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EXISTENCE RESULTS FOR A NEW FRACTIONAL PROBLEM

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Abstract. In the present paper, we prove the existence and uniqueness result of non-trivial weak solutions to a new class of fractional linear and nonlinear fractional problems, the main tool used here is the variational method combined with the theory of new fractional Sobolev spaces.

1. Introduction

This paper is devoted to studying the existence and uniqueness of weak solutions for the fractional Laplacian problem

$$\begin{cases} -div_s(D^s u) = -D^s.(D^s u) = f(x,u) & in\ \Omega, \\ u = 0 & on\ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, s is a fixed number between 0, 1 and $div_s(D^s u)$ is the fractional version of a 2-Laplacian and defined by (see [1])

$$div_s(D^s u) = D^s.(D^s u) = \sum_{i=1}^{N} \frac{\partial^s}{\partial x_i^s}\left(\frac{\partial^s u}{\partial x_i^s}\right)$$

Shieh and Spector have recently studied a novel class of fractional partial differential equations based on the distributional Riesz fractional derivatives in a pair of two fascinating works [14] and [15]. Instead of using the well-known fractional Laplacian, their starting concept is the distributional Riesz fractional gradient of order $s \in (0,1)$, which will be called here the s-gradient D^s, for brevity: for $u \in L^p(\mathbb{R}^N)$, $p \in (1,\infty)$, we set

$$D_j^s u = \frac{\partial^s u}{\partial x_j^s} = \frac{\partial}{\partial x_j}(I_{1-s} * u), \quad 0 < s < 1, \quad j = 1, \cdots, N,$$

where $\dfrac{\partial}{\partial x_j}$ is taken in the distributional sense, for every $v \in C_c^\infty(\mathbb{R}^N)$,

$$\langle \frac{\partial^s u}{\partial x_j^s}, v\rangle = (-1)\langle (I_{1-s} * u), \frac{\partial v}{\partial x_j}\rangle = -\int_{\mathbb{R}^N} (I_{1-s} * u)\frac{\partial v}{\partial x_j} dx,$$

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with I_s denoting the Riesz potential of order s, $0 < s < 1$:

$$I_s(x) := \frac{\gamma(N, 1-s)}{|x|^{N-s}},$$

where

$$\gamma(N, s) := \frac{2^s \Gamma(\frac{N+s+1}{2})}{\pi^{\frac{N}{2}} \Gamma(\frac{1-s}{2})}.$$

Thus, we can write the s-gradient (D^s) and the s-divergence ($D^s.$) for sufficiently regular functions u and vector φ ([7], [14], [15], [11]) in integral form, respectively, by

$$D^s u(x) := \gamma(N, s) \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \frac{z u(x+z)}{|z|^{N+s+1}} \chi_\epsilon(0, z) dz = \gamma(N, s) \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+s}} \frac{x-y}{|x-y|} dy$$

and

$$D^s.\varphi(x) := \gamma(N, s) \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \frac{z.\varphi(x+z)}{|z|^{N+s+1}} \chi_\epsilon(0, z) dz = \gamma(N, s) \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+s}} . \frac{x-y}{|x-y|} dy,$$

where $\chi_\epsilon(x, z)$ is the characteristic function of the set $\{(x, z) : |z - x| > \epsilon\}$ for $\epsilon > 0$. As it was shown in [14], D^s has nice properties for $u \in C_c^\infty(\mathbb{R}^N)$, namely it coincides with the fractional Laplacian as follows:

$$(-\Delta)^s u(x) = -D^s.D^s u,$$

where, $0 < s < 1$,

$$\begin{aligned} (-\Delta)^s u(x) &= \gamma^2(N, s) \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \chi_\epsilon(x, y) dy \\ &= \frac{1}{2} \gamma^2(N, s) \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2S}} dy. \end{aligned}$$

The study of elliptic equations involving fractional operators is an exciting field of nonlinear analysis. These problems have recently received a lot of attention, both for pure mathematical research and for practical applications in the real world. Indeed, this sort of operator appears in a variety of contexts, including the representation of a variety of physical processes, optimization, population dynamics, and mathematics nance. We have already mentioned that the $-D^s.D^s u$ operator coincides with the $(-\Delta)^s u$ operator in the case $u \in C_c^\infty(\mathbb{R}^N)$, and the last operator has many applications in various fields, for example, the fractional Laplacian operator $(-\Delta)^s, 0 < s < 1$, provides a simple model to describe some jump Levy processes in probability theory (see for example [4, 2, 10, 6, 8] and the references therein). As examples of applications of the problem (1.1) ($u \in C_c^\infty(\mathbb{R}^N)$), we state the following two models:

• **Model 1. Filtration in a porous medium.** The filtration phenomena of fluids in porous media are modeled by the following equation,

$$\frac{\partial c(p)}{\partial t} = \nabla a[k(c(p))(\nabla p + e)], \tag{1.2}$$

where p is the unknown pressure, c volumetric moisture content, k the hydraulic conductivity of the porous medium, a the heterogeneity matrix and $-e$ is the direction of gravity.

• **Model 2. Fluid ow through porous media.** This model is governed by the following equation,

$$\frac{\partial\theta}{\partial t} - div(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)) = 0, \tag{1.3}$$

where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydrostatic potential and e is the unit vector in the vertical direction.

The existence and uniqueness of weak solutions to problems involving the fractional Laplacian $(-\Delta)^s$ have been studied in many articles, for example, [13, 12]

Our paper's structure is separated into three sections, as follows: Section 2 introduces some preliminaries on fractional Sobolev spaces, as well as some fundamental tools for proving Theorem 3.1 and Theorem 3.2. We explain the assumptions and define the weak solution of the problem (1.1) in Part 3, and we conclude this section by demonstrating the main result.

2. Preliminaries

We recall in what follows some definitions and basic properties of the Bessel potential spaces $\mathtt{L}^{s,2}(\mathbb{R}^N)$ and spaces $X^{s,p}(\mathbb{R}^N)$. In that context, we refer to the book of E. Stein [5] and the papers of [14] and [15]. We start with the Bessel potentials ϱ_s, for $s \in \mathbb{R}^N_+$. The Bessel potentials ϱ_s are defined by

$$\varrho_s(x) := \frac{1}{(4\pi)^{\frac{s}{2}}\Gamma(\frac{s}{2})}\int_0^\infty e^{\frac{-\pi|x|^2}{t}} e^{\frac{-t}{4\pi}} t^{\frac{s-N}{2}}\frac{dt}{t}.$$

And can be shown to satisfy, for $t, s > 0$

$$\varrho_s * \varrho_t = \varrho_{s+t}$$
$$\widehat{\varrho_s}(\zeta) = (1 + 4\pi^2|\zeta|^2)^{-\frac{s}{2}}$$
$$\|\varrho_s\|_{\mathtt{L}^1(\mathbb{R}^N)}$$

Then the Bessel potential spaces $\mathtt{L}^{s,2}(\mathbb{R}^N)$ are defined as follows.

Definition 2.1. *For $s \in (0, 1)$, we define $\mathtt{L}^{s,2}(\mathbb{R}^N)$ by*

$$\mathtt{L}^{s,2}(\mathbb{R}^N) := \varrho_s(\mathtt{L}^2(\mathbb{R}^N)) = \{\varrho_s * f : f \in \mathtt{L}^2(\mathbb{R}^N\}.$$

with the norm

$$\|u\|_{\mathtt{L}^{s,2}(\mathbb{R}^N)} = \|f\|_{\mathtt{L}^2(\mathbb{R}^N)}.$$

Theorem 2.1. [14] *The following statements hold.*

1) *If* $s \geqslant 0 \Rightarrow \overline{C_0^\infty(\mathbb{R}^N)}^{\mathtt{L}^{s,2}(\mathbb{R}^N)} = \mathtt{L}^{s,2}(\mathbb{R}^N)$.
2) *If* $s \geqslant 0 \Rightarrow \left[\mathtt{L}^{s,2}(\mathbb{R}^N)\right]' = \mathtt{L}^{-s,2}(\mathbb{R}^N)$.
3) *If* $t < s \Rightarrow \mathtt{L}^{s,2}(\mathbb{R}^N) \hookrightarrow \mathtt{L}^{t,2}(\mathbb{R}^N)$.
4) *If* $s \in (0, 1) \Rightarrow \mathtt{L}^{s,2}(\mathbb{R}^N)$ *coincides with the space* $W^{s,2}(\mathbb{R}^N)$*, where*

$$W^{s,2}(\mathbb{R}^N) := \left\{u \in \mathtt{L}^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}dxdy < \infty\right\}.$$

5) *If* $t > 0, s \geqslant 0$ *with* $s \geqslant t \Rightarrow \mathtt{L}^{s+t,2}(\mathbb{R}^N) \hookrightarrow W^{s,2}(\mathbb{R}^N) \hookrightarrow \mathtt{L}^{s-t,2}(\mathbb{R}^N)$.

Theorem 2.2. [14] *Let $s \in (0, 1)$. If $u \in C_0^\infty(\mathbb{R}^N)$, then*

$$D^s u = I_{1-s} * Du.$$

Definition 2.2. *For $s \in (0,1)$. If $u \in C_c^\infty(\mathbb{R}^N)$, we define*

$$X^{s,2}(\mathbb{R}^N) := \overline{C_c^\infty(\mathbb{R}^N)}^{\|.\|_{X^{s,2}(\mathbb{R}^N)}}$$

with the norm

$$\|u\|^2_{X^{s,2}(\mathbb{R}^N)} = \|u\|^2_{\mathrm{L}^2(\mathbb{R}^N)} + \|D^s u\|^2_{\mathrm{L}^2(\mathbb{R}^N)}.$$

Proposition 2.3. [14] *If $s \in (0,1)$, then*

$$X^{s,2}(\mathbb{R}^N) = \mathrm{L}^{s,2}(\mathbb{R}^N).$$

By $\mathrm{L}_0^{s,2}(\Omega)$, we denote the subspace of $\mathrm{L}^{s,2}(\mathbb{R}^N)$ i.e.

$$\mathrm{L}_0^{s,2}(\Omega) := \left\{ u \in \mathrm{L}^{s,2}(\mathbb{R}^N) : u = 0, \ \textit{on } \mathbb{R}^N \setminus \Omega \right\}.$$

Theorem 2.4. [14] *Let $s \in (0,1)$, and $1 \le q < \dfrac{2N}{N-2s}$, then there exists a constant $C = C(\Omega, N, s) > 0$ such that*

$$\left(\int_\Omega |u|^q dx \right)^{\frac{1}{q}} \le C \|D^s u\|_{\mathrm{L}^2(\mathbb{R}^N)}$$

for all $u \in \mathrm{L}^{s,2}(\mathbb{R}^N)$.

Using the Theorem 2.4, we remark that the norm $|u|_{\mathrm{L}^2(\mathbb{R}^N)} + |D^s u|_{\mathrm{L}^2(\mathbb{R}^N)}$ is equivalent to $|D^s u|_{\mathrm{L}^2(\mathbb{R}^N)}$ in $\mathrm{L}_0^{s,2}(\Omega)$. The space $\mathrm{L}_0^{s,2}(\Omega)$ with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} D^s u . D^s v dx,$$

is a Hilbert space.

Next, we recall some embedding results

Theorem 2.5. [14]*(Fractional Sobolev inequality). Let $s \in (0,1)$ be such that $2s < N$. Then there exists a constant $C = C(N,s) > 0$ such that*

$$\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^N)} \leqslant C \|D^s u\|_{\mathrm{L}^2(\mathbb{R}^N)}$$

for all $u \in \mathrm{L}^{s,2}(\mathbb{R}^N)$, where $2^* = \frac{2N}{N-2s}$.

Proposition 2.6. [3] *Let $s \in (0,1)$. Then the embedding*

$$\mathrm{L}_0^{s,2}(\Omega) \hookrightarrow \mathrm{L}^q(\Omega)$$

is compact for $1 \le q < 2^*$.

We recall also the following propositions, which will be needed later:

Lemma 2.7. [3]

$$\int_{\mathbb{R}^N} A D^s u . D^s v = \langle -D^s . A D^s u, v \rangle. \tag{2.1}$$

holds true for $A(x)$ being a bounded measurable function and $u, v \in \mathrm{L}_0^{s,2}(\Omega)$.

Theorem 2.8. [3] *Let $A : \mathbb{R}^N \longrightarrow \mathbb{R}^{N \times N}$ a matrix with coefficients bounded, measurable and strictly elliptic, such that*

$$c_* |y|^2 \le A(x) y . y \quad \textit{and} \quad A(x) y . y^* \le c^* |y| |y^*| \tag{2.2}$$

for some $c_*, c^* > 0$ for all $x \in \mathbb{R}^N$ and all $y, y^* \in \mathbb{R}^N$, there exists $k_A(x,y)$ independent of u, v satisfying

$$\int_{\mathbb{R}^N} A(x)D^s u(x).D^s v(x)dx = P.V.\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} v(x)(u(x)-u(y))k_A(x,y)dydx \quad (2.3)$$

for all $u, v \in \mathrm{L}_0^{s,2}(\Omega)$, where $k_A(x,y)$ is given, for $x \neq y$, by

$$k_A(x,y) = c_{N,s}^2 P.V.\int_{\mathbb{R}^N} A(z)\frac{y-z}{|y-z|^{N+s+1}}.\frac{z-x}{|z-x|^{N+s+1}}dz. \quad (2.4)$$

Theorem 2.9. [9] *Let X be a reflexive real Banach space and $T : X \to X'$ be a bounded operator, hemi-continuous, coercive and monotone on space X . Then, the equation $Tu = f^*$ has at least one solution $u \in X$ for each $f^* \in X'$.*

3. Existence of solutions

We will present the idea of weak solutions for problem (1.1) in this part, as well as the existence and uniqueness results for these solutions. Firstly, we cite the following assumptions:

(h_1) $s \in (0,1)$ with $2s < N$.

(h_2) $f \in \mathrm{L}^\alpha(\Omega)$ with $\alpha > \dfrac{2n}{n+2s}$.

(h_3) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory condition and

$$|f(x,t)| \leqslant a(x) + b|t| \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $a \in \mathrm{L}^2(\Omega)$ and $b \in \mathbb{R}$.

(h_4) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a decreasing function with respect to the second variable.

(h_5) There exists $c_0 \geqslant 0$ such that

$$(f(x,t) - f(x,s))(t-s) \leqslant c_0|t-s|^2$$

for a.a. $x \in \Omega$ and all $(t,s) \in \mathbb{R} \times \mathbb{R}$.

By substituting matrix A with the matrix I in the Lemma 2.7 and Theorem 2.8, we then give the definition of the weak solution to problem (1.1)

Definition 3.1. *We say that $u \in \mathrm{L}_0^{s,2}(\Omega)$ is a weak solution of (1.1), if*

$$\int_{\mathbb{R}^N} D^s u.D^s v dx = \int_\Omega f(x,u)v(x)dx, \quad \forall v \in X := \mathrm{L}_0^{s,2}(\Omega). \quad (3.1)$$

3.1. **f is independent of u.** If f is independent of u, we have one of our main result of this work is the following Theorem

Theorem 3.1. *If $f(x,u) = f(x)$ and if hypotheses (h_1), (h_2) hold, then, the problem (1.1) has a unique weak solution.*

Proof. **Existence part.** Let the operator

$$L : \mathrm{L}_0^{s,2}(\Omega) \longrightarrow (\mathrm{L}_0^{s,2}(\Omega))',$$

such that

$$\begin{aligned} \langle L(u), \eta\rangle &= \int_{\mathbb{R}^N} D^s u.D^s \eta dx - \int_\Omega f(x)\eta(x)dx \\ &= \langle L_1(u), \eta\rangle - \langle L_2(u), \eta\rangle, \end{aligned}$$

where $(\mathrm{L}_0^{s,2}(\Omega))'$ is the dual space of $\mathrm{L}_0^{s,2}(\Omega)$.

The proof of existence part of Theorem 3.1 is divided into several steps.

• **Step 1. The operator L is bounded.**

On the one hand, we use Hölder-type inequality, we have for any $u, \eta \in \mathrm{L}_0^{s,2}(\Omega)$,

$$\begin{aligned} |\langle L_1(u), \eta\rangle| &\leq \int_{\mathbb{R}^N} |D^s u . D^s \eta| dx \\ &\leq |D^s u|_{\mathrm{L}^2(\mathbb{R}^N)} |D^s \eta|_{\mathrm{L}^2(\mathbb{R}^N)} \\ &\leq |u|_{\mathrm{L}_0^{s,2}(\Omega)} |\eta|_{\mathrm{L}_0^{s,2}(\Omega)}. \end{aligned}$$

This implies that L_1 is bounded. On the other hand, using again Hölder-type inequality, hypothesis (h_1) and (h_2), we get

$$\begin{aligned} |\langle L_2(u), \eta\rangle| &\leq \int_{\Omega} |f(x)\eta| dx \\ &\leq |f|_{\mathrm{L}^{\alpha}(\Omega)} |\eta|_{\mathrm{L}^{\alpha'}(\Omega)} \\ &\leq M|f|_{\mathrm{L}^{\alpha}(\Omega)})|\eta|_{\mathrm{L}_0^{s,2}(\Omega)} \end{aligned}$$

where M is constant of continuous embedding given by Theorem 2.4. Hence, the operator L is bounded.

• **Step 2. The operator L is hemi-continuous.**

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{L}_0^{s,2}(\Omega)$ and $u \in \mathrm{L}_0^{s,2}(\Omega)$ such that u_n converges strongly to u in $\mathrm{L}_0^{s,2}(\Omega)$. Firstly, we will prove that L_1 is continuous on $\mathrm{L}_0^{s,2}(\Omega)$, indeed,

$$\begin{aligned} |\langle L_1(u_n) - L_1(u), \eta\rangle| &= |\int_{\mathbb{R}^N} (D^s u_n - D^s u) . D^s \eta dx| \\ &\leq |D^s u_n - D^s u|_{\mathrm{L}^2(\mathbb{R}^N)} |D^s \eta|_{\mathrm{L}^2(\mathbb{R}^N)} \\ &\leq |u_n - u|_{\mathrm{L}_0^{s,2}(\Omega)} |\eta|_{\mathrm{L}_0^{s,2}(\Omega)}. \end{aligned}$$

Consequently

$$L_1(u_n) \longrightarrow L_1(u) \qquad in\ (\mathrm{L}_0^{s,2}(\Omega))^{'}.$$

This implies that the operator L_1 is continuous on $\mathrm{L}_0^{s,2}(\Omega)$. Therefore, L is hemi-continuous on $\mathrm{L}_0^{s,2}(\Omega)$.

• **Step 3 . The operator L is coercive.**

For any $u \in \mathrm{L}_0^{s,2}(\Omega)$, we have

$$\begin{aligned} \langle L(u), u\rangle &= \int_{\mathbb{R}^N} |D^s u|^2 dx - \int_{\Omega} f(x) u dx \\ &\geqslant |u|^2_{\mathrm{L}_0^{s,2}(\Omega)} - |f|_{\mathrm{L}^{\alpha}(\Omega)} |u|_{\mathrm{L}^{\alpha'}(\Omega)} \\ &\geqslant |u|^2_{\mathrm{L}_0^{s,2}(\Omega)} - M|f|_{\mathrm{L}^{\alpha}(\Omega)} |u|_{\mathrm{L}_0^{s,2}(\Omega)}. \end{aligned}$$

Therefore

$$\frac{\langle L(u), u\rangle}{|u|_{\mathrm{L}_0^{s,2}(\Omega)}} \longrightarrow +\infty \quad as\ |u|_{\mathrm{L}_0^{s,2}(\Omega)} \longrightarrow +\infty.$$

Hence, the operator L is coercive.

• **Step 4. The operator L is monotone.**

For that, it suffices to prove that L_1 is monotone

$$\langle L_1(u) - L_1(v), u - v\rangle = \int_{\mathbb{R}^N} |D^s u - D^s v|^2 dx \geqslant 0 \qquad for\ all\ u, v \in \mathrm{L}_0^{s,2}(\Omega).$$

Therefore L is monotone. Hence, the existence of weak solution for problem (1.1) follows from Theorem 2.9.

Uniqueness part. Let u and w be two weak solutions of problem (1.1). As a test function for the solution u, we take $v = u - w$ in equality (3.1) and for the solution w, we take $v = w - u$ as a test function in (3.1), we have

$$\int_{\mathbb{R}^N} D^s u . D^s(u-w)dx = \int_\Omega f(u-w)dx$$

and

$$\int_{\mathbb{R}^N} D^s w . D^s(w-u)dx = \int_\Omega f(w-u)dx.$$

By summing up the two above equalities, we get

$$\int_{\mathbb{R}^N} |D^s u - D^s w|^2 dx = 0.$$

This implies that

$$u = w \quad a.e \ in \ \Omega.$$

□

3.2. f **is dependent of** u. If f is dependent of u, we have

Theorem 3.2. *If hypotheses* $(h_1), (h_2)$ *and* (h_3) *hold, then, the problem (1.1) has a unique weak solution.*

Proof. **Existence part.** Let the operator

$$T : \mathrm{L}_0^{s,2}(\Omega) \longrightarrow (\mathrm{L}_0^{s,2}(\Omega))'$$

such that

$$\begin{aligned} \langle T(u), \eta \rangle &= \int_{\mathbb{R}^N} D^s u . D^s \eta dx - \int_\Omega f(x,u)\eta(x)dx \\ &= \langle \psi(u), \eta \rangle - \langle \Phi(u), \eta \rangle \end{aligned}$$

The proof of existence part of Theorem 3.2 is divided into several steps.

• **Step 1. The operator** T **is bounded.**

From step 1 in the proof of Theorem 3.1 we can see that the operator ψ is bounded.

On the other hand, using again Hölder-type inequality, hypotheses $(h_1) - (h_3)$, we get

$$\begin{aligned} |\langle \Phi(u), \eta \rangle| &\leq \int_\Omega |f(x,u)\eta| dx \\ &\leq |a(x) + b|u||_{\mathrm{L}^2(\Omega)} |\eta|_{\mathrm{L}^2(\Omega)} \\ &\leq M(|a|_{\mathrm{L}^2(\Omega)} + |b||u|_{\mathrm{L}^2(\Omega)})|\eta|_{\mathrm{L}_0^{s,2}(\Omega)}, \end{aligned}$$

where M is constant of continuous embedding given by Theorem 2.4. Hence, the operator T is bounded.

• **Step 2. The operator** T **is hemi-continuous.**

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{L}_0^{s,2}(\Omega)$ and $u \in \mathrm{L}_0^{s,2}(\Omega)$ such that u_n converges strongly to u in $\mathrm{L}_0^{s,2}(\Omega)$.

We conclude by proving step 2 in proving Theorem 3.1 that the operator ψ is continuous in space $\mathrm{L}_0^{s,2}(\Omega)$. We then suffice to prove the continuity of the operator Φ

$$\begin{aligned}|\langle\Phi(u_n)-\Phi(u),\eta\rangle| &= |\int_\Omega (f(x,u_n)-f(x,u))\eta dx| \\ &\leq |f(x,u_n)-f(x,u)|_{\mathrm{L}^2(\Omega)}|\eta|_{\mathrm{L}^2(\Omega)} \\ &\leq M|f(x,u_n)-f(x,u)|_{\mathrm{L}^2(\Omega)}|\eta|_{\mathrm{L}^{s,2}(\Omega)}.\end{aligned}$$

By Theorem 2.4,

$$u_n \longrightarrow u \; in \; \mathrm{L}_0^{s,2}(\Omega) \Rightarrow u_n \longrightarrow u \; in \; \mathrm{L}^2(\Omega). \tag{3.2}$$

Using Lebesgues convergence inverse theorem, we get

$$\begin{cases} u_n(.) \longrightarrow u(.) \qquad a.e. \; on \; \Omega \\ \exists g(.) \in \mathrm{L}^2(\Omega) : |u_n(x)| \leqslant g(x) \;\; \forall n, \; a.e. \; on \; \Omega, \end{cases} \tag{3.3}$$

Now, using Lebesgues convergence theorem and hypothesis (h_3), we derive

$$f(x,u_n) \longrightarrow f(x,u_n) \; in \; \mathrm{L}^2(\Omega). \tag{3.4}$$

So, $\Phi(u_n) \longrightarrow \Phi(u)$ *in* $\mathrm{L}_0^{s,2}(\Omega)$. Then Φ is continuous. Therefore, T is hemi-continuous on $\mathrm{L}_0^{s,2}(\Omega)$.

• **Step 3. The operator T is coercive.**

For any $u \in \mathrm{L}_0^{s,2}(\Omega)$, we have

$$\begin{aligned}\langle T(u),u\rangle &= \int_{\mathbb{R}^N} |D^s u|^2 dx - \int_\Omega f(x,u)u dx \\ &\geqslant |u|^2_{\mathrm{L}_0^{s,2}(\Omega)} - (|a|_{\mathrm{L}^2(\Omega)} + |b||u|_{\mathrm{L}^2(\Omega)})|u|_{\mathrm{L}^2(\Omega)} \\ &\geqslant |u|^2_{\mathrm{L}_0^{s,2}(\Omega)} - M(|a|_{\mathrm{L}^2(\Omega)} + |b||u|_{\mathrm{L}^2(\Omega)})|u|_{\mathrm{L}_0^{s,2}(\Omega)}.\end{aligned}$$

Therefore

$$\frac{\langle T(u),u\rangle}{|u|_{\mathrm{L}_0^{s,2}(\Omega)}} \longrightarrow +\infty \;\; as \; |u|_{\mathrm{L}_0^{s,2}(\Omega)} \longrightarrow +\infty.$$

Hence, the operator T is coercive.

• **Step 4. The operator T is monotone.**

Applying hypothesis (h_4), it is for each $u, v \in \mathrm{L}_0^{s,2}(\Omega)$,

$$\langle T(u)-T(v), u-v\rangle = \int_{\mathbb{R}^N} |D^s u - D^s v|^2 dx - \int_\Omega (f(x,v)-f(x,u))(v-u)dx \geqslant 0.$$

Therefore T is monotone. Hence, the existence of weak solution for problem (1.1) follows from Theorem 2.9.

Uniqueness part. Let $u, v \in \mathrm{L}_0^{s,2}(\Omega)$ be two weak solutions of (1.1). Considering the weak formulation of u and v, by choosing $w = u - v$ as a test function, we have

$$\int_{\mathbb{R}^N} D^s u . D^s(u-v)dx = \int_\Omega f(x,u)(u-v)dx$$

and

$$\int_{\mathbb{R}^N} D^s v . D^s(u-v)dx = \int_\Omega f(x,v)(u-v)dx.$$

Subtracting the above two equations, we have

$$\int_{\mathbb{R}^N} (D^s u - D^s v).D^s(u-v)dx = \int_\Omega (f(x,u)-f(x,v))(u-v)dx. \tag{3.5}$$

For all $x \in \mathbb{R}^N$, then we have the following inequality:

$$(D^s u - D^s v).D^s(u-v) \geqslant \frac{1}{4}|D^s(u-v)|^2. \tag{3.6}$$

Using assumption (h_5) in (3.6), then it follows that

$$\begin{aligned}
\frac{1}{4}\int_{\mathbb{R}^N} |D^s(u-v)|^2 dx &\leqslant \int_{\mathbb{R}^N} (D^s u - D^s v).D^s(u-v) dx \\
&= \int_{\Omega} (f(x,u) - f(x,v))(u-v) dx \\
&\leqslant c_0 \int_{\Omega} |u-v|^2 dx \\
&\leqslant c_0 M \int_{\mathbb{R}^N} |D^s(u-v)|^2 dx.
\end{aligned}$$

Consequently, when $4c_0 M < 1$, it follows from the above inequality that $u = v$ and so the solution of (1.1) is unique. The proof is complete. □

Conclusion

In this work, we have studied the existence and uniqueness of weak solutions to two problems, one of which is linear and the other is non-linear by working on new fractional Sobolev spaces. We hope in the future to generalize this study to fractional spaces with a variable exponent using the $p(x)$-Laplacian fractional operator.

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