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TOPOLOGICAL DEGREE METHOD FOR FRACTIONAL LAPLACIAN SYSTEM

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Abstract. In this paper, we study the existence of weak solutions for a semilinear fractional elliptic system with Dirichlet boundary conditions. We apply the Leray-Schauder degree method in order to obtain a result about the existence of solutions.

1. Introduction

Fractional calculus is the general case of integral and derivatives calculus. Over the last four decades or so, fractional calculus has acquired big attention and gained enormous reputation amid reaserchers and it has developed rapidly, mostly because of its applications in different branches such as: mechanics, engineering, physics, biology, signal and image processing, economic, dynamic systems and other sciences, see for example the work of HongGuang Sun, Yong Zhang, Dumitru Baleanu, Wen Chen and YangQuan Chen in [13], see also [1, 4, 6, 14, 20]. One of the most important branches of fractional calculus is fractional differential equations, which have several applications see for example [5, 15, 10, 18, 22, 24, 25].

The present work studies the existence of weak solutions in fractional Sobolev space for a semilinear fractional elliptic system of non-local equations involving the fractional Laplacian with Dirichlet boundary conditions. This problem can be regarded as the fractional version of the problem in [12] where the study is about establishing the existence of weak solutions in the classical Sobolev space for a semilinear elliptic system of local equations involving the classical Laplacian with Dirichlet boundary conditions. By comparison between the two problems, it appears that the fractional problem is more intresting by his non-local property.

In recent years, fractional elliptic systems have captivated the interest of many reaserchers such as in [17] Manasss de Souza studies the existence and multiplicity of solutions for a class of fractional elliptic systems, in [26] the authors study the multiplicity of solutions for a critical fractional elliptic system involving concave-convex nonlinearities and in [11] Haining Fan sudies the multiplicity of positive solutions for a fractional elliptic system with critical nonlinearities.

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Different methods are used to study the existence of solutions for semilinear fractional elliptic problems for example in [23] Shangjian Liu uses Stampaccia's theorem to study the existence of solutions and in [21] the authors study the existence of solutions for a fractional elliptic problem and they find non-trivial solution using the Mountain Pass theorem.

The study of semilinear elliptic coupled systems involving the Fractional Laplacian is also important in applied sciences see [2, 3, 8, 16, 19, 27].

The fractional Laplacian has a variety of definitions (see [7]) among which following definition: as an integral in the sens of the Cauchy principle value in the real space

$$(-\Delta)^s u(x) := C(n,s)p.v. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, x \in \mathbb{R}^n$$

$\forall u \in \mathscr{S}$, $\forall s \in (0,1)$

$$C(n,s) := \pi^{-(2s+n/2)} \frac{\Gamma(s+n/2)}{\Gamma(-s)},$$

and $\mathscr{S}$ is the Schwartz space.

Our purpose in this paper is to study the existence of weak solutions. For this we will use a topological method which is based on the Leray-Schauder degree. This method is very interesting in solving nonlinear problems because of its homotopy invariance property. This study concerns the following problem

$$\begin{cases} (-\Delta)^s u(x) + g_1(x, u(x), v(x)) = f_1(x) & \text{in } \Omega, \\ (-\Delta)^s v(x) + g_2(x, u(x), v(x)) = f_2(x) & \text{in } \Omega, \\ u = v = 0, & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with a Lipschitz boundary,
with $s \in (0,1)$, $f = (f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$ and $g_1, g_2 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are satisfying the Caratheodory conditions[1] and the following assumptions :

(H_1) The Nemytski operators G and F respectively defined as:
$G(u,v)(x) = g_1(x, u(x), v(x))$ and $F(u,v)(x) = g_2(x, u(x), v(x))$ are continuous and bounded from $L^2(\Omega) \times L^2(\Omega)$ into $L^2(\Omega)$.

(H_2) Sign assumption:

$$\begin{cases} g_1(x,s,p)s & \geq 0 \ \ \forall s,p \in \mathbb{R} \ \text{ and } \ \text{a.a } \ x \in \Omega. \\ g_2(x,s,p)p & \geq 0 \ \ \forall s,p \in \mathbb{R} \ \text{ and } \ \text{a.a } \ x \in \Omega. \end{cases}$$

Remark. *The assumption (H_1) is realised if for example g_1 and g_2 satisfy the following growth conditions:*
there exist $a, b \in L^2(\Omega)$ and $K_1, K_2, r_1, r_2 \in \mathbb{R}_+^*$ such that

$$|g_1(x,s,p)| \leq a(x) + K_1|s| + K_2|p|, \forall s,p \in \mathbb{R} \ \textit{ and } \ a.a \ x \in \Omega.$$
$$|g_2(x,s,p)| \leq b(x) + r_1|s| + r_2|p|, \forall s,p \in \mathbb{R} \ \textit{ and } \ a.a \ x \in \Omega.$$

The organization of this paper is as follows. In section 2, we present some preliminaries and the main result of this paper. Section 3 contains a fixed point

[1] g_1, g_2 satisfy the Caratheodory conditions, i.e. $g_1(.,s), g_2(.,z)$ are measurables for all $s, z \in \mathbb{R}^2$ and $g_1(x,.), g_2(y,.)$ are continuous for almost every $x, y \in \Omega$.

formulation of our problem. Section 4 gives a proof of the main result. Finally, in section 5 we give some conclusions.

2. Preliminaries and main result

In this section, we recall some definitions, properties and propositions about the fractional Sobolev spaces $H^s(\mathbb{R}^n)$ and $D^{s,2}(\Omega)$, which we will use later.
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz[2] open domain, $L^2(\Omega)$ be the classical space of square integrable functions on Ω, which equipped with the usual inner product and norm $< .,. >_{L^2(\Omega)}$, $\|.\|_{L^2(\Omega)}$ is a Hilbert space.

For $s \in (0,1)$, we consider the classical fractional Sobolev space

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\},$$

which equipped with the inner product and norm

$$< u,v >_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)v(x)dx + \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dydx,$$

$$\|u\|_{H^s(\mathbb{R}^n)} := \left(\|u\|^2_{L^2(\mathbb{R}^n)} + [u]^2_{H^s(\mathbb{R}^n)} \right)^{\frac{1}{2}},$$

where

$$[u]_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}$$

is a Hilbert space.
Next, we need to introduce the following fractional space in Ω. Namely, we set

$$D^{s,2}(\Omega) = \overline{\{u \in C_c^\infty(\mathbb{R}^n), supp(u) \subset \Omega\}}^{\|\|_{H^s(\mathbb{R}^n)}}.$$

Here Ω is a bounded Lipschitz open domain, therefore we can set

$$D^{s,2}(\Omega) = \{ u \in H^s(\mathbb{R}^n), \text{ such that } u = 0 \text{ in } \mathbb{R}^n \backslash \Omega \}.$$

Equipped with the inner product and norm inherited from $H^s(\mathbb{R}^n)$

$$\langle u,v \rangle_{D^{s,2}(\Omega)} = C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dydx,$$

$$\|u\|_{D^{s,2}(\Omega)} = \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dydx \right)^{\frac{1}{2}}$$

is a Hilbert space.
$D^{s,2}(\Omega)$ is a closed subspace of $H^s(\mathbb{R}^n)$.
The norm $\|.\|_{D^{s,2}(\Omega)}$ is equivalent to the H^s-norm according to the following proposition.

Proposition 2.1 (see [7]). *Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$ and $s \in (0,1)$ such that $n > 2s$. Let $u : \Omega \to \mathbb{R}$ be a measurable function compactly supported. Then, there exists a positive constant $c_{emb} > 0$ depending on n, s and Ω such that*

$$\|u\|_{L^2(\Omega)} \leq c_{emb} \|u\|_{D^{s,2}(\Omega)}.$$

[2]The domain Ω has to be Lipschitz in order that the fractional Sobolev inequalities hold true for the funtions defined on Ω, for more details see [7].

Proposition 2.2 (see [9]). *Let $s \in (0,1)$, $n \geq 1$, $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded open set and $\mathscr{T}$ be a bounded subset of $L^2(\Omega)$. Suppose that*

$$\sup_{f \in \mathscr{T}} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}} dxdy < +\infty.$$

Then $\mathscr{T}$ is precompact in $L^2(\Omega)$.

Now, we take the space $L^2(\Omega) \times L^2(\Omega)$, with the norm

$$\|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)} = \|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}.$$

Next, we consider the space $D^{s,2}(\Omega) \times D^{s,2}(\Omega)$, with the norm that we will denote by $\|(\cdot,\cdot)\|_{D^{s,2}(\Omega)\times D^{s,2}(\Omega)}$

$$\|(u,v)\|_{D^{s,2}(\Omega)\times D^{s,2}(\Omega)} = \|u\|_{D^{s,2}(\Omega)} + \|v\|_{D^{s,2}(\Omega)}.$$

Throughout the paper, we always assume that $n > 2s$.
The following theorem is the main result of this paper.

Theorem 2.3. *Under the assumptions (H_1) and (H_2), the problem* (1.1) *has at least one weak solution $(u,v) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega)$.*

3. Fixed point formulation of the problem (1.1)

In this section, we are interested to the following problem to define a certain map, which is used to formulate a fixed point problem.
For $(u,v) \in L^2(\Omega) \times L^2(\Omega)$, we consider the linear problem

$$\begin{cases} (-\Delta)^s \varphi(x) + t g_1(x, u(x), v(x)) = t f_1(x) & \text{in } \Omega, \\ (-\Delta)^s \phi(x) + t g_2(x, u(x), v(x)) = t f_2(x) & \text{in } \Omega, \\ \varphi = \phi = 0 & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{3.1}$$

where $(f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$.

Lemma 3.1. *For $(f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$ and under the assumption (H_1), the problem* (3.1) *has a unique weak solution $(\varphi, \phi) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega)$.*

Proof. For all $(u,v) \in L^2(\Omega) \times L^2(\Omega)$, we have $g_1(.,u,v), g_2(.,u,v) \in L^2(\Omega)$ from the assumption (H_1).
To prove the existence and uniqueness of weak solution for the problem (3.1) we will use the Lax-Milgram theorem.
First of all, we multiply the first and second equation of the problem (3.1) by $\psi_1 \in D^{s,2}(\Omega)$ and $\psi_2 \in D^{s,2}(\Omega)$ respectively, integrate over $\mathbb{R}^n$, use the definition of the fractional Laplacian $(-\Delta)^s$ and obtain the following weak formulation of the problem (3.1)

$$\begin{cases} (\varphi, \phi) \in D^{s,2}(\Omega) \times D^{s,2}(\Omega), \\ a_1(\varphi, \psi_1) = l_1(\psi_1) \quad \forall \psi_1 \in D^{s,2}(\Omega), \\ a_2(\phi, \psi_2) = l_2(\psi_2) \quad \forall \psi_2 \in D^{s,2}(\Omega), \end{cases} \tag{3.2}$$

with

$$a_1(.,.) : D^{s,2}(\Omega) \times D^{s,2}(\Omega) \longrightarrow \mathbb{R}$$
$$(\varphi, \psi_1) \mapsto a_1(\varphi, \psi_1) = C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(\varphi(x) - \varphi(y))(\psi_1(x) - \psi_1(y))}{|x-y|^{n+2s}} dydx,$$

$$\begin{aligned}
l_1 : D^{s,2}(\Omega) &\longrightarrow \mathbb{R}\\
&\psi_1 \mapsto\ l_1(\psi_1) = \int_\Omega tf_1(x)\psi_1 dx - \int_\Omega tg_1(x,u(x),v(x))\psi_1 dx,\\
a_2(.,.) : D^{s,2}(\Omega)\times D^{s,2}(\Omega) &\longrightarrow \mathbb{R}\\
&(\phi,\psi_2) \mapsto\ a_2(\phi,\psi_2) = C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\phi(x)-\phi(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}}dydx,\\
l_2 : D^{s,2}(\Omega) &\longrightarrow \mathbb{R}\\
&\psi_2 \mapsto\ l_2(\psi_2) = \int_\Omega tf_2(x)\psi_2 dx - \int_\Omega tg_2(x,u(x),v(x))\psi_2 dx.
\end{aligned}$$

Let us prove that the bilinear forms $a_1(.,.)$ and $a_2(.,.)$ are continuous and coercives in $D^{s,2}(\Omega)\times D^{s,2}(\Omega)$. In addition, we prove that the linear forms l_1 and l_2 are continuous.

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|a_1(\varphi,\psi_1)| &= \left|C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\varphi(x)-\varphi(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}}dydx\right|\\
&\leq C(n,s)\iint_{\mathbb{R}^{2n}} \frac{|\varphi(x)-\varphi(y)||\psi_1(x)-\psi_1(y)|}{|x-y|^{n+2s}}dydx\\
&\leq C(n,s)\|\varphi\|_{D^{s,2}(\Omega)}\|\psi_1\|_{D^{s,2}(\Omega)},
\end{aligned}$$

hence the bilinear form $a_1(.,.)$ is continuous.

Applying the Cauchy-Schwarz inequality and Proposition (2.1), we obtain

$$\begin{aligned}
|l_1(\psi_1)| &= \left|\int_\Omega tf_1(x)\psi_1 dx - \int_\Omega tg_1(x,u(x),v(x))\psi_1 dx\right|\\
&\leq \|f_1\|_{L^2(\Omega)}\|\psi_1\|_{L^2(\Omega)} + \|g_1(.,u,v)\|_{L^2(\Omega)}\|\psi_1\|_{L^2(\Omega)}\\
&\leq \left[\|f_1\|_{L^2(\Omega)} + \|g_1(.,u,v)\|_{L^2(\Omega)}\right]\|\psi_1\|_{L^2(\Omega)}\\
&\leq c_{emb}\left[\|f_1\|_{L^2(\Omega)} + \|g_1(.,u,v)\|_{L^2(\Omega)}\right]\|\psi_1\|_{D^{s,2}(\Omega)},
\end{aligned}$$

therfore l_1 is continuous.

Similar to the calculus of $a_1(.,.)$ and l_1, we can obtain that $a_2(.,.)$ and l_2 are continuous.

Moreover

$$\begin{aligned}
a_1(\psi_1,\psi_1) =& C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\psi_1(x)-\psi_1(y))(\psi_1(x)-\psi_1(y))}{|x-y|^{n+2s}}dydx\\
&= C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\psi_1(x)-\psi_1(y))^2}{|x-y|^{n+2s}}dydx\\
&= C(n,s)\|\psi_1\|^2_{D^{s,2}(\Omega)},
\end{aligned}$$

thus, $a_1(.,.)$ is coercive.

$$\begin{aligned}
a_2(\psi_2,\psi_2) =& C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\psi_2(x)-\psi_2(y))(\psi_2(x)-\psi_2(y))}{|x-y|^{n+2s}}dydx\\
&= C(n,s)\iint_{\mathbb{R}^{2n}} \frac{(\psi_2(x)-\psi_2(y))^2}{|x-y|^{n+2s}}dydx\\
&= C(n,s)\|\psi_2\|^2_{D^{s,2}(\Omega)},
\end{aligned}$$

hence, $a_2(.,.)$ is coercive.

Consequently, we may apply the Lax-Milgram theorem and we conclude that the problem (3.1) has a unique weak solution $(\varphi,\phi)\in D^{s,2}(\Omega)\times D^{s,2}(\Omega)$.

□

According to the previous lemma, we can define the following map noted A by

$$A : [0,1] \times L^2(\Omega) \times L^2(\Omega) \longrightarrow D^{s,2}(\Omega) \times D^{s,2}(\Omega)$$

$$(t,u,v) \mapsto A(t,u,v) = (\varphi, \phi),$$

where (φ, ϕ) is a weak solution to the problem (3.1).

(u,v) is a weak solution for the problem (1.1) if and only if (u,v) is a solution to the following problem

$$\begin{cases} (u,v) \in L^2(\Omega) \times L^2(\Omega), \\ (u,v) = A(1,u,v). \end{cases} \tag{3.3}$$

We will prove using the Leray-Schauder degree that the problem (3.3) has at least one solution.

4. Proof of main result

This section is devoted to prove our main result, using some lemmas which give results about the conditions of the Leray-Schauder degree method.

Firstly, we want to construct a ball which contains any possible solution, this is what gives us the following lemma.

Lemma 4.1 (Priori estimate)**.** *Under the assumption* (H_2)*, there exists* $R > 0$*,* $\forall (u,v) \in L^2(\Omega) \times L^2(\Omega)$ *such that*

$$\left\{ \begin{array}{l} A(t,u,v) = (u,v) \\ t \in [0,1], (u,v) \in L^2(\Omega) \times L^2(\Omega) \end{array} \right\} \Rightarrow \|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)} < R+1.$$

Proof. Let $A(t,u,v) = (\varphi, \phi) = (u,v)$, we obtain

$$\begin{cases} (-\Delta)^s u(x) + t g_1(x,u(x),v(x)) = t f_1(x) & \text{in } \Omega, \\ (-\Delta)^s v(x) + t g_2(x,u(x),v(x)) = t f_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{4.1}$$

Let us multiply the first and second equation in (4.1) by $u(x)$ and $v(x)$ respectively, integrate over $\mathbb{R}^n$ and obtain

$$\begin{cases} C(n,s) \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dydx &= t\int_\Omega f_1(x)u(x)dx - t\int_\Omega g_1(x,u(x),v(x))u(x)dx, \\ C(n,s) \iint_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dydx &= t\int_\Omega f_2(x)v(x)dx - t\int_\Omega g_2(x,u(x),v(x))v(x)dx. \end{cases}$$

Applying the assumption (H_2) and the Cauchy-Schwarz inequality, we have

$$\begin{cases} C(n,s) \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dydx &\leq \|f_1\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}, \\ C(n,s) \iint_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dydx &\leq \|f_2\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}, \end{cases}$$

and the proposition 2.1 implies that

$$\begin{cases} \frac{C(n,s)}{c_{emb}^2} \|u\|^2_{L^2(\Omega)} &\leq \|f_1\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}, \\ \frac{C(n,s)}{c_{emb}^2} \|v\|^2_{L^2(\Omega)} &\leq \|f_2\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}. \end{cases} \tag{4.2}$$

Adding the two inequalities of (4.2), we get

$$\|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)} \leq \frac{c_{emb}^2}{C(n,s)} \|f_1\|_{L^2(\Omega)} + \frac{c_{emb}^2}{C(n,s)} \|f_2\|_{L^2(\Omega)}.$$

If we set $R = \frac{c_{emb}^2}{C(n,s)}\|f_1\|_{L^2(\Omega)} + \frac{c_{emb}^2}{C(n,s)}\|f_2\|_{L^2(\Omega)}$, we obtain

$$\|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)} < R+1.$$

We deduce that there isn't any solution (u,v) for the equation $A(t,u,v) = (u,v)$ in the boundary of $B_{R+1} = \{(u,v) \in L^2(\Omega)\times L^2(\Omega); \|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)} < R+1\}$ and that's for all $t \in [0,1]$. □

Lemma 4.2. *Under the assumption* (H_1), $\left\{A(t,u,v), t \in [0,1], (u,v) \in \bar{B}_{R+1}\right\}$ *is relatively compact in* $L^2(\Omega)\times L^2(\Omega)$.

Proof. Let $(t_n)_{\in\mathbb{N}} \subset [0,1]$ and $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset \bar{B}_{R+1}$. So using the assumption (H_1), the sequences $\{g_1(.,u_n,v_n)\}_{n\in\mathbb{N}}$ and $\{g_2(.,u_n,v_n)\}_{n\in\mathbb{N}}$ are bounded in $L^2(\Omega)$. Firstly, setting $A(t_n,u_n,v_n) = (\varphi_n,\phi_n)$, we get

$$\left\{\begin{array}{ll} (-\Delta)^s\varphi_n(x) + t_n g_1(x,u_n(x),v_n(x)) = t_n f_1(x) & \text{in } \Omega, \\ (-\Delta)^s\phi_n(x) + t_n g_2(x,u_n(x),v_n(x)) = t_n f_2(x) & \text{in } \Omega, \\ \varphi_n = \phi_n = 0 & \text{on } \mathbb{R}^n\backslash\Omega. \end{array}\right. \tag{4.3}$$

As in the previous proof of the lemma 4.1, we multiply the first and second equation in the problem (4.3) by $\varphi_n(x)$ and $\phi_n(x)$ respectively, integrate over $\mathbb{R}^n$, apply the Cauchy-Schwarz inequality and obtain

$$\begin{cases} C(n,s)\iint_{\mathbb{R}^{2n}} \frac{|\varphi_n(x)-\varphi_n(y)|^2}{|x-y|^{n+2s}}dydx & \leq \|f_1\|_{L^2(\Omega)}\|\varphi_n\|_{L^2(\Omega)} + \|g_1(.,u_n,v_n)\|_{L^2(\Omega)}\|\varphi_n\|_{L^2(\Omega)}, \\ C(n,s)\iint_{\mathbb{R}^{2n}} \frac{|\phi_n(x)-\phi_n(y)|^2}{|x-y|^{n+2s}}dydx & \leq \|f_2\|_{L^2(\Omega)}\|\phi_n\|_{L^2(\Omega)} + \|g_2(.,u_n,v_n)\|_{L^2(\Omega)}\|\phi_n\|_{L^2(\Omega)}. \end{cases}$$

Then, using the fact that $\{g_1(.,u_n,v_n)\}_{n\in\mathbb{N}}$ and $\{g_2(.,u_n,v_n)\}_{n\in\mathbb{N}}$ are bounded in $L^2(\Omega)$ and the proposition 2.1, we obtain

$$\begin{cases} \|\varphi_n\|^2_{D^{s,2}(\Omega)} & \leq \frac{c_{emb}}{C(n,s)}[\|f_1\|_{L^2(\Omega)} + M_1]\|\varphi_n\|_{D^{s,2}(\Omega)}, \\ \|\phi_n\|^2_{D^{s,2}(\Omega)} & \leq \frac{c_{emb}}{C(n,s)}[\|f_2\|_{L^2(\Omega)} + M_2]\|\phi_n\|_{D^{s,2}(\Omega)}. \end{cases} \tag{4.4}$$

By simplification and adding the two inequalities of (4.4), we arrive at

$$\|(\varphi_n,\phi_n)\|_{D^{s,2}(\Omega)\times D^{s,2}(\Omega)} \leq \frac{c_{emb}}{C(n,s)}[\|f_1\|_{L^2(\Omega)} + M_1] + \frac{c_{emb}}{C(n,s)}[\|f_2\|_{L^2(\Omega)} + M_2].$$

Then,

$$\|(\varphi_n,\phi_n)\|_{D^{s,2}(\Omega)\times D^{s,2}(\Omega)} \leq K,$$

where $K = \frac{c_{emb}}{C(n,s)}[\|f_1\|_{L^2(\Omega)} + M_1] + \frac{c_{emb}}{C(n,s)}[\|f_2\|_{L^2(\Omega)} + M_2]$.

Therfore $\{(\varphi_n,\phi_n)\}_{n\in\mathbb{N}}$ is bounded in $D^{s,2}(\Omega)\times D^{s,2}(\Omega)$, using the compact embedding theorem (proposition 2.2), we deduce that there is a subsequence of $\{(\varphi_{n_k},\phi_{n_k})\}_{k\in\mathbb{N}}$ which converges to (φ,ϕ) strongly in $L^2(\Omega)\times L^2(\Omega)$. □

Lemma 4.3. *Under the assumption* (H_1), A *is continuous from* $[0,1]\times L^2(\Omega)\times L^2(\Omega)$ *into* $L^2(\Omega)\times L^2(\Omega)$.

Proof. Let $\{(t_n,u_n,v_n)\}_{n\in\mathbb{N}} \subset [0,1]\times L^2(\Omega)\times L^2(\Omega)$ be a sequence which converges to (t,u,v) in $[0,1]\times L^2(\Omega)\times L^2(\Omega)$ when $n \to +\infty$. We pose for all $n \in \mathbb{N}$ that $A(t_n,u_n,v_n) = (\varphi_n,\phi_n)$ and $A(t,u,v) = (\varphi,\phi)$, we obtain

$$\left\{\begin{array}{ll} (-\Delta)^s\varphi_n(x) + t_n g_1(x,u_n(x),v_n(x)) = t_n f_1(x) & \text{in } \Omega, \\ (-\Delta)^s\phi_n(x) + t_n g_2(x,u_n(x),v_n(x)) = t_n f_2(x) & \text{in } \Omega, \\ \varphi_n = \phi_n = 0 & \text{on } \mathbb{R}^n\backslash\Omega, \end{array}\right.$$

and

$$\begin{cases} (-\Delta)^s\varphi(x) + tg_1(x,u(x),v(x)) = tf_1(x) & \text{in } \Omega, \\ (-\Delta)^s\phi(x) + tg_2(x,u(x),v(x)) = tf_2(x) & \text{in } \Omega, \\ \varphi = \phi = 0 & \text{on } \mathbb{R}^n\backslash\Omega. \end{cases}$$

Making the difference between the two previous systems, we get

$$\begin{cases} (-\Delta)^s(\varphi_n(x) - \varphi(x)) = tg_1(x,u(x),v(x)) - t_ng_1(x,u_n(x),v_n(x)) + (t_n - t)f_1(x), \\ (-\Delta)^s(\phi_n(x) - \phi(x)) = tg_2(x,u(x),v(x)) - t_ng_2(x,u_n(x),v_n(x) + (t_n - t)f_2(x). \end{cases}$$

As in the proof of the lemma 4.1, we multiply the first and second equation of the previous system by $(\varphi_n(x) - \varphi(x))$ and $(\phi_n(x) - \phi(x))$, integrate over $\mathbb{R}^n$, apply the Cauchy-Schwarz inequality and the proposition 2.1 and obtain

$$\begin{cases} \|\varphi_n(x) - \varphi(x)\|_{L^2(\Omega)} & \leq \frac{c^2_{emb}}{C(n,s)}[\|tg_1(x,u(x),v(x)) - t_ng_1(x,u_n(x),v_n(x))\|_{L^2(\Omega)} + |t_n - t|\|f_1\|_{L^2(\Omega)}], \\ \|\phi_n(x) - \phi(x)\|_{L^2(\Omega)} & \leq \frac{c^2_{emb}}{C(n,s)}[\|tg_2(x,u(x),v(x)) - t_ng_2(x,u_n(x),v_n(x))\|_{L^2(\Omega)} + |t_n - t|\|f_2\|_{L^2(\Omega)}]. \end{cases}$$

We have that $(t_n)_{n\in\mathbb{N}}$ converges to t and $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ converges to (u,v) in $L^2(\Omega)\times L^2(\Omega)$ when $n \to +\infty$ and from the assumption (H_1), we deduce that $\{g_1(.,u_n,v_n)\}_{n\in\mathbb{N}}, \{g_2(.,u_n,v_n)\}_{n\in\mathbb{N}}$ converge respectively to $g_1(.,u,v), g_2(.,u,v)$. Therfore $\{(\varphi_n,\phi_n)\}_{n\in\mathbb{N}}$ converges to (φ,ϕ) in $L^2(\Omega)\times L^2(\Omega)$. We conclude that A is continuous from $[0,1]\times L^2(\Omega)\times L^2(\Omega)$ into $L^2(\Omega)\times L^2(\Omega)$. □

Proof of the main result. We have according to the lemma 4.1 that there is no solution of the equation $Id(u,v) - A(t,u,v) = 0$ in the boundary of the ball B_{R+1} and the lemma 4.2 gives us that $\left\{A(t,u,v), t\in[0,1], (u,v)\in \bar{B}_{R+1}\right\}$ is relatively compact in $L^2(\Omega)\times L^2(\Omega)$. Furthermore, according to the lemma 4.3, the map A is continuous from $[0,1]\times L^2(\Omega)\times L^2(\Omega)$ into $L^2(\Omega)\times L^2(\Omega)$. Consequently, we can define the degree $d(Id - A(t,\cdot,\cdot), B_{R+1}, 0)$ and with the homotopy invariance property we have

$$d(Id - A(t,\cdot,\cdot), B_{R+1}, 0) = d(Id - A(0,\cdot,\cdot), B_{R+1}, 0) = d(Id, B_{R+1}, 0) = 1 \neq 0,$$

for all $t \in [0,1]$.

Therfore, there exists $(u,v) \in B_{R+1}$ such that

$$Id(u,v) - A(1,u,v) = 0,$$

which is equivalent to

$$A(1,u,v) = (u,v),$$

and proves that the problem (1.1) has at least one weak solution $(u,v)\in D^{s,2}(\Omega)\times D^{s,2}(\Omega)$. □

Conclusion

We can say that the theory of topological degrees is an efficient tool to solve nonlinear systems, not only elliptic, but also fractional. We hope in a future work to solve other similar problems in spaces with variable exponents under adapted conditions (growth, monotony, coercivity, ...) on the p-Laplacian fractional operator.

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