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ON APPROXIMATION OF ABSTRACT FIRST ORDER DIFFERENTIAL EQUATION WITH AN INTEGRAL CONDITION

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Abstract. We apply an iteration approximation method to approximate an initial condition of a boundary value problem for an abstract first order homogeneous linear differential equation with an integral boundary condition on a Banach space.

1. Introduction

Let $B(E)$ denote the Banach algebra of all linear bounded operators on a complex Banach space E. The set of all linear closed densely defined operators in E will be denoted by $\mathcal{C}(E)$. We denote by $\sigma(B)$ the spectrum of the operator B; by $\rho(B)$ the resolvent set of B; by $\mathcal{N}(B)$ the null space of B and by $\mathcal{R}(B)$ the range of B.
Let A be a generator of analytic C_0-semigroup $U(t)$, defined on a Banach space E, which means that $A : D(A) \subseteq E \to E$ is a closed linear operator, such that

$$\left\| (\lambda I - A)^{-1} \right\|_{B(E)} \leq \frac{1}{1+|\lambda|}, \text{ for any } \operatorname{Re}\lambda \geq 0. \tag{1.1}$$

Consider in a Banach space E the equation

$$u^{'}(t) = Au(t), t \in [0,T] \tag{1.2}$$

Definition 1.1. *The vector fonction $u(t) = U(t)f$; $0 \leq t \leq T$, corresponding to some element $f \in E$ is called a generalized solution of (1.2). If, in addition, $f \in D(A)$, then the solution $u(t) = U(t)f$ is said to be classical.*

Remark. *In the case, when $f \in D(A)$ obviously f coincides with the initial state $u(0)$ of the corresponding solution $u(t)$.*

Suppose that the initial state f is unknown, and consider the additional relation

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$$\int_0^T w(t)\,u(t)\,dt = g; \tag{1.3}$$

where $g \in E$ is a given element in E and $w(t)$ is scalar measurable function of bounded variation on the segment $[0,\ T]$

Remark. *The integral occurring in (1.3) is well-defined in the sense of Bochner for any function $u(t) = U(t)\,f$.*

Definition 1.2. *A generalized solution of the problem (1.2), (1.3) is defined to be a function $u(t) = U(t)\,f$; $0 \le t \le T$, corresponding to some element $f \in E$ and reducing relation (1.3) to a valid identity. If, in addition $f \in D(A)$, then the corresponding solution $u(t) = U(t)\,f$ of the problem (1.2), (1.3) is called a classical solution.*

From Definition 1, the solution of (1.2) is given in the form $u(t) = U(t)\,f$. Therefore, the function $u(t) = U(t)\,f$ satisfies the condition (1.3) if and only if f satisfies the equation

$$\int_0^T w(t)\,U(t)\,f dt = g. \tag{1.4}$$

So, for $f \in E$, we have the operator equation $Bf = g$, where,

$$Bf = \int_0^T w(t)\,U(t)\,f dt.$$

Lemma 1.3. *([1]). The operator B maps E into $D(A)$.*

Remark. *If $g \in E \backslash D(A)$, then the problem (1.2), (1.3) is unsolvable in the sense of the Definition 1.2.*

Now applying the operator A in (1.4) and integrating by parts, we get the Fredholm second order equation in the form

$$(I - K)\,f = G, \tag{1.5}$$

where,

$$Kf = \left(\frac{w(T)}{w(0)} U(T) + \frac{1}{w(0)} \int_0^T U(t)\,d(-w(t)) \right) f, \tag{1.6}$$

and,

$$G = -\frac{1}{w(0)} Ag. \tag{1.7}$$

In such settings one would say that the problem (1.2), (1.3) is well posed if the element g is given in the space $D(A)$ and the unknow element f is considered as an element from the space E. From 1.1 it follows that the resolvent $(\lambda I - A)^{-1}$ exists for $\lambda = 0$ and is positive operator, and therefore A^{-1} exists, which implies equivalence of the problem (1.2), (1.3) to the Fredholm second order equation 1.5. Using ideas from [8] the aim of this paper is the construction of an algorithm for the approximation of an element f, which solves the problem (1.2), (1.3) or, in other words, we want to solve equation 1.5. We present the algorithm as a general approximation scheme, which includes finite element methods and finite difference methods and projection methods.

The main question is a solvability of the problem (1.2), (1.3). It is clear that in the case of compact operator K the operator $(I-K)$ is Fredholm operator of index 0. Most of the results on the existence of solution of the problem (1.2), (1.3) are concerned to compactness or positivity property of resolvent of operator A. So the existence of bounded inverse operator $(I-K)^{-1}$ follows practically from condition $\mathcal{N}(I-K)=\{0\}$ and compact convergence of resolvent, see Theorem 2.9 and Step 4 of the proof of Theorem 5.1. There are some theorems proved, say, in [1, 2], which guarantee that condition $\mathcal{N}(I-K)=\{0\}$ holds. Namely, let us list some results which could be applied here.
Consider in a Banach space E the problem of finding an element f from relations

$$u^{'}(t)=Au(t)\,,t\in[0,\ T]\,, \tag{1.8}$$

with

$$\int_0^T w(t)\,u(t)\,dt=g, \tag{1.9}$$

where $g\in E$ is a given element in E and $w(t)$ is scalar measurable function of bounded variation on the segment $[0,\ T]$.

Theorem 1.4. *([1, 2]). Let $w(t)$ be a nonnegative non increasing function for $t\in[0,T]$ such that $w(t)>0$ as $t\to 0^+$, and let the semigroup $U(t)$ generated by the operator A satisfy the estimate $\|U(t)\|\le M\exp(-\beta t)$ with constants $M\ge 1$, $\beta>0$. Then the problem (1.8)-(1.9) is well-posed.*

If E is a Banach lattice. We recall that an order set $(E,\preceq)$ is called a lattice if for any pair of elements $x,y\in E$ the elements $\sup(x,y)$ and $\inf(x,y)$ exist in E. Moreover, for any $x\in E$ we define $x^+=\sup(x,0)$, $x^-=\inf(-x,0)$ which called positive and negative parts, respectively. The following relation is valid, $x=x^+-x^-$.

Definition 1.5. *Let B be a linear operator on E. The operator B is called positive if $Bx\succeq 0$ for all $x\succeq 0$.*

Definition 1.6. *A C_0- semigroup $\exp(tA)\,,t\ge 0$, is called positive in a Banach space with a cone E^+ if $\exp(tA)\,E^+\subseteq E^+$ for any $t\ge 0$.*

Definition 1.7. *A C_0- semigroup $\exp(tA)\,,t\ge 0$, is positive iff resolvent $(\lambda I-A)^{-1}E^+\subseteq E^+$ for any $\lambda>w(A)$.*

Definition 1.8. *A linear $A:D(A)\subseteq E\to E$ is said to have the positive off-diagonal (**POD**) property if $\langle Au,\phi\rangle\ge 0$ whenever $0\preceq u\in D(A)$ and $0\preceq\phi\in E^*$ with $\langle u,\phi\rangle=0$.*

Theorem 1.9. *([1, 2]). Let $w(t)$ be a nonnegative non increasing function for $t\in[0,T]$ such that $w(t)>0$ as $t\to 0^+$, and let the semigroup $U(t)$ generated by the operator A be positive and compact for $t>0$. Assume that the spectrum of A lies in the half-plane $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda<0\}$. Then the problem (1.8)-(1.9) is well-posed.*

2. General approximation scheme

Now we give the algorithm on general approximation scheme which includes finite element and finite difference methods and projection methods.

The general approximation scheme, due to [5, 13, 15] can be described in the following way. Let E_n and E be Banach spaces and $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \to E_n, p_n \in B(E; E_n), n \in IN = \{1, 2, ...\}$, with the property: $\|p_n x\|_{E_n} \to \|x\|_E$ as $n \to \infty$ for any $x \in E$.

Definition 2.1. *The sequence of elements $\{x_n\}, x_n \in E_n, n \in IN$; is said to be $\mathcal{P}$-convergent to $x \in E$ iff $\|x_n - p_n x\|_{E_n} \to 0$ as $n \to \infty$ and we write this $x_n \overset{\mathcal{P}}{\to} x$.*

Definition 2.2. *The sequence of bounded linear operators $B_n \in B(E_n), n \in IN$, is said to be $\mathcal{PP}$-convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in IN$; such that $x_n \overset{\mathcal{P}}{\to} x$ one has $Bx_n \overset{\mathcal{P}}{\to} Bx$. We write then $B_n \overset{\mathcal{PP}}{\to} B$.*

For general examples of notions of $\mathcal{P}$-convergence see for instance [12].

Remark. *If we put $E_n = E$ and $p_n = I$ for each $n \in IN$, where I is the identity operator on E, then Definition 2.1 leads to the traditional pointwise convergent bounded linear operators which we denote by $B_n \to B$.*

Definition 2.3. *A sequence of elements $\{x_n\}, x_n \in E_n, n \in IN$, is said to be $\mathcal{P}$-compact if for any $IN' \subset IN$ there exist $IN'' \subset IN'$ and $x \in E$ such that $x_n \overset{\mathcal{P}}{\to} x$, as $n \to \infty$ in IN''.*

Definition 2.4. *A system $\{p_n\}$ is said to be discrete order preserving if for all sequences $\{x_n\}, x_n \in E_n$, and any element $x \in E$, the following implication holds:*

$$x_n \overset{\mathcal{P}}{\to} x \text{ implies } x_n^+ \overset{\mathcal{P}}{\to} x^+.$$

It is know [6] that $\{p_n\}$ preserves the order iff $\|p_n x^+ - (p_n x)^n\|_{E_n} \to 0$ as $n \to \infty$ for any $x \in E$. If $B_n \overset{\mathcal{PP}}{\to} B$ and $B_n \succeq 0$ for $n \geq n_0$ and the system $\{p_n\}$ is order preserving, then [10] $B \succeq 0$. However, the inverse statement does not hold in general and we need to assume positiveness of $B_n \succeq 0$.

Definition 2.5. *A sequence of operators $\{B_n\}, B_n \in B(E_n), n \in IN$, converges compactly to an operator $B \in B(E)$ if $B_n \overset{\mathcal{PP}}{\to} B$ and the following compactness condition holds:*

$$\|x_n\|_{E_n} = O(1), \{B_n x_n\} \text{ is } \mathcal{P} - \text{compact.}$$

Let us mention that the last implication could be writtten as $\mu(\{B_n x_n\}) = 0$ as $\|x_n\| \leq$constant for mesure of noncompactness $\mu(.)$. The main property of $\mu(.)$ is that$\mu(\{y_n\}) = 0$ iff $\{y_n\}$ is $\mathcal{P}$-compact. It is also easy to check that $\mu(\{x_n + y_n\}) \leq \mu(\{x_n\}) + \mu(\{y_n\})$ and $\mu(\{D_n x_n\}) \leq \overline{\lim}_{n\to\infty} \|D_n\| \|x_n\|$ for any operators $D_n \in B(E_n)$ and any sequences $\{x_n\}, \{y_n\}$.

Definition 2.6. *A sequence of operators $\{B_n\}, B_n \in B(E_n), n \in IN$, is said to be stably convergent to an operator $B \in B(E)$ iff $B_n \overset{\mathcal{PP}}{\to} B$ and $\|B_n^{-1}\|_{E_n} = O(1), n \to \infty$. We will write this as: $B_n \overset{\mathcal{PP}}{\to} B$ stably.*

Definition 2.7. *A sequence of operators* $\{B_n\}, B_n \in B(E_n)$, $n \in IN$, *is called regularly convergent to the operator* $B \in B(E)$ *iff* $B_n \overset{\mathcal{PP}}{\to} B$ *and the following implication holds:* $\|x_n\|_{E_n} = O(1)$ *and* $\{B_n x_n\}$ *is* $\mathcal{P}$*-compact,* $\{x_n\}$ *is* $\mathcal{P}$*-compact. We write this as:* $B_n \overset{\mathcal{PP}}{\to} B$ *regularly.*

Theorem 2.8. *([15]). Let* $C_n, Q_n \in B(E_n), C, Q \in B(E)$ *and* $R(Q) = E$. *Assume also that* $C_n \overset{\mathcal{PP}}{\to} C$ *compactly and* $Q_n \overset{\mathcal{PP}}{\to} Q$ *stably. Then* $Q_n + C_n \overset{\mathcal{PP}}{\to} Q + C$ *converge regularly.*

Theorem 2.9. *([15]). For* $B_n \in B(E_n)$ *and* $B \in B(E)$ *the following conditions are equivalent.*

- **(i):** $B_n \overset{\mathcal{PP}}{\to} B$ *regularly,* B_n *are Fredholm operators of index* 0 *and* $\mathcal{N}(B) = \{0\}$*;*
- **(ii):** $B_n \overset{\mathcal{PP}}{\to} B$ *stably and* $\mathcal{R}(B) = E$*;*
- **(iii):** $B_n \overset{\mathcal{PP}}{\to} B$ *stably and regularly;*

If one of conditions (i)–(iii) holds, then there exist $B_n^{-1} \in B(E_n)$, $B^{-1} \in B(E)$, *and* $B_n^{-1} \overset{\mathcal{PP}}{\to} B^{-1}$ *regularly and stably.*

Definition 2.10. *The region of stability* $\Delta_s = \Delta_s(\{A_n\}), A_n \in \mathcal{C}(E_n)$, *is defined as the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda \in \rho(A_n)$ *for almost all* n *and such that the sequence* $\left\{\left\|(\lambda I_n - A_n)^{-1}\right\|\right\}_{n\in\mathbb{N}}$ *is bounded. The region of convergence* $\Delta_c = \Delta_c(\{A_n\}), A_n \in \mathcal{C}(E_n)$, *is defined as the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda \in \Delta_s(\{A_n\})$ *and such that the sequence of operators* $\left\{(\lambda I_n - A_n)^{-1}\right\}_{n\in\mathbb{N}}$ *is* $\mathcal{PP}-$ *convergent to some operator* $S(\lambda) \in B(E)$.

Definition 2.11. *A sequence of operators* $\{L_n\}, L_n \in \mathcal{C}(E_n)$, *is said regularly compatible with an operator* $L \in \mathcal{C}(E)$ *if* (L_n, L) *are compatible and, for any bounded sequence* $\|x_n\|_{E_n} = O(1)$ *such that* $x_n \in D(L_n)$ *and* $\{L_n x_n\}$ *is* $\mathcal{P}$*-compact, it follows that* $\{x_n\}$ *is* $\mathcal{P}$*-compact, and the* $\mathcal{P}$*- convergence of* $\{x_n\}$ *to some element* x *and convergence of* $\{L_n x_n\}$ *to some element* y *as* $n \to \infty$ *in* $\mathbb{N}' \subseteq \mathbb{N}$ *imply that* $x \in D(L)$ *and* $Lx = y$.

Definition 2.12. *The region of regularity* $\Delta_r = \Delta_r(\{A_n\}, A)$, *is defined as the set of all* $\lambda \in \mathbb{C}$ *such that* $(L_n(\lambda), L(\lambda))$ *are regularly compatible, where* $L_n(\lambda) = \lambda I_n - A_n$ *and* $L(\lambda) = \lambda I - A$.

The relationships between these regions are given by the following statement.

Proposition 2.13. *([14]). Suppose that* $\Delta_c \neq \emptyset$ *and* $\mathcal{N}(S(\lambda)) = \{0\}$ *at least for one point* $\lambda \in \Delta_c$, *so that* $S(\lambda) = (\lambda I - A)^{-1}$. *Then* (A_n, A) *are compatible and*

$$\Delta_c = \Delta_s \cap \rho(A) = \Delta_s \cap \Delta_r = \Delta_r \cap \rho(A).$$

Definition 2.14. *The region of compact convergence of resolvent,* $\Delta_{cc} = \Delta_{cc}(A_n, A)$, *where* $A_n \in \mathcal{C}(E_n)$ *and* $A \in \mathcal{C}(E)$ *is defined as the set of all* $\lambda \in \Delta_c \cap \rho(A)$ *such that* $(\lambda I_n - A_n)^{-1} \overset{\mathcal{PP}}{\to} (\lambda I - A)^{-1}$ *compactly.*

Theorem 2.15. *([4]). Assume that* $\Delta_{cc} \neq \emptyset$. *Then for any* $\mu \in \Delta_s$ *the following implication holds:*

$$\|x_n\|_{E_n} = O(1) \text{ and } \|(\mu I_n - A_n)x_n\|_{E_n} = O(1) \Rightarrow \{A_n\} \text{ is } \mathcal{P} - \text{compact} \quad (2.1)$$

Conversely, if for some $\mu \in \Delta_c \cap \rho(A)$ implication (2.1) holds, then $\Delta_{cc} \neq \emptyset$.

Corollary 2.16. *([4]). Assume that $\Delta_{cc} \neq \emptyset$. Then $\Delta_{cc} = \Delta_c \cap \rho(A)$.*

Theorem 2.17. *([4]). Assume that $\Delta_{cc} \neq \emptyset$. Then $\Delta_r = \mathbb{C}$.*

In the case of unbounded operators, and we know in general infinitesimal generators are unbounded, we consider the notion of compatibility.

Definition 2.18. *The sequence of closed linear operators $\{A_n\}, A_n \in \mathcal{C}(E_n), n \in IN$, are said to be compatible with a closed linear operator $A \in \mathcal{C}(E)$ iff for each $x \in D(A)$ there is a sequence $\{x_n\}$, $x_n \in D(A_n) \subseteq E_n, n \in IN$, such that $x_n \stackrel{\mathcal{P}}{\to} x$ and $A_n x_n \stackrel{\mathcal{P}}{\to} Ax$. We write (A_n, A) are compatible.*

Note, that (A_n, A) are compatible if resolvent converge $(\lambda I_n - A_n)^{-1} \stackrel{\mathcal{PP}}{\to} (\lambda I - A)^{-1}$. Usually in practice Banach spaces E_n are finite dimensional, although, in general, say for the case of a closed operator A, we have $dim E_n \to \infty$ and $\|A_n\|_{B(E_n)} \to \infty$ as $n \to \infty$.

3. Discretization of semigroups

Let us consider the well-posed Cauchy problem in the Banach space E with operator $A \in \mathcal{C}(E)$

$$\begin{cases} u'(t) = Au(t); t \in [0;\infty), \\ u(0) = u^0 \in E, \end{cases} \tag{3.1}$$

where operator A generates C_0-semigroup $U(t)$. It is well-known that the C_0-semigroup gives the solution of (3.1) by the formula $u(t) = U(t)u^0$ for $t \geq 0$. The theory of well-posed problems and numerical analysis of these problems have been developed extensively, see [4, 7]. Let us consider on the general discretization scheme the semidiscrete approximation of the problem (3.1) in the Banach spaces E_n,

$$\begin{cases} u_n'(t) = A_n u_n(t); t \in [0;\infty), \\ u_n(0) = u_n^0 \in E_n, \end{cases} \tag{3.2}$$

with the operators $A_n \in \mathcal{C}(E_n)$, such that they generate C_0-semigroups, which are consistent with the operator $A \in \mathcal{C}(E)$ and $u_n^0 \stackrel{\mathcal{P}}{\to} u^0$.

4. The simplest discretization schemes

We have the following version of Trotter-Kato's Theorem on general approximation scheme.

Theorem 4.1. *([12, Theorem ABC]). Assume that $A \in C(E); A_n \in C(E_n)$ and they generate C_0-semigroups. The following conditions (A) and (B) are equivalent to condition (C).*

- **(A):** *Consistency. There exists $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ such that the resolvents converge $(\lambda I_n - A_n)^{-1} \stackrel{\mathcal{PP}}{\to} (\lambda I - A)^{-1}$;*
- **(B):** *Stability. There are some constants $M \geq 1$ and ω; which are not depending on n and such that $\|U_n(t)\| \leq M \exp(\omega t)$ for $t \geq 0$ and any $n \in \mathbb{N}$;*

(C): *Convergence. For any finite $T>0$ one has* $\max\limits_{t\in[0;T]}\left\|U_n(t)u_n^0 - p_nU(t)u^0\right\| \to 0$ *as* $n\to\infty$*; whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$ *for any* $u_n^0\in E_n$*;* $u^0\in E$.

Remark. *The condition (A) in the contents of these Theorems is equivalent to compatibility of operators (A_n, A).*

Theorem 4.2. *(*[4]*) Let operators A and A_n generate analytic C_0-semigroup. The following conditions (A) and (B_1) are equivalent to condition (C_1).*

(A): *Consistency. There exists $\lambda\in\rho(A)\cap\cap_n\rho(A_n)$ such that the resolvents converge* $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$*;*

(B$_1$): *Stability. There are some constants $M_1\geq 1$ and ω_1 independent of n such that for any $Re\lambda>\omega_1$,* $\left\|(\lambda I_n - A_n)^{-1}\right\| \leq \frac{M_1}{|\lambda-\omega_1|}$ *for all $n\in\mathbb{N}$;*

(C$_1$): *Convergence. For any finite $\mu>0$ and some $0<\theta<\frac{\pi}{2}$ we have* $\max\limits_{\eta\in\Sigma(\theta,\mu)}\left\|U_n(\eta)u_n^0 - p_nU(\eta)u^0\right\| \to 0$ *as $n\to\infty$ whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$*.*
Here $\Sigma(\theta,\mu) = \{z\in\Sigma(\theta) : |z|\leq\mu\}$ *and* $\Sigma(\theta) = \{z\in\mathbb{C} : |\arg z|\leq\theta\}$*.*

Definition 4.3. *An element $e\in E^+$ is said to be an order unit in a Banach lattice E if for every $x\in E$ there exists $0\leq\lambda\in\mathbb{R}$ such that $-\lambda e\preceq x\preceq\lambda e$. For $e\in intE^+$ we can define the order unit norm by*

$$\|x\|_e = \inf\{\lambda\geq 0 : -\lambda e\preceq x\preceq\lambda e\}.$$

An order Banach space E is callad an order unit space if there exists $e\in intE^+$ such that $\|.\|_E = \|.\|_e$.

The following version of the Trotter-Kato's Theorem for positive C_0- semigroup holds.

Theorem 4.4. *(*[10]*) Let the operators A_n and A from (3.1) and (3.2) be compatible, let E, E_n be order unit spaces, and let $e_n\in D(A_n)\cap intE_n^+$. Assume that the operators A_n have the POD property and $A_ne_n\preceq 0$ for sufficiently large n. Then* $\exp(tA_n) \xrightarrow{\mathcal{PP}} \exp(tA)$ *uniformly in $t\in[0,T]$.*

We can assume that conditions (A) and (B) for the corresponding C_0- semigroups case are satisfied without any restriction of generality if any discretization processes in time are considered.

We denote by $T_n(.)$ a family of discrete semigroups as in [7], i.e. $T_n(t) = T_n(\tau_n)^{k_n}$, where $k_n = \left[\frac{t}{\tau_n}\right]$, as $n\to 0$, $n\to\infty$. The generator of discrete semigroups is defined by $\breve{A}_n = \frac{1}{\tau_n}(T_n(\tau_n) - I_n) \in B(E_n)$ and so $T_n(t) = \left(I_n + \tau_n\breve{A}_n\right)^{k_n}$; where $t = k_n\tau_n$.

Theorem 4.5. *(*[12, Theorem ABC- discr]*). The following conditions (A) and (B_0) are equivalent to condition (C_0): (A) Consistency. There exists* $\lambda\in\rho(A)\cap\cap_n\rho\left(\breve{A}_n\right)$ *such that the resolvents converge* $\left(\lambda I_n - \breve{A}_n\right)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$, *$(B_0)$ Stability. There are some constants $M_2\geq 1$ and $\omega_2\in\mathbb{R}$ such that* $\|T_n(t)\| \leq M_2\exp(\omega_2 t)$ *for $t\in IR+$, $n\in IN$, (C_0) Convergence. For any finite $T>0$ one* *has* $\max_{t\in[0;T]} \left\| T_n(t)\, u_n^0 - p_n \exp(tA)\, u^0 \right\| \to 0$ *as* $n \to \infty$, *whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$ *for any* $u_n^0 \in E_n$, $u^0 \in E$.

Theorem 4.6. *([12]). Assume that* $A \in \mathcal{C}(E)$, $A_n \in \mathcal{C}(E_n)$ *and they generate* C_0*-semigroup. Assume also that conditions* (A) *and* (B) *of Theorem 4.1 holds. Then the implicit difference scheme*

$$\frac{\overline{U_n}(t+\tau_n) - \overline{U_n}(t)}{\tau_n} = A_n \overline{U_n}(t+\tau), \overline{U_n}(0) = u_n^0, \tag{4.1}$$

is stable, i.e. $\left\| (I_n - \tau_n A_n)^{-k_n} \right\| \le M_2 \exp(\omega_2 t)$, $t = k_n \tau_n \in IR+$*; and gives an approximation to the solution of the problem (3.1), i.e.* $\overline{U_n}(t) \equiv (I_n - \tau_n A_n)^{-k_n} u_n^0 \xrightarrow{\mathcal{P}} \exp(tA)\, u_n^0$ *$\mathcal{P}$-converges uniformly with respect to* $t = k_n \tau_n \in [0;T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0$, $n \to \infty$, $k_n \to \infty$, $\tau_n \to 0$.

Theorem 4.7. *([4]). Assume that conditions* (A) *and* (B_1) *of Theorem 4.2 hold and condition*

$$\|\tau_n A_n\| \le \frac{1}{(M+2)}, n \in IN, \tag{4.2}$$

is fulfilled. Then the difference scheme

$$\frac{U_n(t+\tau_n) - U_n(t)}{\tau_n} = A_n U_n(t), U_n(0) = u_n^0, \tag{4.3}$$

is stable and gives an approximation to the solution of the problem (1.2), i.e. $U_n(t) \equiv (I_n + \tau_n A_n)^{k_n} u_n^0 \xrightarrow{\mathcal{P}} u(t)$ *discretely $\mathcal{P}$-converge uniformly with respect to* $t = k_n \tau_n \in [0;T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0$, $n \to \infty$, $k_n \to \infty$, $\tau_n \to 0$.

Let us introduce the following equivalent conditions:
(B_1') Stability. There are constants M', ω' such that

$$\|\exp(tA_n)\| \le M' \exp\left(\omega' t\right), \|A_n \exp(tA_n)\| \le \frac{M'}{t} \exp\left(\omega' t\right), t \in IR+.$$

(B_1'') Stability. There are constants M'', ω'' and $\tau^* > 0$ such that

$$\left\| (I_n - \tau_n A_n)^{-k} \right\| \le M'' \exp\left(\omega'' k \tau_n\right),$$
$$\left\| k\tau_n A_n (I_n - \tau_n A_n)^{-k} \right\| \le M'' \exp\left(\omega'' k \tau_n\right) \text{ for } 0 < \tau_n < \tau^*, n, k \in IN.$$

Theorem 4.8. *The conditions* (A) *and* (B_1') *are equivalent to the condition* (C_1).

Proof. See ([12]). □

Remark. *Conditions* (B_1), (B_1') *and* (B_1'') *are equivalent, see ([11])*

5. Main results

Let A_n be a generator of compact analytic C_0-semigroup $U_n(t)$. Consider in a Banach space E_n the equations

$$u_n'(t) = A_n u_n(t), t \in [0,T] \tag{5.1}$$

with the integral conditions

$$\int_0^T w_n(t)\, u_n(t)\, dt = g_n. \tag{5.2}$$

The solution of the problem (5.1), (5.2) is given by the formula $u_n(t) = U_n(t) f_n$, where $f_n = (I - K_n)^{-1} G_n$ and corresponding second order Fredholm equation can be written in the form:

$$(I_n - K_n) f_n = G_n, \tag{5.3}$$

where

$$K_n f_n = \left(\frac{w_n(T)}{w_n(0)} U_n(T) + \frac{1}{w_n(0)} \int_0^T U_n(t)\, d(-w_n(t))\, t \right) f_n, \tag{5.4}$$

and

$$G_n = -\frac{1}{w_n(0)} A_n g_n$$

Before we formulate our main results just recall that condition $\mathcal{N}(I - K) = \{0\}$ could be obtained from Theorems in Section 2.

Theorem 5.1. *Let $w(t)$ be a nonnegative non increasing function for $t \in [0, T]$ such that $w(t) > 0$ as $t \to 0^+$, $w_n(t)$ be a nonnegative non increasing function for $t \in [0, T]$ such that $w_n(t) > 0$ as $t \to 0^+$, and they converge $w_n(t) \to w(t)$ uniformly in $t \in [0; T]$. Let conditions (A); (B_{01}) be satisfied and $G_n \to G$. Assume also that $\mathcal{N}(I - K) = \{0\}$; operator $(\lambda I - A)^{-1}$ is compact and $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$ compactly. Then solutions of the problems (5.3) exist and converge to the solution of the problem (1.5); i.e. $f_n \to f$.*

Proof. The proof is done in four steps.

Step 1. First, let us show that the com
pact convergence of resolvents $R(\lambda; A_n) \to R(\lambda; A)$ is equivalent to the compact convergence of C_0-semigroups $U_n(t) \to U(t)$ for any $t > 0$. Let $\|x_n\| = O(1)$. Then from the estimate $\|A_n U_n(t)\| \le \frac{M}{t} \exp(\omega t)$; which is exactly condition (B_1'), we obtain the boundedness in n of the sequence $\{(A_n - \lambda I_n) U_n(t) x_n\}$ for any fixed $t > 0$. Because of the compact convergence of resolvent, we obtain the compactness of the sequence $\{U_n(t) x_n\}$.

The necessity will be proved if for the measure of noncompactness $\mu(.)$ (for the definition, see [15]), we establish that $\mu\left(\left\{(\lambda I_n - A_n)^{-1} x_n\right\}\right) = 0$ for any $\|x_n\| = O(1)$.We have

$$\mu\left(\left\{(\lambda I_n - A_n)^{-1} x_n\right\}\right) = \mu\left(\left\{(\int_0^\infty \exp(-t\lambda) U_n(t) x_n dt\right\}\right)$$
$$\leq \mu\left(\left\{(\int_0^q \exp(-t\lambda) U_n(t) x_n dt\right\}\right)$$
$$+ \mu\left(\left\{(\int_p^\infty \exp(-t\lambda) U_n(t) x_n dt\right\}\right)$$
$$+ \mu\left(\left\{U_n(\varepsilon) \int_q^p \exp(-t\lambda) U_n(t-\varepsilon) x_n dt\right\}\right).$$

Two first terms can be made less than ε by the choice of q, p. The last term is equal to zero because of the compact convergence $U_n(\varepsilon) \to U(\varepsilon)$ for any $0 < \varepsilon < q$.

Step 2. Consider the operators K and K_n defined by (1.6) and (5.4) on the spaces E and E_n. The operator K defined by (1.6) is compact in E. Indeed, we obtain that the

$$K_\varepsilon = \left(\frac{w(T)}{w(0)} U(T) + \frac{1}{w(0)} \int_\varepsilon^T U(t)\, d(-w(t))\right)$$

is a product of compact and bounded operators. Moreover $\|K_\varepsilon - K\| \leq C\varepsilon$, where

$$K = \left(\frac{w(T)}{w(0)} U(T) + \frac{1}{w(0)} \int_0^T U(t)\, d(-w(t))\right)$$

and $\varepsilon > 0$. Then it follows that the operator $K : E \to E$ is compact.

Step 3. It is easy to see that $K_n \to K$. To show that $K_n \to K$ compactly, we assume that $\|f_n\|_{E_n} = O(1)$. Now $\{K_n f_n\}$ is $\mathcal{P}$-compact because of representation

$$K_{\varepsilon,n} = \left(\frac{w_n(T)}{w_n(0)} U_n(T) + \frac{1}{w_n(0)} \int_\varepsilon^T U_n(t)\, d(-w_n(t))\right)$$

and one can easy verify the vanishing of the noncompactness measure $\mu(\{K_n f_n\}) = 0$ for all $n \in \mathbb{N}$, taking into an account that $\|K_{\varepsilon,n} - K_n\| \leq C\varepsilon$.

Step 4. Now $I_n \to I$ stably and $K_n \to K$ compactly. Hence it follows from Theorem 2.8 that $I_n - K_n \to I - K$ regularly. Moreover, the nullspace $\mathcal{N}(I-K) = \{0\}$ and the operators $I_n - K_n$ are Fredholm of index zero. Then it follows from Theorem 2.9 that $I_n - K_n \to I - K$ stably, i.e. $(I_n - K_n)^{-1} \to (I-K)^{-1}$.

Since $G_n \to G$, one gets $f_n = (I_n - K_n)^{-1} G_n \to (I-K)^{-1} G = f$. The Theorem is proved. □

One can find that solution of the problem (5.3) according to Theorem 5.2, and under the assumption that functions $w_n(t), w(t) \in C^1([0;T])$ and they converge $w_n(t) \to w(t)$ uniformly in $t \in [0;T]$.

Theorem 5.2. *Let C_0-semigroups $U_n(t)$ be positive and compact for $t>0$. Assume that the spectrum of A_n lies in the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ and $w_n(t) \geq 0, w_n(0) > 0; w_n^{'}(t) \leq 0$ for any $t \in [0;T]$. Define the operator K_n as in (5). Then $r(K_n) < 1$.*

We are recalling that $r(A)$ is the spectral radius of $A \in B(E)$. The spectral radius of A, denoted by $r(A)$, is the radius of the smallest disk centered at zero that contains $\sigma(A)$,

$$r(A) = \{|\lambda| : \lambda \in \sigma(A)\}.$$

It is well known that for every $A \in B(E)$, we have

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}, \text{ and } r(A) \leq \|A\|.$$

Proof. The proof of the Theorem 5.2 is similar to the proof of the Theorem 5.4. □

As a consequence of the Theorem 5.2, we have the following

Theorem 5.3. *Let C_0-semigroup $U_n(t)$ be positive and compact for $t>0$. Assume that the spectrum of A_n lies in the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ and $w_n(t) \geq 0, w_n(0) > 0; w_n^{'}(t) \leq 0$ for any $t \in [0;T]$. Then for any $g \in D(A_n)$, there is unique solution of the problem (5.1), (5.2).*

Since $r(K_n) < 1$, could be organized as follows

$$f_{n,j+1} = K_n f_{n,j} - \frac{1}{w_n(0)} A_n g_n, \qquad n, j = 0; 1..., \tag{5.5}$$

with initial condition $f_{n,0} = 0$. The value $K_n f_{n,j}$ is nothing else as a solution of Cauchy problem

$$v_n'(t) = A_n v_n(t) - \frac{w_n^{'}(T-t)}{w_n(0)} f_{n;j}, \quad v_n(0) = \frac{w_n(T)}{w_n(0)} f_{n;j}$$

at the point T; i.e.

$$K_n f_{n,j} = v_n(T, f_{n,j}) = \frac{w_n(T)}{w_n(0)} U_n(T) f_{n,j} + \frac{1}{w_n(0)} \int_0^T - w_n^{'}(t) U_n(t) f_{n,j} dt.$$

So (5.5) could be written in the form, starting from

$$f_{n,0} = 0, \quad f_{n,j+1} = v_n(T, f_{n,j}) - \frac{1}{w_n(0)} A_n g_n, \qquad n, j = 0; 1; ......$$

Moreover, $f_{n,j} \to f_n$ as $j \to \infty$ since $r(K_n) < 1$.

There are different ways how one can calculate $v_n(T, f_{n,j})$ One can use directly Theorems 4.6, 4.7 or maybe some higher order difference schemes for approximation of $U_n(T)$;

say as in [4, 9], and then apply some quadrature formula for approximation the term

$$\frac{1}{w_n(0)} \int_0^T - w_n^{'}(t) U_n(t) f_{n,j} dt$$

In this paper we consider just the simplest way which comes from Theorem 4.6. In case of Theorem 4.7 we have to assume stability condition, but the other considerations are the same. So following the scheme (4.1) we consider approximation of the equation (1) by

$$\frac{\overline{U_n}\left(t+\tau_n\right)-\overline{U_n}\left(t\right)}{\tau_n}=A_n\overline{U_n}\left(t+\tau\right),$$

and approximation of the condition (5.2) by

$$\sum_{j=0}^{k-1} w_n\left(j\tau_n\right)u_n\left(j\tau_n+\tau_n\right)\tau_n=g_n. \tag{5.6}$$

The solution of the scheme (5) can be written in the form

$$\overline{U_n}\left(t\right)=\left(I_n-\tau_nA_n\right)^{-k}u_n^0;t=k\tau_n$$

To construct approximation of operator K_n in (5.4),we just consider the simplest formula ($T=k_n\tau_n$):

$$\breve{K}_n=\left(I_n-\tau_nA_n\right)^{-k_n}\frac{w_n\left(T\right)}{w_n\left(0\right)} - \frac{1}{w_n\left(0\right)}\sum_{l=0}^{k_n-1}\left(I_n-\tau_nA_n\right)^{-l}\frac{w_n\left(l\tau_n+\tau_n\right)-w_n\left(l\tau_n\right)}{\tau_n}\tau_n.$$

Theorem 5.4. *Let C_0-semigroup $U_n\left(t\right)$ be positive and compact for $t>0$. Assume that the spectrum of A_n lies in the half-plane $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda<0\}$ and $w_n\left(t\right)\geq 0, w_n\left(0\right)>0; w_n^{'}\left(t\right)\leq 0$ for any $t\in[0;T]$. Define the operator $\breve{K}_n$ as in (5). Then $r\left(\breve{K}_n\right)<1$.*

Proof. The operator $\breve{K}_n$ is positive and compact, so by Krein-Rutman Theorem there are $\lambda_0\geq 0$ and $0\leq f_n^0\neq 0$ such that $\breve{K}_n\ f_n^0=\lambda_0 f_n^0$, and moreover, $r\left(\breve{K}_n\right)=\lambda_0$. Assume now in contradiction that $\lambda_0\geq 1$.
Substutting $\overline{U_n}\left(t\right)=\left(I_n-\tau_nA_n\right)^{-k}f_n$ in

$$\sum_{l=0}^{k-1} w_n\left(l\tau_n\right)u_n\left(l\tau_n+\tau_n\right)\tau_n. \tag{5.7}$$

with $f_n=f_n^0$. One gets that

$$\sum_{l=0}^{k-1} w_n\left(l\tau_n\right)\left(I_n-\tau_nA_n\right)^{-l-1}f_n^0\tau_n. \tag{5.8}$$

is positive for positive f_n^0. Putting

$$\varphi_n=\sum_{l=0}^{k-1} w_n\left(l\tau_n\right)\left(I_n-\tau_nA_n\right)^{-l-1}f_n^0\tau_n. \tag{5.9}$$

So, applying the operator A_n to (5), and using the formula of summation by parts. $\tau_n \sum_{l=0}^{k-1} \frac{v_{l+1}-v_l}{\tau_n} y_l = (y_k v_k - y_0 v_0) - \tau_n \sum_{l=0}^{k-1} v_{l+1} \frac{y_{l+1}-y_l}{\tau_n}$. We obtain that $A_n \varphi_n = -w_n(0) f_n^0 + w_n(0) \widetilde{K}_n f_n^0 = -w_n(0) f_n + w_n(0) \lambda_0 f_n = w_n(0)(\lambda_0 - 1) f_n^0 \geq 0$, since $w_n(0) > 0$, $\lambda_0 \geq 1$, and $f_n^0 \geq 0$. So, if we apply $(-A_n)^{-1}$; then because of positiveness of C_0-semigroup $U_n(t)$, the resolvent $(-A_n)^{-1}$ is also positive and $(-A_n)^{-1} A_n \varphi_n \geq 0$; which means that $0 \geq \varphi_n$. From the other hand from (5) it follows that $\varphi_n \geq 0$ for $f_n^0 \geq 0$. This means that $\varphi_n = 0$; which means that $w_n(l\tau_n)(I_n - \tau_n A_n)^{-l-1} f_n = 0$ for all $l = 0, ...k-1$, in particular for $l = 0$ we have $w_n(0)(I_n - \tau_n A_n)^{-1} f_n = 0$, because $Ker(I_n - \tau_n A_n)^{-1} = \{0\}$, and $w_n(0) \neq 0$, one gets that f_n^0. But this contradicts to $f_n^0 \neq 0$. The Theorem is proved. □

From Theorem 5.4 it follows that one can organize the process $\breve{f}_{n,j+1} = \breve{K}_n \breve{f}_{n,j} - \frac{1}{w_n(0)} A_n g_n$, $n; j = 0; 1$, which converges $\breve{f}_{n,j} \to \breve{f}_n$ as $j \to \infty$; where $\breve{f}_n$ is a solution of the problem $\breve{f}_n = \breve{K}_n \breve{f}_n - \frac{1}{w_n(0)} A_n g_n$.

Theorem 5.5. *Let C_0-semigroups $U_n(t)$ be positive and analytic. Assume also that functions $w_n(t), w(t) \in C^1([0;T])$ and they converge $w_n'(t) \to w'(t)$ uniformly in $t \in [0;T]$. Let conditions (A); (B_{01}) be satisfied and $G_n \to G$. Assume also that $\mathcal{N}(I-K) = \{0\}$, operator $(\lambda I - A)^{-1}$ is compact and $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$ compactly and $w_n(t) \in C^3([0;T])$ and $\left|w_n'''(t)\right| \leq$constant; $t \in [0;T]$. Then solutions of the problems (5) exist and converge to the solution of the problem (1.5); i.e. $\breve{f}_n \to f$ as $n \to \infty$.*

Proof. If $\breve{K}_n \to K$ compactly, then the statement of the Theorem 5.5 follows the same way as in the **Step 4** of Theorem 5.1. So, we are going to show that $\breve{K}_n \to K$ compactly. To do this it is enough to prove that $\left\|\breve{K}_n - K\right\| \to 0$ as $n \to \infty$; since the statement $K_n \to K$ compactly is already proved in Theorem 5.1. One can write

$$K_n - \breve{K}_n = \frac{w_n(T)}{w_n(0)} U_n(T) - (I_n - \tau_n A_n)^{-k_n} \frac{w_n(T)}{w_n(0)} + \frac{1}{w_n(0)} \int_0^T -w_n'(t) U_n(t)\, dt -$$

$$\frac{1}{w_n(0)} \sum_{l=0}^{k_n-1} (I_n - \tau_n A_n)^{-l-1} \frac{w_n(l\tau_n + \tau_n) - w_n(l\tau_n)}{\tau_n} \tau_n$$

where $k_n \tau_n = T$. In [3], it is proved under condition (B_1) that

$$\left\|U_n(t) - (I_n - \tau_n A_n)^{-k_n}\right\| \leq C \frac{\tau_n}{t} \exp(\omega t)$$

as $k_n \to \infty$ and $t = k\tau_n$.Let us consider now the difference

$$\sum_{l=0}^{k_n-1} \frac{1}{w_n(0)} \int\limits_{l\tau_n}^{(l+1)\tau_n} -w_n'(t)\, U_n(t)\, dt$$
$$-\frac{1}{w_n(0)} \sum_{l=0}^{k_n-1} (I_n - \tau_n A_n)^{-l-1} \frac{w_n(l\tau_n + \tau_n) - w_n(l\tau_n)}{\tau_n} \tau_n.$$

To finish with the demonstration we have to use

$$\pm \sum_{l=0}^{k_n-1} \frac{-1}{w_n(0)} \int\limits_{l\tau_n}^{(l+1)\tau_n} U_n(t) \frac{w_n(l\tau_n + \tau_n) - w_n(l\tau_n)}{\tau_n} \tau_n dt$$

terms.Indeed, it is easy to show that difference

$$\sum_{l=0}^{k_n-1} \frac{1}{w_n(0)} \int\limits_{l\tau_n}^{(l+1)\tau_n} -w_n'(t)\, U_n(t)\, dt$$
$$-\sum_{l=0}^{k_n-1} \frac{-1}{w_n(0)} \int\limits_{l\tau_n}^{(l+1)\tau_n} U_n(t) \frac{w_n(l\tau_n + \tau_n) - w_n(l\tau_n)}{\tau_n} \tau_n dt$$

converge to zero as $k_n \to \infty$ and $T = k_n \tau_n$; since

$$\frac{-1}{w_n(0)} \int\limits_{l\tau_n}^{(l+1)\tau_n} U_n(t) \left(w_n'(t) - \frac{w_n(l\tau_n + \tau_n) - w_n(l\tau_n)}{\tau_n} \tau_n \right) dt$$

is estimated by

$$C \frac{1}{w_n(0)} \int\limits_{l\tau_n}^{(l+1)\tau_n} U_n(t) \left| \left(w_n'(t) - \frac{w_n(l\tau_n + \tau_n) - w_n(l\tau_n)}{\tau_n} \right) \right| dt = O\left(\tau_n^2\right).$$

The second term from $\pm$ construction could be estimated as

$$\left\| \sum_{l=0}^{k_n-1} \frac{-1}{w_n(0)} \int\limits_{l\tau_n}^{(l+1)\tau_n} U_n(t) - (I_n - \tau_n A_n)^{-l-1} dt \frac{w_n(l\tau_n + \tau_n) - w_n(l\tau_n)}{\tau_n} \right\|$$

$$\leq C \sum_{l=1}^{k_n-1} \frac{1}{w_n(0)} \int\limits_{l\tau_n}^{(l+1)\tau_n} \|U_n(t) - U_n(l\tau_n + \tau_n)\| \, dt + C\tau_n$$
$$+ C \sum_{l=0}^{k_n-1} \left\| U_n(l\tau_n + \tau_n) - (I_n - \tau_n A_n)^{-l-1} \right\| \tau_n$$
$$\leq C \left(\sum_{l=1}^{k_n-1} \frac{\tau_n}{l} + \sum_{l=0}^{k_n-1} \frac{\tau_n}{l+1} \right) + \tau_n.$$

Where we used the fact that for any $t \in [j\tau_n, (j+1)\tau_n], 1 \le j \le k_n - 1$,

$$\|U_n(t) - U_n(j\tau_n + \tau_n)\| \le C\frac{\tau_n}{j\tau_n}.$$

The Theorem is proved.

□

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