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SOME FIXED POINTS OF $\alpha-\psi-K$-CONTRACTIVE MAPPINGS IN PARTIAL METRIC SPACES

MUHAMMAD ARSHAD, SHAIF ALSHORAIFY AND ABDULLAH SHOAIB

Abstract. In this paper, we study the existence and uniqueness of fixed points for new class of contractive mapping involving rational expressions in complete partial metric space. We obtain fixed point results on partial metric spaces endowed with a partial order and that for cyclic contractive mappings. We give examples to illustrate the usability of our results. Our results extended and generalized some well known results in the literature.

1. Introduction

The notion of partial metric is one of the most useful and interesting generalizations of the classical concept of metric. Matthews [13] introduced the concept of a partial metric space and obtained a Banach-type fixed point theorem on complete partial metric spaces. [2]. Infect, a partial metric space is a generalization of usual metric spaces in which $d\,(a,a)$ are no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (see, [1, 6, 7, 4, 5, 16, 17, 18, 19]). Samet et al. [15] introduced the notion of $(\alpha-\psi)$ contractive mapping and established some fixed point results in the setting of complete metric space (see also, [8, 20, 21]). Very recently, Karapinar [11] introduced the existence and uniqueness of fixed points for a new class of contractive mappings involving rational expressions in metric space. In this paper, we introduce the notion of $\alpha-\psi-K$-contractive mappings and give fixed point results for this class in the setting of partial metric spaces. Also, we deduce fixed point results in ordered partial metric spaces and in the context of cyclic mapping beside the partial metric space by using the auxiliary function $\alpha\,(a,y)$. Our results extend and generalize theorems of Karpinar et al [11]. Throughout this paper $\mathbb{N}$ and R^+ denote the set of all natural numbers and the set of all non-negative real numbers, respectively. We begin with a brief recollection of basic notions that will be useful in the paper.

Definition 1.1.[13] Let X be a non empty set and $p: X\times X\to[0,\infty)$, if the following conditions hold:

(pm1) $a=y\Leftrightarrow p(a,a)=p(a,y)=p(y,y)$;

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(pm2) $p(a,a) \leq p(a,y)$;

(pm3) $p(a,y) = p(y,a)$;

(pm4) $p(a,y) \leq p(a,z) + p(z,y) - p(z,z)$, for all $a, y, z \in X$. Then the pair (X,p) is called a partial metric space and p is called a partial metric on X.

If p is a partial on X, then the function $p^s : X \times X \to R^+$ defined by $p^s(a,y) = 2p(a,y) - p(a,a) - p(y,y)$ satisfies the conditions of a metric space on X and hence it is a usual metric on X.

Definition 1.2. [13] Let (X,p) be a partial metric space:

(i) A sequence $\{a_n\}$ is converges to $a \in X$ if and only if $p(a,a) = \lim_{n\to\infty} p(a,a_n)$.

(ii) A sequence $\{a_n\}$ in X is a Cauchy sequence whenever $\lim_{n,m\to\infty} p(a_n,a_m)$ exists and finite.

(iii) A partial metric space (X,p) is said to be a complete if every Cauchy sequence $\{a_n\}$ in X is convergent with respect to τ_p.

Lemma 1.3. [1] A sequence $\{a_n\}$ in a partial metric space (X,p) is:

(i) A Cauchy sequence if and only if it is a Cauchy sequence in a metric space (X,p^s).

(ii) A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Moreover,

$$\lim_{n\to\infty} p^s(a,a_n) = 0 \text{ if and only if } \lim_{n\to\infty} p(a,a_n) = \lim_{n,m\to\infty} p(a_n,a_m) = p(a,a).$$

Lemma 1.4. [1] Let (X,p) be a partial metric space and $T : X \to X$ be a given mapping. T is said to be continuous at $a_0 \in X$ if it is sequentially continuous at a_0, that is, if and only if

$$\forall \{a_n\} \subset X : \lim_{n\to+\infty} a_n = a_0 \Rightarrow \lim_{n\to+\infty} Ta_n = Ta_0.$$

Lemma 1.5. [1]. Assume that $a_n \to z$ as $n \to \infty$ in a partial metric space (X,p) such that $p(z,z) = 0$. Then

$$\lim_{n\to\infty} p(a_n,y) = p(z,y), \text{ for all } y \in X.$$

Definition 1.6.[15]. Let $T : X \to X$. The mapping T is said to be an α-admissible, if

$$\text{for all } a,y \in X,\ \alpha(a,y) \geq 1 \text{ implies } \alpha(Ta,Ty) \geq 1.$$

Definition 1.7. [11] Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is $\alpha - \psi - K$-contractive if there exists two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that for all $a,y \in X$, $a \neq y$, we have

$$\alpha(a,y)d(T(a),T(y)) \leq \psi(K(a,y))$$

where,

$$K(a,y) = \max\left\{d(a,y), \frac{d(a,Ta)+d(y,Ty)}{2}, \frac{d(a,Ty)+d(y,Ta)}{2}, \right.$$
$$\left. \frac{d(a,Ta)\,d(y,Ty)}{d(a,y)}, \frac{d(y,Ty)\,[1+d(a,Ta)]}{[1+d(a,y)]}\right\}.$$

Theorem 1.8. [11] Let (X,d) be a complete metric space. Suppose that $T:X\to X$ is $\alpha-\psi-K$-contractive satisfies the following conditions:
(i) T is α-admissible.
(ii) there exists $a_0\in X$ such that $\alpha(a_0,Ta_0)\geq 1$.
(iii) T is a continuous.
Then $u\in X$ such that $T(u)=u$.

Let Ψ denoted the family of nondecreasing functions $\psi:[0,+\infty)\to[0,+\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t)<+\infty$ for all $t>0$, where ψ^n is the n-th iterate of ψ.

2. Main results

In this section, we are starting to introduce the generalized of $\alpha-\psi-K$-contractive in the context of partial metric spaces as follows:

Definition 2.1. Let (X,p) be a partial metric space and $T:X\to X$ be a given mapping. We say that T is $\alpha-\psi-K$-contractive if there exists two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that for all $a,y\in X$, $a\neq y$, we have

$$\alpha(a,y)p(T(a),T(y))\leq\psi(K(a,y)) \tag{2.1}$$

whenever,

$$K(a,y)=\max\left\{p(a,y),\frac{p(a,Ta)+p(y,Ty)}{2},\frac{p(a,Ty)+p(y,Ta)}{2},\right.$$
$$\left.\frac{p(a,Ta)\,p(y,Ty)}{p(a,y)},\frac{p(y,Ty)\left[1+p(a,Ta)\right]}{\left[1+p(a,y)\right]}\right\}.$$

Theorem 2.2. Let (X,p) be a complete partial metric space and $T:X\to X$ is $\alpha-\psi-K$-contractive mapping satisfies the following conditions:
(i) T is α-admissible.
(ii) there exists $a_0\in X$ such that $\alpha(a_0,Ta_0)\geq 1$.
(iii) T is a continuous. Then $u\in X$ such that $T(u)=u$.

Proof. From condition (ii), there exists $a_0\in X$ such that $\alpha(a_0,Ta_0)\geq 1$. We construct a sequence $\{a_n\}$ in X as follows: $a_{n+1}=Ta_n$, for all $n\in\mathbb{N}\cup\{0\}$. If $a_0=a_{0+1}$ for some $n\in\mathbb{N}_0$, then $u=a_{n_0}$ is a fixed point of T. Assume that $a_n\neq a_{n+1}$ for all $n\in\mathbb{N}\cup\{0\}$. By (i), we have

$$\alpha(a_0,a_1)=\alpha(a_0,Ta_0)\geq 1\Rightarrow\alpha(Ta_0,Ta_1)=\alpha(a_1,a_2)\geq 1,$$
$$\alpha(a_1,a_2)=\alpha(a_1,Ta_1)\geq 1\Rightarrow\alpha(Ta_1,Ta_2)=\alpha(a_2,a_3)\geq 1.$$

By induction, we get

$$\alpha(a_n,a_{n+1})\geq 1. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned}p(a_{n+1},a_{n+2}) &= p(Ta_n,Ta_{n+1})\\ &\leq \alpha(a_n,a_{n+1})\,p(Ta_n,Ta_{n+1})\leq\psi(K(a_n,a_{n+1}))\end{aligned}$$
$$\leq\psi\max\left\{p(a_n,a_{n+1}),\frac{p(a_n,Ta_n)+p(a_{n+1},Ta_{n+1})}{2},\right.$$
$$\frac{p(a_n,Ta_{n+1})+p(a_{n+1},Ta_n)}{2},\frac{p(a_n,Ta_n)\,p(a_{n+1},Ta_{n+1})}{p(a_n,a_{n+1})},$$

$$\left.\frac{p\left(a_{n+1},Ta_{n+1}\right)\left[1+p\left(a_n,Ta_n\right)\right]}{\left[1+p\left(a_n,a_{n+1}\right)\right]}\right\}$$

$$=\psi\max\left\{p\left(a_n,a_{n+1}\right),\frac{p\left(a_n,a_{n+1}\right)+p\left(a_{n+1},a_{n+2}\right)}{2},\frac{p\left(a_n,a_{n+2}\right)+p\left(a_{n+1},a_{n+1}\right)}{2},\right.$$

$$\left.\frac{p\left(a_n,a_{n+1}\right)p\left(a_{n+1},a_{n+2}\right)}{p\left(a_n,a_{n+1}\right)},\frac{p\left(a_{n+1},a_{n+2}\right)\left[1+p\left(a_n,a_{n+1}\right)\right]}{\left[1+p\left(a_n,a_{n+1}\right)\right]}\right\}$$

$$\leq\psi\max\left\{p\left(a_n,a_{n+1}\right),p\left(a_{n+1},a_{n+2}\right)\right\},$$

$$p\left(a_{n+1},a_{n+2}\right)\leq\psi\left(\max\left\{p\left(a_n,a_{n+1}\right),p\left(a_{n+1},a_{n+2}\right)\right\}\right),\text{ for all }n.$$

If $\max\left\{p\left(a_n,a_{n+1}\right),p\left(a_{n+1},a_{n+2}\right)\right\}=p\left(a_{n+1},a_{n+2}\right)$, then

$$p\left(a_{n+1},a_{n+2}\right)\leq\psi\left(p\left(a_{n+1},a_{n+2}\right)\right)<p\left(a_{n+1},a_{n+2}\right),$$

which is contradiction. Thus, $\max\left\{p\left(a_n,a_{n+1}\right),p\left(a_{n+1},a_{n+2}\right)\right\}=p\left(a_n,a_{n+1}\right)$, for all $n\in\mathbb{N}$. Hence,

$$p\left(a_{n+1},a_{n+2}\right)\leq\psi\left(p\left(a_n,a_{n+1}\right)\right). \tag{2.3}$$

Continuing in this process inductively, we obtain

$$p\left(a_n,a_{n+1}\right)\leq\psi^n\left(p\left(a_0,a_1\right)\right),\text{ for all }n\in\mathbb{N}. \tag{2.4}$$

Using the definition of partial metric, we have

$$\max\left\{p\left(a_n,a_n\right),p\left(a_{n+1},a_{n+1}\right)\right\}\leq p\left(a_n,a_{n+1}\right). \tag{2.5}$$

Using inequality (2.4), we have

$$\max\left\{p\left(a_n,a_n\right),p\left(a_{n+1},a_{n+1}\right)\right\}\leq\psi^n\left(a_0,a_1\right). \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned}p^s\left(a_n,a_{n+1}\right)&=2p\left(a_n,a_{n+1}\right)-p\left(a_n,a_n\right)-p\left(a_{n+1},a_{n+1}\right)\\&\leq 2p\left(a_n,a_{n+1}\right)+p\left(a_n,a_n\right)+p\left(a_{n+1},a_{n+1}\right)\end{aligned}$$

$$p^s\left(a_n,a_{n+1}\right)\leq 4\psi^n\left(a_0,a_1\right). \tag{2.7}$$

Using inequality (2.7), we have

$$\begin{aligned}p^s\left(a_{n+k},a_n\right)&\leq p^s\left(a_{n+k},a_{n+k-1}\right)+...+p^s\left(a_{n+1},a_n\right)\\&\leq 4\psi^{n+k-1}p\left(a_0,a_1\right)+...+4\psi^n p\left(a_0,a_1\right)\\&\leq 4\sum_{i=n}^{n+k-1}\psi^i p\left(a_0,a_1\right).\end{aligned}$$

As $\sum_{i=0}^{\infty}\psi^i p\left(a_0,a_1\right)$ is a convergent. We obtain that $\{a_n\}$ is a Cauchy sequence in a metric space (X,p^s). Now, by Lemma 1.3 and the completeness of (X,p), we conclude the completeness of (X,p^s). Therefore the sequence $\{a_n\}$ is a convergent in (X,p^s), say $\lim_{n\to\infty}p^s\left(a_n,u\right)=0$. A gain by Lemma 1.3, we have

$$p\left(u,u\right)=\lim_{n\to\infty}p\left(a_n,u\right)=\lim_{m,n\to\infty}p\left(a_m,a_n\right). \tag{2.8}$$

Now, since $\{a_n\}$ is a Cauchy sequence in the metric space (X,p^s), we have

$$\lim_{m,n\to\infty}p^s\left(a_m,a_n\right)=0. \tag{2.9}$$

View of (2.6) ,we have

$$\lim_{n\to\infty}p\left(a_n,a_n\right)=0. \tag{2.10}$$

From (2.9), (2.10) and definition of p^s, we conclude that

$$\lim_{m,n\to\infty} p\left(a_m, a_n\right) = 0.$$

On using (2.8), we have

$$p\left(u,u\right) = \lim_{n\to\infty} p\left(a_n, u\right) = \lim_{m,n\to\infty} p\left(a_m, a_n\right) = 0. \tag{2.11}$$

Now, we show that $Tu = u$. By Lemma 1.4, we have

$$p\left(Tu,Tu\right) = \lim_{n\to\infty} p\left(Ta_n, Tu\right) = \lim_{m,n\to\infty} p\left(Ta_m, Ta_n\right), \tag{2.12}$$

that is,

$$p\left(Tu,Tu\right) = \lim_{m,n\to\infty} p\left(a_{m+1}, a_{n+1}\right) = 0. \tag{2.13}$$

By (2.11) and (2.13) , we have

$$p\left(u,u\right) = p\left(Tu,Tu\right) = 0. \tag{2.14}$$

By Lemma 1.5, we have

$$\lim_{n\to\infty} p\left(a_n, Tu\right) = p\left(u, Tu\right). \tag{2.15}$$

Therefore, using (2.12) , (2.14) and (2.15) , we have

$$p\left(Tu,Tu\right) = p\left(u,u\right) = p\left(u,Tu\right) = 0.$$

Thus $Tu = u$. Hence u is a fixed point of T.

Theorem 2.3. Let (X, p) be a complete partial metric space and let $T : X \to X$ be an $\alpha - \psi - K$-contractive mapping. Assume that:

(i) T is α-admissible.

(ii) there exists $a_0 \in X$ such that $\alpha\left(a_0, \mathrm{T}\left(a_0\right)\right) \geq 1$.

(iii) If $\{a_n\}$ is a sequence in X such that $\alpha\left(a_n, a_{n+1}\right) \geq 1$, for all $n \in \mathbb{N}$ and $a_n \to a \in X$ as $n \to \infty$, then there exists a subsequence $\left\{a_{n_{(k)}}\right\}$ such that $\alpha\left(a_{n_{(k)}}, a\right) \geq 1$, for all $k \in \mathbb{N}$. Then T has a fixed point in X.

*Proof***.** Following the proof of Theorem 2.2, we know that the sequence $\{a_n\}$ given by $a_{n+1} = Ta_n$ for all $n \geq 0$, converges to some $u \in X$. From (2.2) and given hypotheses there exists a subsequence $\left\{a_{n_{(k)}}\right\}$ of $\{a_n\}$ such that

$$\alpha\left(a_{n_{(k)}}, a\right) \geq 1, \text{ for all } k. \tag{2.16}$$

Now, we proceed to show that u is a fixed point of T. Suppose the contrary $p(u, Tu) > 0$. Therefore, from (2.1) and (2.16), we have

$$\begin{aligned} p(u,Tu) &\leq p(u, a_{n_{(k)}+1}) + p(a_{n_{(k)}+1}, Tu) - p(a_{n_{(k)}+1}, a_{n_{(k)}+1}) \\ &\leq p(u, a_{n_{(k)}+1}) + p(a_{n_{(k)}+1}, Tu) \\ &= p(u, a_{n_{(k)}+1}) + p(Ta_{n_{(k)}}, Tu) \\ &\leq p(u, a_{n_{(k)}+1}) + \alpha(a_{n_{(k)}}, u)p(Ta_{n_{(k)}}, Tu) \\ &\leq p(u, a_{n_{(k)}+1}) + \psi\left(K(a_{n(k)}, u)\right), \end{aligned}$$

$$p(u,Tu) \leq p(u, a_{n_{(k)}+1}) + \psi \max\left\{p\left(a_{n_{(k)}}, u\right),\right.$$
$$\frac{p\left(a_{n_{(k)}}, a_{n_{(k)}+1}\right) + p(u,Tu)}{2}, \frac{p\left(a_{n_{(k)}}, Tu\right) + p\left(u, a_{n_{(k)}+1}\right)}{2},$$

$$\left.\frac{p\left(a_{n_{(k)}}, a_{n_{(k)}+1}\right) p\left(u, Tu\right)}{p\left(a_{n_{(k)}}, u\right)}, \frac{p\left(u, Tu\right)\left[1+p\left(a_{n_{(k)}}, a_{n_{(k)}+1}\right)\right]}{\left[1+p\left(a_{n_{(k)}}, u\right)\right]}\right\}. \tag{2.17}$$

Letting $k \to \infty$ in (2.17), we get

$$p(u, Tu) \leq \psi\left(p(u, Tu)\right) < p(u, Tu).$$

Which is a contradiction. Therefore, $p(u, Tu) = 0$ and $Tu = u$.

Corollary 2.4.[11] Let (X, d) be a complete metric space and $T : X \to X$ be an $\alpha-\psi-K$-contractive mapping. Assume that:

(i) T is α-admissible.

(ii) there exists $a_0 \in X$ such that $\alpha\left(a_0, T\left(a_0\right)\right) \geq 1$.

(iii) T is continuous. Then T has a fixed point in X.

Corollary 2.5 [16] Let (X, p) be a complete partial metric space, $\alpha : X \times X \to [0, \infty)$ be an a function $\alpha \in \Psi$ and T be generalized $\alpha-\psi$ contractive type mapping on X. Assume that:

(i) T is α-admissible and continuous.

(ii) there exists $a_0 \in X$ such that $\alpha\left(a_0, T\left(a_0\right)\right) \geq 1$.

Then, there exists $u \in X$ such that $T(u) = u$.

Corollary 2.6 [10] Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Assume that there exists $\psi \in \Psi$ such that

$$d(T(a), T(y)) \leq \psi(M(a, y)),$$

for all $(a, y) \in X \times X$, where

$$M(a, y) = \max\left\{d\left(a, y\right), \frac{d\left(a, Ta\right)+d\left(y, Ty\right)}{2}, \frac{d\left(a, Ty\right)+d\left(y, Ta\right)}{2}\right\}.$$

Then T has a unique fixed point.

Example 2.7. Let $X = R^{+}$ where, (X, p) is a complete partial metric space with partial metric p given by $p(a, y) = \max\{a, y\}$. Defined $T : X \to X$ by $T(a) = \begin{cases} \frac{a}{6}, \text{ if } a \in [0,1], \\ \frac{a}{3}-\frac{1}{6}, \text{ if } a \in (1,2], \\ a-\frac{3}{2}, \ a > 2. \end{cases}$. Let $\alpha\left(a, y\right) = \begin{cases} 2, \text{ if } a, y \in [0,1], \\ \frac{4}{3}, \text{ if } a, y \in (1,2], \\ 0, \text{ otherwise.} \end{cases}$

Clearly $T(a)$ is a continuous for all $a \in X$. To show that T is α-admissible: Let $a, y \in X$ such that $\alpha\left(a, y\right) \geq 1$, by definition of α we have $a, y \in [0,1]$ implies $\alpha\left(Ta, Ty\right) = \alpha\left(a, y\right) \geq 1$. Similarly in the case $a, y \in (1,2]$. Now, the cases arrises:

(i) In the case $a, y \in [0,1]$, we have

$$\begin{aligned} \alpha\left(a, y\right) p\left(Ta, Ty\right) &= \alpha\left(a, y\right) p\left(\frac{a}{6}, \frac{y}{6}\right) = 2\max\{\frac{a}{6}, \frac{y}{6}\} \leq \frac{1}{3}p(a, y) \\ &\leq \psi(K(a, y)). \end{aligned}$$

(ii) Also, holds in the case $a, y \in (1,2]$.

(iii) In the case $a \in [0,1]$, $y \in (1,2]$, we have

$$\alpha\left(a, y\right) p\left(Ta, Ty\right) = \alpha\left(a, y\right) p\left(\frac{a}{6}, \frac{y}{3}-\frac{1}{6}\right) = 0.$$

(iv) For $y \in [0,1]$, $a \in (1,2]$, we have

$$\alpha\left(a, y\right) p\left(Ta, Ty\right) = \alpha\left(a, y\right) p\left(\frac{a}{3}-\frac{1}{6}, \frac{y}{6}\right) = 0.$$

(v) If a or y is not in $[0,2]$, then $\alpha(a,y)=0$, that is

$$\alpha(a,y)\,p(Ta,Ty)\leq\psi(K(a,y)).$$

Thus, T is $\alpha-\psi-K$-contractive mapping with $\psi(t)=\frac{t}{3}$, for all $t\geq 0$. Indeed, for $a_0=1$, we have

$$\alpha(a_0,T(a_0))=\alpha(1,T1)=\alpha\left(1,\frac{1}{6}\right)=2.$$

Now, all the hypotheses of Theorem 2.2 are satisfied. Consequently, T has a fixed point. In fact, 0 is a fixed point.

We can not apply Theorem 1.8 because, $\frac{2}{9}\nleq\frac{1}{6}$. Indeed, let $a=\frac{3}{2}$, $y=2$, we have $\alpha\left(\frac{3}{2},2\right)p^s\left(\frac{1}{2}-\frac{1}{6},\frac{2}{3}-\frac{1}{6}\right)=\frac{4}{3}p^s\left(\frac{1}{3},\frac{1}{2}\right)=\frac{4}{3}\left[2p\left(\frac{1}{3},\frac{1}{2}\right)-p\left(\frac{1}{3},\frac{1}{3}\right)-p\left(\frac{1}{2},\frac{1}{2}\right)\right]=\frac{2}{9}$ and $p^s\left(\frac{3}{2},2\right)=2p\left(\frac{3}{2},2\right)-p\left(\frac{3}{2},\frac{3}{2}\right)-p(2,2)=\frac{1}{2}$, $\psi\left(\frac{1}{2}\right)=\frac{1}{6}$.

Example 2.8. Let $X=\{1,2,3,4\}$ and the function $p:X\times X\to[0,\infty)$ defined as

$p(1,2)=p(3,4)=3$, $p(1,3)=p(2,4)=5$, $p(1,4)=p(2,3)=4$, $p(a,y)=p(y,a)$,

$p(1,1)=0$, $p(2,2)=p(3,3)=p(4,4)=2$. Define a self mapping T as: $T(1)=4$, $T(3)=3$, $T(2)=4$, $T(4)=3$, for all $a\in X$. Obviously p is a partial metric on X, but not metric (since $p(a,a)\neq 0$, for $a\in\{2,3,4\}$). Clearly T is an $\alpha-\psi-K$-contractive mapping with $\psi(t)=\frac{4}{5}t$, $t\geq 0$. In fact, for all $a,y\in X$, we have

$$\alpha(a,y)p(T(a),T(y))\leq\psi(K(a,y)),\ \alpha(a,y)=\begin{cases}1, \text{ if } (a,y)\neq(3,3),\\ \frac{1}{4}, \text{ if } (a,y)=(3,3).\end{cases}$$

Moreover, there exists $a_0\in X$ such that $\alpha(a_0,Ta_0)=1$. In fact, for $a_0=1$, we have

$$\alpha(1,T1)=\alpha(1,4)=1.$$

Let $\{a_n\}$ be a sequence in X such that $\alpha(a_n,a_{n+1})\geq 1$, for all $n\in\mathbb{N}$ and $a_n\to a\in X$, as $n\to\infty$, for some $a\in X$. From the definition of α, for all n, we have $a_n\neq 0$. Thus, $a\neq 0$ and we have $\alpha(a_n,a)\geq 1$ for all n. Also, T is α-admissible. For this, we have

$$\alpha(a,y)\geq 1\Rightarrow a\neq 3,\ y\neq 3\Rightarrow Ta\neq 3,\ Ty\neq 3\Rightarrow\alpha(Ta,Ty)\geq 1.$$

Consequently, T has a fixed point. In this case, 3 is a fixed point.

We denote by $\mathrm{fix}(T)$ the set of fixed points of T, for the uniqueness, we need the follows additional condition:

(H) For all $a,y\in\mathrm{fix}(T)$, there exists $z\in X$ such that $\alpha(z,a)\geq 1$ and $\alpha(z,y)\geq 1$.

Theorem 2.9. Adding the condition (H) to the hypotheses of Theorem 2.3, one obtains that u is the unique fixed point of T.

Proof. Suppose that v is another fixed point of T. From (H), there exists $z\in X$ such that

$$\alpha(z,u)\geq 1,\ \alpha(z,v)\geq 1. \tag{2.18}$$

Since T is α-admissible, from (2.18), we have

$$\alpha(T^n(z),u)\geq 1,\ \alpha(T^n(z),v)\geq 1, \tag{2.19}$$

for all $n\geq 0$. Define the sequence $\{z_n\}$ in X by $z_{n+1}=T(z_n)$ for all $n\geq 0$ and $z_0=z$ and assume that $d(z_n,u)>0$. From (2.19), for all n, we get

$$p(z_{n+1},u)=p(T(z_n),T(u))\leq\alpha(z_n,u)p(T(z_n),T(u))\leq\psi(K(z_n,u)). \tag{2.20}$$

On the other hand, we infer that

$$\begin{aligned}K(z_n,u) &= \max\left\{p(z_n,u),\frac{p(z_n,z_{n+1})}{2},\frac{p(z_n,u)+p(u,z_{n+1})}{2}\right\}\\ &= \max\left\{p(z_n,u),\frac{p(z_n,u)+p(u,z_{n+1})}{2}\right\}\\ &\le \max\left\{p(z_n,u),p\left(u,z_{n+1}\right)\right\}.\end{aligned}$$

Using (2.20) and the fact that ψ is non decreasing, we have

$$p(z_{n+1},u)\le\psi\max\left\{p(z_n,u),p\left(u,z_{n+1}\right)\right\}, \text{ for all } n. \tag{2.21}$$

If $\max\left\{p(z_n,u),p\left(u,z_{n+1}\right)\right\}=p\left(u,z_{n+1}\right)$, then from (2.21) we obtain that

$$p(z_{n+1},u)\le\psi\left(p(z_{n+1},u)\right)<p(z_{n+1},u),$$

which is contradiction. Thus, we have

$$\max\left\{p(z_n,u),p\left(u,z_{n+1}\right)\right\}=p(z_n,u).$$

Also,

$$p(z_{n+1},u)\le\psi\left(p(z_n,u)\right), \text{ for all } n.$$

This implies that

$$\left(p(z_n,u)\right)\le\psi^n\left(p\left(z_0,u\right)\right),$$

for all $n\ge 0$. Letting $n\to\infty$ in the a above equality, we conclude that

$$\lim_{n\to\infty}p\left(z_n,u\right)=0. \tag{2.22}$$

Similarly, one can show that

$$\lim_{n\to\infty}p\left(z_n,v\right)=0. \tag{2.23}$$

From (2.22) and (2.23), we deduce that $u=v$.

3. Fixed points partial metric spaces endowed with a partial order

Before presenting our results, we collect relevant concepts which will be needed in the proof of our results.

Definition 3.1. Let $(X,\preceq)$ be a partially ordered set and $T:X\to X$ be a given mapping. We say that T is non decreasing with respect to $\preceq$ if

$$a,y\in X,\ a\preceq y\Rightarrow Ta\preceq Ty.$$

Definition 3.2. [10] Let $(X,\preceq)$ be a partially ordered set and p be a partial metric on X. We say that $(X,\preceq,p)$ is regular if for every nondecreasing sequence $\{a_n\}\subset X$ such that $a_n\to a$ as $n\to\infty$, there exists a subsequence $\left\{a_{n(k)}\right\}$ of $\{a_n\}$ such that $a_{n(k)}\preceq a$ for all k.

Now, we have the following result.

Corollary 3.3. Let $(X,\preceq)$ be a partially ordered set and p be a partial metric on X such that (X,p) is complete. Let $T:X\to X$ be a nondecreasing mapping with respect to $\preceq$ and satisfies the following inequality

$$p(T(a),T(y))\le\psi(K(a,y)),\ \text{for all } a,y\in X \text{ with } a\succeq y,$$

where

$$K\left(a,y\right)=\max\left\{p\left(a,y\right),\frac{p\left(a,Ta\right)+p\left(y,Ty\right)}{2},\frac{p\left(a,Ty\right)+p\left(y,Ta\right)}{2},\right.$$

$$\left.\frac{p\left(a,Ta\right)p\left(y,Ty\right)}{p\left(a,y\right)},\frac{p\left(y,Ty\right)\left[1+p\left(a,Ta\right)\right]}{\left[1+p\left(a,y\right)\right]}\right\}.$$

Suppose also that the following conditions hold:

(i) there exists $a_0 \in X$ such that $a_0 \preceq Ta_0$.

(ii) T is continuous or $(X, \preceq, p)$ is regular. Then T has a fixed point. Moreover if for $a, y \in X$, there exists $z \in X$ such that $a \preceq z$ and $y \preceq z$, we have the uniqueness of the fixed point.

Proof. The proof follows easily from the following. Define a mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha\left(a,y\right)=\left\{\begin{array}{l}1,\text{ if } a\preceq y \text{ or } a\succeq y,\\ 0,\ \text{otherwise.}\end{array}\right.$$

Corollary 3.4. Let $(X, \preceq)$ be a partially ordered set and p be a partial metric on X such that (X, p) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1)$ such that

$$p(T(a),T(y))\leq\lambda\max\left\{p\left(a,y\right),\frac{p\left(a,Ta\right)+p\left(y,Ty\right)}{2},\frac{p\left(a,Ty\right)+p\left(y,Ta\right)}{2}\right\},$$

for all $a, y \in X$ with $a \succeq y$. Suppose also that the following conditions hold:

(i) there exists $a_0 \in X$ such that $a_0 \preceq Ta_0$.

(ii) T is continuous or $(X, \preceq, p)$ is regular. Then T has a fixed point.

Corollary 3.5. Let $(X, \preceq)$ be a partially ordered set and p be a partial metric on X such that (X, p) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in \left(0, \frac{1}{2}\right)$ such that

$$p(T(a),T(y))\leq\lambda[p\left(a,Ta\right)+p\left(y,Ty\right)],$$

for all $a, y \in X$ with $a \succeq y$. Suppose also that the following conditions hold:

(i) there exists $a_0 \in X$ such that $a_0 \preceq Ta_0$.

(ii) T is continuous or $(X, \preceq, p)$ is regular. Then T has a fixed point.

4. Fixed point for cyclic contractive mappings

As a generalization of the Banach contraction principle, Kirk-Srinivasan-Veeramani [12] developed the cyclic contraction. A contraction $T : A \cup B \to A \cup B$ on nonempty sets A, B is called cyclic if $T(A) \subset B$ and $T(B) \subset A$ hold for closed subsets A, B of a complete metric space X. In the last decade, several authors have used the cyclic representations and cyclic contractions to obtain various fixed point results (see, [9, 14]). In this section, we will show that, from our Theorem 2.2 and 2.3, we can deduce some fixed point theorems for cyclic contractive mappings. Now, we have the following result.

Corollary 4.1. Let $\{A_i\}$ be nonempty closed subsets of a complete partial metric space (X, p) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:

(i) $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;

(ii) there exists a function $\psi \in \Psi$ such that

$$p(T(a),T(y))\leq\psi(K(a,y)),\text{ for all } (a,y)\in A_1\times A_2.$$

Then T has a fixed point that belongs to $A_1 \cap A_2$.

Proof. Due to the fact that A_1 and A_2 are closed subsets of the complete metric space (X,d), we get completeness of the space (Y,d). Let us define the mapping $\alpha : X\times X\to[0,\infty)$ by

$$\alpha(a,y)=\begin{cases}1, \text{ if } (a,y)\in(A_1\times A_2)\cup(A_2\times A_1),\\ 0, \text{ otherwise.}\end{cases}$$

Notice that in view of definition α and condition (ii), we infer that

$$\alpha(a,y)\,p(Ta,Ty)\leq\psi(K(a,y)), \text{ for all } a,y\in X.$$

Thus T is $\alpha-\psi-K$-contractive mapping. Now, we proceed to show that T is α-admissible. For this, let $(a,y)\in Y\times Y$ such that $\alpha(a,y)\geq 1$. If $(a,y)\in A_1\times A_2$, then from (i), we have $(Ta,Ty)\in A_2\times A_1$, thereby implies $\alpha(Ta,Ty)\geq 1$. Again from (i), we obtain that $(a,y)\in A_2\times A_1$ implies that $(Ta,Ty)\in A_1\times A_2$, which further implies that $\alpha(Ta,Ty)\geq 1$. Thus, we have $\alpha(Ta,Ty)\geq 1$, in all the cases. Therefore, we obtain that T is α- admissible. Also, in view of (i), for any $u\in A_1$, we have $(u,Tu)\in A_1\times A_2$, which suggest that $\alpha(u,Tu)\geq 1$.

Now, we consider that $\{a_n\}$ be a sequence in X such that $\alpha(a_n,a_{n+1})\geq 1$ for all n and $a_n\to a\in X$ as $n\to\infty$. This suggest from the definition of α that

$$(a_n,a_{n+1})\in(A_1\times A_2)\cup(A_2\times A_1), \text{ for all } n.$$

Since $(A_1\times A_2)\cup(A_2\times A_1)$ is a closed set with respect to the Euclidean metric, we obtain that $(a,a)\in(A_1\times A_2)\cup(A_2\times A_1)$, which refer that $a\in A_1\cap A_2$. Consequently, we get from the definition of α that $\alpha(a_n,a)\geq 1$, for all n. Therefore, all the hypotheses of Theorem 2.4 are satisfied and we conclude that T has a fixed point that belongs to $A_1\cap A_2$(from (i)).

The following results are immediate consequences of Corollary 4.1.

Corollary 4.2. Let $T:Y\to Y$ be a given mapping and $\{A_i\}_{i=1}^2$ be non empty closed subsets of a complete partial metric space (X,p), where $Y=A_1\cup A_2$. Suppose that the following conditions hold:

(i) $T(A_1)\subseteq A_2$ and $T(A_2)\subseteq A_1$.

(ii) there exists a function $\psi\in\Psi$ such that

$$p(T(a),T(y))\leq\psi(K(a,y)), \text{ for all } (a,y)\in A_1\times A_2.$$

Then T has a fixed point that belongs to $A_1\cap A_2$.

Corollary 4.3. Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete partial metric space (X,p) and $T:Y\to Y$ be a given mapping, where $Y=A_1\cup A_2$. Suppose that the following conditions hold:

(i) $T(A_1)\subseteq A_2$ and $T(A_2)\subseteq A_1$.

(ii) there exists a constant $\lambda\in(0,1)$ such that

$$p(T(a),T(y))\leq\lambda\max\left\{p(a,y),\frac{p(a,Ta)+p(y,Ty)}{2},\frac{p(a,Ty)+p(y,Ta)}{2},\right.$$

$$\left.\frac{p(a,Ta)\,p(y,Ty)}{p(a,y)},\frac{p(y,Ty)\left[1+p(a,Ta)\right]}{\left[1+p(a,y)\right]}\right\},$$

for all $(a,y)\in A_1\times A_2$. Then T has a fixed point that belongs to $A_1\cap A_2$.

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Muhammad Arshad

Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad 44000, Pakistan

E-mail address: `marshadzia@iiu.edu.pk`

Shaif Alshoraify

Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad 44000, Pakistan

E-mail address: `shaif.phdma61@iiu.edu.pk`

Abdullah Shoaib
Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan
E-mail address: abdullahshoaib15@yahoo.com