4}{2(1+2\lambda)^{2n}} + \frac{p_1^4}{12(1+\lambda)^n(1+3\lambda)^n} + \frac{p_1^2xF}{6(1+\lambda)^n(1+3\lambda)^n} \\
&\quad - \frac{p_1^2Fx^2}{6(1+\lambda)^n(1+3\lambda)^n} + \frac{p_1FG}{6(1+\lambda)^n(1+3\lambda)^n} + \frac{p_1^4}{4(1+\lambda)^n(1+3\lambda)^n} \\
&\quad + \frac{p_1^2xF}{4(1+\lambda)^n(1+3\lambda)^n} + \frac{p_1^4}{6(1+\lambda)^n(1+3\lambda)^n}.
\end{aligned}$$

Choosing $p_1 = p \in [0,2]$, applying triangle inequality and simplifying, we obtain

$$\begin{aligned}
\left|a_2^2 - 2a_3^2 + a_2a_4\right| &\leq \left[\frac{\left(4-p^2\right)^2}{8(1+2\lambda)^{2n}} + \frac{\left(p^2-2p\right)\left(4-p^2\right)}{12(1+\lambda)^n(1+3\lambda)^n}\right]|x|^2 \\
&\quad + \left[\frac{5}{12(1+\lambda)^n(1+3\lambda)^n} - \frac{3}{4(1+2\lambda)^{2n}}\right]p^2\left(4-p^2\right)|x| + \frac{p\left(4-p^2\right)}{6(1+\lambda)^n(1+3\lambda)^n} \\
&\quad + \left|\frac{p^2}{(1+\lambda)^{2n}} - \frac{9p^4}{8(1+2\lambda)^{2n}} + \frac{p^4}{2(1+\lambda)^n(1+3\lambda)^n}\right| \\
&= \Gamma(p,|x|)
\end{aligned}$$

We need to find the maximum value of $\Gamma(p,|x|)$ on$[0,2]\times[0,1]$. First, assume that there is a maximum at an interior point $\Gamma(p_0,|x_0|)$ of $[0,2]\times[0,1]$. Differentiating $\Gamma(p,|x|)$ with respect to $|x|$ and equating it to zero implies that $p = p_0 = 2$, which is a contradiction. Thus for the maximum of $\Gamma(p,|x|)$, we need only to consider the end points of $[0,2]\times[0,1]$.For $p = 0$ we obtain

$$\Gamma(0,|x|) = \frac{2}{(1+2\lambda)^{2n}}|x|^2 \leq \frac{2}{(1+2\lambda)^{2n}}$$

For $p = 2$ we have

$$\Gamma(2,|x|) = \left|\frac{4}{(1+\lambda)^{2n}} - \frac{18}{(1+2\lambda)^{2n}} + \frac{8}{(1+\lambda)^n(1+3\lambda)^n}\right|$$

For $|x| = 0$ we obtain

$$\Gamma(p,0) = \frac{p\left(4-p^2\right)}{6(1+\lambda)^n(1+3\lambda)^n} + \left|\frac{p^2}{(1+\lambda)^{2n}} - \frac{9p^4}{8(1+2\lambda)^{2n}} + \frac{p^4}{2(1+\lambda)^n(1+3\lambda)^n}\right|$$

which has maximum

$$\Gamma(p,0) = \left|\frac{4}{(1+\lambda)^{2n}} - \frac{18}{(1+2\lambda)^{2n}} + \frac{8}{(1+\lambda)^n(1+3\lambda)^n}\right|$$

on $[0,2]$.

For $|x|=1$we obtain

$$\Gamma(p,1)=\left[\frac{\left(4-p^2\right)^2}{8(1+2\lambda)^{2n}}+\frac{\left(p^2-2p\right)\left(4-p^2\right)}{12(1+\lambda)^n(1+3\lambda)^n}\right]+\left[\frac{5}{12(1+\lambda)^n(1+3\lambda)^n}-\frac{3}{4(1+2\lambda)^{2n}}\right]p^2\left(4-p^2\right)$$
$$+\frac{p\left(4-p^2\right)}{6(1+\lambda)^n(1+3\lambda)^n}+\left|\frac{p^2}{(1+\lambda)^{2n}}-\frac{9p^4}{8(1+2\lambda)^{2n}}+\frac{p^4}{2(1+\lambda)^n(1+3\lambda)^n}\right|$$

which has maximum value $\Gamma(p,1)=\frac{2}{(1+2\lambda)^{2n}}$ for $p=0$ and $\Gamma(p,1)=\left|\frac{4}{(1+\lambda)^{2n}}-\frac{18}{(1+2\lambda)^{2n}}+\frac{8}{(1+\lambda)^n(1+3\lambda)^n}\right|$ for $p=2$.

Thus, we have

$$\left|a_2^2-2a_3^2+a_2a_4\right|\leq\max\left\{\frac{2}{(1+2\lambda)^{2n}},\left|\frac{4}{(1+\lambda)^{2n}}-\frac{18}{(1+2\lambda)^{2n}}+\frac{8}{(1+\lambda)^n(1+3\lambda)^n}\right|\right\}.$$

If expressed as

$$\left|a_2^2-2a_3^2+a_2a_4\right|\leq\max\left\{\frac{2}{(1+2\lambda)^{2n}},\left|\frac{4}{(1+\lambda)^{2n}}-\frac{18}{(1+2\lambda)^{2n}}+\frac{8}{(1+\lambda)^n(1+3\lambda)^n}\right|\right\}=\delta_1$$

and

$$|a_2-a_4|\leq\max\frac{2}{(1+\lambda)^n}\left\{\frac{1}{3}\left(\frac{1+\lambda}{1+3\lambda}\right)^n,\left|1-2\left(\frac{1+\lambda}{1+3\lambda}\right)^n\right|\right\}=\delta_2,$$

we obtain

$$|T_3(2)|=\left|(a_2-a_4)\left(a_2^2-2a_3^2+a_2a_4\right)\right|=|a_2-a_4|\left|a_2^2-2a_3^2+a_2a_4\right|\leq\delta_1.\delta_2.$$

□

Remark. *For $n=0$, we get the sharp bound as*

$$|T_3(2)|=\left|(a_2-a_4)\left(a_2^2-2a_3^2+a_2a_4\right)\right|\leq 12.$$

Theorem 3.5. *Let$\lambda\geq 0$, and if the function $f(z)$ be of the form (1.1) belongs to the class$M(\lambda,n)$, then*

$$|T_3(1)|\leq\max\left\{1+\frac{1}{(1+2\lambda)^n},\left|1+\frac{24(1+2\lambda)^{2n}-9(1+\lambda)^{2n}-8(1+2\lambda)^{2n}}{(1+\lambda)^{2n}(1+2\lambda)^n}\right|\right\}$$

Proof. With a simple calculation, we can write

$$T_3(1)=\begin{vmatrix}1 & a_2 & a_3\\ a_2 & 1 & a_2\\ a_3 & a_2 & 1\end{vmatrix}=1-a_2^2+a_3a_2^2-a_2^2+a_3a_2^2-a_3^2=1-2a_2^2(a_3-1)-a_3^2.$$

Expanding the determinant by using equations (3.1) and (3.2) and applying Lemma 2.2, with $F=4-p_1^2$, we write

$$\begin{aligned}T_3(1)&=1+2a_2^2(a_3-1)-a_3^2=1+2\left(\frac{p_1}{(1+\lambda)^n}\right)^2\left[\frac{p_2+p_1^2}{2(1+2\lambda)^n}-1\right]-\left(\frac{p_2+p_1^2}{2(1+2\lambda)^n}\right)^2\\&=1+\frac{24(1+2\lambda)^n-9(1+\lambda)^{2n}}{16(1+\lambda)^{2n}(1+2\lambda)^{2n}}p_1^4-\frac{2}{(1+\lambda)^{2n}}p_1^2+\frac{4(1+2\lambda)^n-3(1+\lambda)^{2n}}{8(1+\lambda)^{2n}(1+2\lambda)^{2n}}p_1^2xF\\&\quad-\frac{x^2F^2}{16(1+2\lambda)^{2n}}\end{aligned}$$

Without loss of generality, we let $0 \leq p_1 = p \leq 2$. Now substituting this into the above equation and applying the triangle inequality, we obtain the following quadratic equation in terms of x.

$$\begin{aligned} |T_3(1)| &\leq \frac{\left(4-p^2\right)^2}{16(1+2\lambda)^{2n}} + \frac{4(1+2\lambda)^n + 3(1+\lambda)^{2n}\left(4-p^2\right)p^2}{8(1+\lambda)^{2n}(1+2\lambda)^{2n}} \\ &\quad + \left[1 + \frac{\left[24(1+2\lambda)^n + 9(1+\lambda)^{2n}\right]p^2 + 32(1+2\lambda)^{2n}}{16(1+\lambda)^{2n}(1+2\lambda)^{2n}}p^2\right] \\ &= \Theta(p,\lambda) \end{aligned}$$

Differentiating $\Theta(p,\lambda)$ with respect to p we obtain

$$\frac{\partial\Theta}{\partial p} = \frac{p\left[p^2\left((1+\lambda)^{2n} + 4(1+2\lambda)^n\right) + 2(1+\lambda)^{2n} + 4(1+2\lambda)^n + 4(1+2\lambda)^{2n}\right]}{(1+\lambda)^{2n}(1+2\lambda)^{2n}}$$

Equating to 0 we have $\frac{\partial\Theta}{\partial p} = 0 \Rightarrow p = 0$ and

$$p^2 = -\frac{2(1+\lambda)^{2n} + 4(1+2\lambda)^n + 4(1+2\lambda)^{2n}}{(1+\lambda)^{2n} + 4(1+2\lambda)^n}.$$

Since $\Theta'(p) > 0$on $0 \leq p_1 = p \leq 2$ and so $\Theta(p) \leq \Theta(2)$. For $p_1 = 0$ we have

$$|T_3(1)| \leq \left|1 + \frac{24(1+2\lambda)^{2n} - 9(1+\lambda)^{2n} - 8(1+2\lambda)^{2n}}{(1+\lambda)^{2n}(1+2\lambda)^n}\right|.$$

□

Remark. *For $n = 0$, we get the sharp bound as* $|T_3(1)| = \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| \leq 8$.

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