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REGULARLY IDEAL CONVERGENCE OF DOUBLE SEQUENCES IN FUZZY NORMED SPACES

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Abstract. In this study, we introduce the notions of regularly $(\mathcal{I}_2, \mathcal{I})$-convergence, regularly $(\mathcal{I}_2^*, \mathcal{I}^*)$-convergence, regularly $(\mathcal{I}_2, \mathcal{I})$-Cauchy and regularly $(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy double sequences in fuzzy normed linear spaces. Also, we establish some basic results related to these notions.

1. Introduction and Background

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [19] and Schoenberg [37]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [24] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Das et al. [5] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this type convergence. Dündar [14] introduces the notions of regularly $(\mathcal{I}_2, \mathcal{I})$-convergence and $(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequences of real valued functions.

The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka [27] and proved some basic theorems for sequences of fuzzy numbers. Nanda [30] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. Dündar and Talo [11,12] investigated $\mathcal{I}_2$-convergence, $\mathcal{I}_2^*$-convergence and $\mathcal{I}_2$-Cauchy sequence of fuzzy numbers and Dündar et al. [13] introduced regularly $(\mathcal{I}_2, \mathcal{I})$-convergence and regularly $(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequences of fuzzy numbers. Hazarika [21] studied the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence and $\mathcal{I}$-Cauchy sequence in a fuzzy normed linear space. Also, Hazarika and Kumar [22] defined the concepts of $\mathcal{I}_2$-convergence, $\mathcal{I}_2^*$-convergence and $\mathcal{I}_2$-Cauchy sequence in a fuzzy normed linear space. Dündar and Türkmen [15, 16] studied $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy double sequences in fuzzy normed spaces. A lot of developments have been made in this area after the works of [17, 23, 29, 35, 36, 39–42, 45].

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Now, we recall the concept of ideal convergence, double sequence and fuzzy normed space and some basic definitions (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13–16, 18, 20–22, 24–28, 32–34, 38, 43, 44])

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0,1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh [46], a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \to [0,1]$. The function u itself is often used for the fuzzy set.

A fuzzy set u on $\mathbb{R}$ is called a fuzzy number if it has the following properties:

1. u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

2. u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$;

3. u is upper semicontinuous;

4. The set $[u]_0 = cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact.

Let $L(\mathbb{R})$ be set of all fuzzy numbers. If $u \in L(\mathbb{R})$ and $u(t) = 0$ for $t < 0$, then u is called a non-negative fuzzy number. We denote the set of all non-negative fuzzy numbers by $L^*(\mathbb{R})$. We can say that $u \in L^*(\mathbb{R})$ iff $u_\alpha^- \geq 0$ for each $\alpha \in [0,1]$. Clearly we have $\widetilde{0} \in L(\mathbb{R})$. For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ cl\{x \in \mathbb{R} : u(x) > 0\}, & \text{if } \alpha = 0. \end{cases}$$

A partial ordering $\preceq$ on $L(\mathbb{R})$ is defined by $u \preceq v$ if $u_\alpha^- \leq v_\alpha^-$ and $u_\alpha^+ \leq v_\alpha^+$ for all $\alpha \in [0,1]$.

Some arithmetic operations for α-level sets are defined as follows:

$u, v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$, $\alpha \in (0,1]$. Then,

$[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$, $[u \ominus v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-]$,

$[u \odot v]_\alpha = [u_\alpha^-.v_\alpha^-, u_\alpha^+.v_\alpha^+]$ and $\left[\tilde{1} \oslash u\right]_\alpha = \left[\frac{1}{u_\alpha^+}, \frac{1}{u_\alpha^-}\right]$, $u_\alpha^- > 0$.

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ defined as

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} \max\left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\}.$$

It is known that D is a metric on $L(\mathbb{R})$ and $(L(\mathbb{R}), D)$ is a complete metric space.

A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to the fuzzy number x_0, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $D(x_k, x_0) < \varepsilon$ for $k > k_0$ and a sequence $x = (x_k)$ of fuzzy numbers level-wise converges to x_0 iff $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^-$ and $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^+$, where $[x_k]_\alpha = \left[(x_k)_\alpha^-, (x_k)_\alpha^+\right]$ and $[x_0]_\alpha = \left[(x_0)_\alpha^-, (x_0)_\alpha^+\right]$, for every $\alpha \in (0,1)$.

Let X be a vector space over $\mathbb{R}$, $\|.\| : X \to L^*(\mathbb{R})$ and the mappings $L; R$ (respectively, left norm and right norm) : $[0,1] \times [0,1] \to [0,1]$ be symetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$.

The quadruple $(X, \|.\|, L, R)$ is called fuzzy normed linear space (briefly $(X, \|.\|)$ FNS) and $\|.\|$ a fuzzy norm if the following axioms are satisfied

(1) $\|x\| = \widetilde{0}$ iff $x = 0$,

(2) $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,

(3) For all $x, y \in X$
(a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$, whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s+t \leq \|x+y\|_1^-$,
(b) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$, whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s+t \geq \|x+y\|_1^-$.

In the sequel we take $L(p,q) = \min(p,q)$ and $R(p,q) = \max(p,q)$ for all $p, q \in [0,1]$. So, we get triangle inequality as $\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$ and $\|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$, for all $\alpha \in (0,1)$ and $x, y \in X$. Then, we say that $\|.\|_\alpha^-$ and $\|.\|_\alpha^+$ are norms in the usual sense on X.

Let $(X, \|.\|_C)$ be an ordinary normed linear space. Then, a fuzzy norm $\|.\|$ on X can be obtained by

$$\|x\|(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq a\|x\|_C \text{ or } t \geq b\|x\|_C \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a}, & a\|x\|_C \leq t \leq \|x\|_C \\ \frac{-t}{(b-1)\|x\|_C} + \frac{b}{b-1}, & \|x\|_C \leq t \leq b\|x\|_C \end{cases} \tag{1}$$

where $\|x\|_C$ is the ordinary norm of $x\ (\neq \theta)$, $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \widetilde{0}$. Hence, $(X, \|.\|)$ is a fuzzy normed linear space.

Let us consider the topological structure of an $FNS\ (X, \|.\|)$. For any $\varepsilon > 0, \alpha \in [0,1]$ and $x \in X$, the $(\varepsilon, \alpha)-$ neighborhood of x is the set $\mathcal{N}_x(\varepsilon, \alpha) = \{y \in X : \|x-y\|_\alpha^+ < \varepsilon\}$. Throughout the paper, we let $(X, \|.\|)$ be an FNS.

A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $L \in X$ with respect to the fuzzy norm on X and we denote by $x_n \overset{FN}{\to} L$ or $FN - \lim_{n\to\infty} x_n = L$, provided that $(D) - \lim_{n\to\infty} \|x_n - L\| = \widetilde{0}$; i.e., for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $D\left(\|x_n - L\|, \widetilde{0}\right) < \varepsilon$, for all $n \geq N(\varepsilon)$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$, $\sup_{\alpha\in[0,1]} \|x_n - L\|_\alpha^+ = \|x_n - L\|_0^+ < \varepsilon$.

If $K \subseteq \mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of K_n. The natural density of K is given by $d(K) = \lim_{n\to\infty} \frac{1}{n}|K_n|$, if it exists.

The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ we have $d(K(\varepsilon)) = 0$, where $K = K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. In this case, we write $st - \lim x = L$.

A double sequence $x = (x_{mn})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case, we shall write this as $\lim_{m,n\to\infty} x_{mn} = L$.

A double sequence $x = (x_{mn})$ is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$ for all $m, n \in \mathbb{N}$, that is, $\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty$. We let the set of all bounded double sequences by L_∞.

A double sequence (x_{mn}) is said to be convergent to $L \in X$ (in Pringsheim's sense) with respect to the fuzzy norm on X if for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that $D\left(\|x_{mn} - L\|, \widetilde{0}\right) < \varepsilon$, for all $m, n \geq N$. In this case, we write $x_{mn} \overset{FN}{\to} L$. This means that, for every $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$

such that $\sup_{\alpha\in[0,1]} \|x_{mn}-L\|_{\alpha}^{+} = \|x_{mn}-L\|_{0}^{+} < \varepsilon$, for all $m,n \geq N$. In terms of neighborhoods, we have $x_{mn} \xrightarrow{FN} L$ provided that for any $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that $x_{mn} \in \mathcal{N}_x(\varepsilon, 0)$, whenever $m,n \geq N$.

Let $K \subset \mathbb{N}\times\mathbb{N}$. Let K_{mn} be the number of $(j,k) \in K$ such that $j \leq m$, $k \leq n$. If the sequence $\left\{\frac{K_{mn}}{m.n}\right\}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by $d_2(K) = \lim_{m,n\to\infty} \frac{K_{mn}}{m.n}$.

A double sequence $x = (x_{mn})$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : |x_{mn} - L| \geq \varepsilon\}$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$, (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, (iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Let $\mathcal{I}$ is a nontrivial ideal in X, then $\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X\backslash A)\}$ is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout the paper we take $\mathcal{I}$ as an admissible ideal in $\mathbb{N}$.

If we take $\mathcal{I} = \mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$, then $\mathcal{I} = \mathcal{I}_d$ is a non-trivial admissible ideal of $\mathbb{N}$ and the ideal convergence coincides with statistical convergence with respect to the fuzzy norm on $\mathbb{N}$.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the property (AP), if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is also admissible.

Let $\mathcal{I}_2^0 = \{A \subset \mathbb{N}\times\mathbb{N} : (\exists m(A), (i,j) \geq m(A) \Rightarrow (i,j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.

If we take $\mathcal{I}_2 = \mathcal{I}_{d_2} = \{A \subset \mathbb{N}\times\mathbb{N} : d_2(A) = 0\}$, then $\mathcal{I}_2 = \mathcal{I}_{d_2}$ is a nontrivial strongly admissible ideal of $\mathbb{N}\times\mathbb{N}$ and the ideal convergence coincides with statistical convergence with respect to the fuzzy norm on $\mathbb{N}$.

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2), if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

A sequence $x = (x_m)_{m\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $L \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$, the set $A(\varepsilon) = \left\{m \in \mathbb{N} : \|x_m - L\|_0^{+} \geq \varepsilon\right\}$ belongs to $\mathcal{I}$. In this case, we write $x_m \xrightarrow{F\mathcal{I}} L$ or $F\mathcal{I} - \lim_{m\to\infty} x_m = L$. The element L is called the $\mathcal{I}$-limit of (x_m) in X.

A sequence (x_m) in X is said to be $\mathcal{I}^*$ convergent to L in X with respect to the fuzzy norm on X if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \cdots\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} \|x_{m_k} - L\| = 0$. In this case, we write $x_m \xrightarrow{F\mathcal{I}^*} L$ or $F\mathcal{I}^* - \lim_{m\to\infty} x_m = L$.

A sequence (x_m) in X is said to be $\mathcal{I}$-Cauchy with respect to the fuzzy norm on X if for every $\varepsilon > 0$, there exists an integer $n = n(\varepsilon)$ in $\mathcal{N}$ such that $\{m \in \mathcal{N} : \|x_m - x_n\|_0^+ \geq \varepsilon\} \in \mathcal{I}$.

A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$- convergent to $L \in X$ with respect to fuzzy norm on X if for every $\varepsilon > 0$, $A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - L\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2$. In this case, we write $x_{mn} \xrightarrow{F\mathcal{I}_2} L$ or $x_{mn} \to L\,(F\mathcal{I}_2)$ or $F\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L$.

A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2^*$-convergent to L in X with respect to the fuzzy norm on X if there exists a set $M \in \mathcal{F}(\mathcal{I}_2), M = \{m_1 < \cdots < m_k < \cdots ; n_1 < \cdots < n_l < \cdots\} \subset \mathbb{N}\times\mathbb{N}$ such that $\lim_{k,l\to\infty} \|x_{m_k n_l} - L\|$.

A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-Cauchy with respect to the fuzzy norm on X if for each $\varepsilon > 0$, there exists integers $s = s(\varepsilon)$ and $t = t(\varepsilon)$ such that $\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - x_{st}\|_0^+ \geq \varepsilon\right\} \in \mathcal{I}_2$.

A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2^*$-Cauchy double sequence with respect to fuzzy norm on X, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) and $k_0 = k_0(\varepsilon)$ such that for every $\varepsilon > 0$ and for $(m,n),(s,t) \in M$, $\|x_{mn} - x_{st}\|_0^+ < \varepsilon$, whenever $m,n,s,t > k_0$. In this case we write

$$\lim_{m,n,s,t\to\infty} \|x_{mn} - x_{st}\|_0^+ = 0.$$

Lemma 1.1. *[15] Let $(X, \|.\|)$ be a fuzzy normed space, (x_{mn}) be a double sequence in X and $L_1 \in X$. Then, $FP - \lim_{m,n\to\infty} x_{mn} = L_1 \Rightarrow F\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L_1$.*

Lemma 1.2. *[21] Let $(X, \|.\|)$ be a fuzzy normed space, $x = (x_{mn})$ be a double sequence in X and $L_1 \in X$. If $x = (x_{mn})$ is $\mathcal{I}_2^*$-convergent to L_1 then it is $\mathcal{I}_2$-convergent to L_1.*

Lemma 1.3. *[21] Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with property (AP2), $(X, \|.\|)$ be a fuzzy normed space, $x = (x_{mn})$ be a double sequence in X and $L_1 \in X$. If $x = (x_{mn})$ is $\mathcal{I}_2$-convergent to L_1 then it is $\mathcal{I}_2^*$-convergent to L_1.*

Lemma 1.4. *[16] Let $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N}\times\mathbb{N}$. If a double sequence (x_{mn}) in X is an $F\mathcal{I}_2^*$-Cauchy sequence, then it is $F\mathcal{I}_2$-Cauchy.*

Lemma 1.5. *[21] Let $(X, \|.\|)$ be a fuzzy normed space, $x = (x_{mn})$ be a double sequence in X. If $x = (x_{mn})$ is $\mathcal{I}_2$-convergent, then it is $\mathcal{I}_2$-Cauchy sequence in X.*

Lemma 1.6. *[31] Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}$ such that $P_i \in \mathcal{F}(\mathcal{I})$ for each i, where $\mathcal{F}(\mathcal{I})$ is a filter associated with a strongly admissible ideal $\mathcal{I}$ with the property (AP). Then, there exists a set $P \subset \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I})$ and the set $P\backslash P_i$ is finite for all i.*

Lemma 1.7. *[16] Let $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N}\times\mathbb{N}$ with the property $(AP2)$ and (x_{mn}) be a double sequence in X. Then, the concepts $\mathcal{I}_2-$Cauchy double sequence with respect to fuzzy norm on X and $\mathcal{I}_2^*$-Cauchy double sequence with respect to fuzzy norm on X coincide.*

2. Main Results

In this section, we introduce the notions of regularly $(\mathcal{I}_2, \mathcal{I})$-convergence, regularly $(\mathcal{I}_2^*, \mathcal{I}^*)$-convergence, regularly $(\mathcal{I}_2, \mathcal{I})$-Cauchy and regularly $(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy double sequences in fuzzy normed linear spaces. Also, we establish some basic results related to these notions.

Definition 2.1. A double sequence (x_{mn}) in X is said to be regularly convergent with respect to fuzzy norm on X, if it is convergent in Pringsheim's sense and the limits

$$FN - \lim_{m\to\infty} x_{mn},\ (n \in \mathbb{N}) \text{ and } FN - \lim_{n\to\infty} x_{mn},\ (m \in \mathbb{N}),$$

exist for each fixed $n \in \mathbb{N}$ and each fixed $m \in \mathbb{N}$, respectively. Note that if (x_{mn}) is regularly convergent to L in X, then the limits

$$FN - \lim_{n\to\infty} \lim_{m\to\infty} x_{mn} \text{ and } FN - \lim_{m\to\infty} \lim_{n\to\infty} x_{mn}$$

exist and are equal to L. In this case we write

$$Fr - \lim_{m,n\to\infty} x_{mn} = L \quad \text{or} \quad x_{mn} \xrightarrow{Fr} L.$$

In terms of neighborhoods, we have $x_{mn} \xrightarrow{Fr} L$ if for every $\varepsilon > 0$, there exists an integer $k = k_0(\varepsilon) \in \mathbb{N}$ such that $x_{mn} \in \mathcal{N}_L(\varepsilon, 0)$, whenever $m, n \geq k$, $x_{mn} \in \mathcal{N}_L(\varepsilon, 0)$, whenever $m \geq k$ and for each fixed $n \in \mathbb{N}$ and $x_{mn} \in \mathcal{N}_L(\varepsilon, 0)$, whenever $n \geq k$ and for each fixed $m \in \mathbb{N}$.

Definition 2.2. A double sequence (x_{mn}) in X is said to be regularly $(\mathcal{I}_2, \mathcal{I})$-convergent ($Fr(\mathcal{I}_2, \mathcal{I})$-convergent) with respect to fuzzy norm on X, if it is $F\mathcal{I}_2$-convergent in Pringsheim's sense and for each $\varepsilon > 0$, the following statements hold:

$$\{m \in \mathbb{N} : \|x_{mn} - L_n\|_0^+ \geq \varepsilon\} \in \mathcal{I} \tag{2}$$

for some $L_n \in X$ and each fixed $n \in \mathbb{N}$ and

$$\{n \in \mathbb{N} : \|x_{mn} - K_m\|_0^+ \geq \varepsilon\} \in \mathcal{I} \tag{3}$$

for some $K_m \in X$ and each fixed $m \in \mathbb{N}$.

If (x_{mn}) is $Fr(\mathcal{I}_2, \mathcal{I})$-convergent to $L \in X$, then the limits

$$F\mathcal{I} - \lim_{n\to\infty} \lim_{m\to\infty} x_{mn} \text{ and } F\mathcal{I} - \lim_{m\to\infty} \lim_{n\to\infty} x_{mn}$$

exist and are equal to L. In this case we write

$$Fr(\mathcal{I}_2, \mathcal{I}) - \lim_{m,n\to\infty} x_{mn} = L \quad \text{or} \quad x_{mn} \xrightarrow{Fr(\mathcal{I}_2, \mathcal{I})} L.$$

In terms of neighborhoods, we have $x_{mn} \xrightarrow{Fr(\mathcal{I}_2, \mathcal{I})} L$ if for every $\varepsilon > 0$,

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : x_{mn} \notin \mathcal{N}_L(\varepsilon, 0)\} \in \mathcal{I}_2$$

and

$$\{m \in \mathbb{N} : x_{mn} \notin \mathcal{N}_L(\varepsilon, 0)\} \in \mathcal{I} \text{ and } \{n \in \mathbb{N} : x_{mn} \notin \mathcal{N}_L(\varepsilon, 0)\} \in \mathcal{I}$$

for each fixed $n \in \mathbb{N}$ and each fixed $m \in \mathbb{N}$, respectively.

A useful interpretation of the above definition is the following;

$$x_{mn} \xrightarrow{Fr(\mathcal{I}_2, \mathcal{I})} L \Leftrightarrow F\mathcal{I}_2 - \lim_{m,n\to\infty} \|x_{mn} - L\|_0^+ = 0,$$

$$F\mathcal{I}-\lim_{m\to\infty}\|x_{mn}-L\|_0^+=0,\ \text{(for each fixed } n\in\mathbb{N})$$

and

$$F\mathcal{I}-\lim_{n\to\infty}\|x_{mn}-L\|_0^+=0,\ \text{(for each fixed } m\in\mathbb{N}).$$

Note that $Fr(\mathcal{I}_2,\mathcal{I})-\lim_{m,n\to\infty}\|x_{mn}-L\|_0^+=0$ implies that

$$F\mathcal{I}_2-\lim_{m,n\to\infty}\|x_{mn}-L\|_\alpha^-=F\mathcal{I}_2-\lim_{m,n\to\infty}\|x_{mn}-L\|_\alpha^+=0,$$

$$F\mathcal{I}-\lim_{m\to\infty}\|x_{mn}-L\|_\alpha^-=F\mathcal{I}_2-\lim_{m\to\infty}\|x_{mn}-L\|_\alpha^+=0,\ \text{(for each fixed } n\in\mathbb{N})$$

and

$$F\mathcal{I}-\lim_{n\to\infty}\|x_{mn}-L\|_\alpha^-=F\mathcal{I}_2-\lim_{n\to\infty}\|x_{mn}-L\|_\alpha^+=0,\ \text{(for each fixed } m\in\mathbb{N})$$

for each $\alpha\in[0,1]$, since

$$0\le\|x_{mn}-L\|_\alpha^-\le\|x_{mn}-L\|_\alpha^+\le\|x_{mn}-L\|_0^+,\ \text{(for each } m,n\in\mathbb{N}),$$

$$0\le\|x_{mn}-L\|_\alpha^-\le\|x_{mn}-L\|_\alpha^+\le\|x_{mn}-L\|_0^+,\ \text{(for each } m\in\mathbb{N} \text{ and fixed } n\in\mathbb{N})$$

and

$$0\le\|x_{mn}-L\|_\alpha^-\le\|x_{mn}-L\|_\alpha^+\le\|x_{mn}-L\|_0^+,\ \text{(for each } n\in\mathbb{N} \text{ and fixed } m\in\mathbb{N})$$

holds for each $\alpha\in[0,1]$.

Example 2.1. Let $\mathcal{I}=\mathcal{I}_d$, $\mathcal{I}_2=\mathcal{I}_{d_2}$, $(\mathbb{R}^m,\|.\|)$ be a FNS and $(x_{kn})_{k,n=1}^m\in\mathbb{R}^m$ be a fixed nonzero vector, where the fuzzy norm on $\mathbb{R}^m$ is defined as in (1) such that $\|x\|_C=\left(\sum_{k=1}^m\sum_{n=1}^m|x_{kn}|^2\right)^{1/2}$. Now we define the double sequence (x_{kn}) in $\mathbb{R}^m$ as

$$x_{kn}=\begin{cases} n, & if\ k\le 2\\ x, & if\ n=k=j^2,\ j\in\mathbb{N}\ and\ k\ge 3\\ \theta, & otherwise.\end{cases}$$

It is clear that for any ε satisfying $0<\varepsilon\le b\|x\|_C$, where $1<b<\infty$. Then, for $k\ge3$ we have

$$K(\varepsilon)=\{(n,k)\in\mathbb{N}\times\mathbb{N}:\|x_{nk}-\theta\|_0^+\ge\varepsilon\}=\{(9,9),(16,16),\cdots\},$$

$$K_1(\varepsilon)=\{n\in\mathbb{N}:\|x_{nk}-\theta\|_0^+\ge\varepsilon\}=\{9,16,\cdots\},$$

for each $k\in\mathbb{N}$ and

$$K_2(\varepsilon)=\{k\in\mathbb{N}:\|x_{nk}-\theta\|_0^+\ge\varepsilon\}=\{9,16,\cdots\}$$

for each $n\in\mathbb{N}$ and so, $d_2(K(\varepsilon))=0$, $d(K_1(\varepsilon))=0$ and $d(K_2(\varepsilon))=0$. If we choose $\varepsilon>b\|x\|_C$ then $K(\varepsilon)=\emptyset$, $K_1(\varepsilon)=\emptyset$ and $K_2(\varepsilon)=\emptyset$ and so, $d_2(K(\varepsilon))=0$, $d(K_1(\varepsilon))=0$ and $d(K_2(\varepsilon))=0$. It is clear that (x_{kn}) is $\mathcal{I}_2$-convergent to 0 but (x_{kn}) is not $Fr(\mathcal{I}_2,\mathcal{I})$-convergent in $(\mathbb{R}^m,\|.\|)$.

Theorem 2.1. *If a double sequence (x_{mn}) in X is Fr-convergent, then (x_{mn}) is $Fr(\mathcal{I}_2,\mathcal{I})$-convergent.*

Proof. Let (x_{mn}) be any double sequence in X and suppose that (x_{mn}) be Fr-convergent. Then, (x_{mn}) is convergent in Pringsheim's sense and the limits

$$FN-\lim_{m\to\infty} x_{mn},\ (n\in\mathbb{N}) \text{ and } FN-\lim_{n\to\infty} x_{mn},\ (m\in\mathbb{N}),$$

exist for each fixed $n\in\mathbb{N}$ and each fixed $m\in\mathbb{N}$, respectively. By Lemma 1.1, (x_{mn}) is $\mathcal{I}_2$-convergent. Also, for each $\varepsilon>0$ there exist $m=m_0(\varepsilon)$ and $n=n_0(\varepsilon)$ such that for all $m>m_0$

$$\|x_{mn}-L_n\|_0^+<\varepsilon,$$

for some L_n and each fixed $n\in\mathbb{N}$ and also, for all $n>n_0$

$$\|x_{mn}-K_m\|_0^+<\varepsilon,$$

for some K_m and each fixed $m\in\mathbb{N}$. Then, since $\mathcal{I}$ is an admissible ideal so for each $\varepsilon>0$, we have

$$\{m\in\mathbb{N}:\|x_{mn}-L_n\|_0^+\geq\varepsilon\}\subset\{1,2,\ldots,m_0\}\in\mathcal{I},$$

$$\{n\in\mathbb{N}:\|x_{mn}-K_m\|_0^+\geq\varepsilon\}\subset\{1,2,\ldots,n_0\}\in\mathcal{I}.$$

Hence, (x_{mn}) is $Fr(\mathcal{I}_2,\mathcal{I})$-convergent in X. □

The opposite of this theorem is not always true. Let's see this with an example.

Example 2.2. Let $\mathcal{I}=\mathcal{I}_d$, $\mathcal{I}_2=\mathcal{I}_{d_2}$, $(\mathbb{R}^m,\|.\|)$ be a FNS and $(x_{kn})_{k,n=1}^m\in\mathbb{R}^m$ be a fixed nonzero vector, where the fuzzy norm on $\mathbb{R}^m$ is defined as in (1) such that $\|x\|_C=\left(\sum\limits_{k=1}^m\sum\limits_{n=1}^m|x_{kn}|^2\right)^{1/2}$. Now we define a double sequence (x_{kn}) in $\mathbb{R}^m$ as

$$x_{kn}=\begin{cases} x, & if\ n,k=j^3, j\in\mathbb{N}\\ \theta, & otherwise.\end{cases}$$

It is clear that for any ε satisfying $0<\varepsilon\leq b\|x\|_C$, where $1<b<\infty$. Then, we have

$$K(\varepsilon)=\{(n,k)\in\mathbb{N}\times\mathbb{N}:\|x_{nk}-\theta\|_0^+\geq\varepsilon\}=\{(1,1),(8,8),(27,27),\cdots\},$$

$$K_1(\varepsilon)=\{n\in\mathbb{N}:\|x_{nk}-\theta\|_0^+\geq\varepsilon\}=\{1,8,27,\cdots\},$$

for each $k\in\mathbb{N}$ and

$$K_2(\varepsilon)=\{k\in\mathbb{N}:\|x_{nk}-\theta\|_0^+\geq\varepsilon\}=\{1,8,27,\cdots\}$$

for each $n\in\mathbb{N}$ and so, $d_2(K(\varepsilon))=0$, $d(K_1(\varepsilon))=0$ and $d(K_2(\varepsilon))=0$. If we choose $\varepsilon>b\|x\|_C$ then $K(\varepsilon)=\emptyset$, $K_1(\varepsilon)=\emptyset$ and $K_2(\varepsilon)=\emptyset$ and so, $d_2(K(\varepsilon))=0$, $d(K_1(\varepsilon))=0$ and $d(K_2(\varepsilon))=0$. Hence, (x_{kn}) is $Fr(\mathcal{I}_2,\mathcal{I})$-convergent in $(\mathbb{R}^m,\|.\|)$. But (x_{kn}) is not Fr-convergent in $(\mathbb{R}^m,\|.\|)$.

Definition 2.3. A double sequence (x_{mn}) in X is said to be $Fr(\mathcal{I}_2^*,\mathcal{I}^*)$-convergent with respect to fuzzy norm on X, if there exist the sets $M\in\mathcal{F}(\mathcal{I}_2)$, $M_1\in\mathcal{F}(\mathcal{I})$ and $M_2\in\mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N}\times\mathbb{N}\setminus M\in\mathcal{I}_2$, $\mathbb{N}\setminus M_1\in\mathcal{I}$ and $\mathbb{N}\setminus M_2\in\mathcal{I}$) such that the limits

$$FN-\lim_{\substack{m,n\to\infty\\(m,n)\in M}} x_{mn},\quad FN-\lim_{\substack{m\to\infty\\m\in M_1}} x_{mn} \text{ and } FN-\lim_{\substack{n\to\infty\\n\in M_2}} x_{mn}$$

exist for each fixed $n\in\mathbb{N}$ and each fixed $m\in\mathbb{N}$, respectively.

Theorem 2.2. *If a double sequence* (x_{mn}) *in* X *is* $Fr(\mathcal{I}_2^*, \mathcal{I}^*)$*-convergent, then it is* $Fr(\mathcal{I}_2, \mathcal{I})$*-convergent.*

Proof. Let (x_{mn}) in X be $Fr(\mathcal{I}_2^*, \mathcal{I}^*)$-convergent. Then, it is $\mathcal{I}_2^*$-convergent and so, by Lemma 1.2, it is $\mathcal{I}_2$-convergent. Also, there exist the sets $M_1, M_2 \in \mathcal{F}(\mathcal{I})$ such that

$$(\forall \varepsilon > 0)\ (\exists m_0 = m_0(\varepsilon) \in \mathbb{N})\ (\forall m \geq m_0)\ (m \in M_1)\ \|x_{mn} - L_n\|_0^+ < \varepsilon,\ (n \in \mathbb{N})$$

for some $L_n \in X$ and

$$(\forall \varepsilon > 0)\ (\exists n_0 = n_0(\varepsilon) \in \mathbb{N})\ (\forall n \geq n_0)\ (n \in M_2)\ \|x_{mn} - K_m\|_0^+ < \varepsilon,\ (m \in \mathbb{N})$$

for some $K_m \in X$. Hence, for each $\varepsilon > 0$ we have

$$\begin{aligned} A(\varepsilon) &= \{m \in \mathbb{N} : \|x_{mn} - L_n\|_0^+ \geq \varepsilon\} \subset H_1 \cup \{1, 2, \ldots, m_0 - 1\},\ (n \in \mathbb{N}), \\ B(\varepsilon) &= \{n \in \mathbb{N} : \|x_{mn} - K_m\|_0^+ \geq \varepsilon\} \subset H_2 \cup \{1, 2, \ldots, n_0 - 1\},\ (m \in \mathbb{N}), \end{aligned}$$

for $H_1, H_2 \in \mathcal{I}$. Since $\mathcal{I}$ is an admissible ideal we get

$$H_1 \cup \{1, 2, \ldots, (m_0 - 1)\} \in \mathcal{I},\ \ H_2 \cup \{1, 2, \ldots, n_0 - 1\} \in \mathcal{I}$$

and therefore $A(\varepsilon), B(\varepsilon) \in \mathcal{I}$. This shows that the double sequence (x_{mn}) is $Fr(\mathcal{I}_2, \mathcal{I})$-convergent in X. □

Theorem 2.3. *Let* $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ *be a strongly admissible ideal with property (AP2),* $\mathcal{I} \subset 2^{\mathbb{N}}$ *be an admissible ideal with property (AP). If a double sequence* (x_{mn}) *is* $Fr(\mathcal{I}_2, \mathcal{I})$*-convergent, then* (x_{mn}) *is* $Fr(\mathcal{I}_2^*, \mathcal{I}^*)$*-convergent in* X*.*

Proof. Let a double sequence (x_{mn}) in X be $Fr(\mathcal{I}_2, \mathcal{I})$-convergent. Then, (x_{mn}) is $\mathcal{I}_2$-convergent and so (x_{mn}) is $\mathcal{I}_2^*$-convergent by Lemma 1.3. Also, for each $\varepsilon > 0$ we have

$$A(\varepsilon) = \{m \in \mathbb{N} : \|x_{mn} - L_n\|_0^+ \geq \varepsilon\} \in \mathcal{I}$$

for some $L_n \in X$ and for each fixed $n \in \mathbb{N}$ and

$$C(\varepsilon) = \{n \in \mathbb{N} : \|x_{mn} - K_m\|_0^+ \geq \varepsilon\} \in \mathcal{I}$$

for some $K_m \in X$ and for each fixed $m \in \mathbb{N}$.

Now put

$$\begin{aligned} A_1 &= \{m \in \mathbb{N} : \|x_{mn} - L_n\|_0^+ \geq 1\}, \\ A_k &= \left\{m \in \mathbb{N} : \frac{1}{k} \leq \|x_{mn} - L_n\|_0^+ < \frac{1}{k-1}\right\} \end{aligned}$$

for $k \geq 2$, for some $L_n \in X$ and for each fixed $n \in \mathbb{N}$. It is clear that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i \in \mathcal{I}$ for each $i \in \mathbb{N}$. By the property (AP) there is a countable family of sets $\{B_1, B_2, \ldots\}$ in $\mathcal{I}$ such that $A_j \triangle B_j$ is a finite set for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

We prove that

$$FN - \lim_{\substack{m\to\infty \\ m\in M}} x_{mn} = L_n,$$

for some L_n, each fixed $n \in \mathbb{N}$ and $M = \mathbb{N}\backslash B \in \mathcal{F}(\mathcal{I})$. Let $\delta > 0$ be given. Choose $k \in \mathbb{N}$ such that $1/k < \delta$. Then, we have

$$\{m \in \mathbb{N} : \|x_{mn} - L_n\|_0^+ \geq \delta\} \subset \bigcup_{j=1}^{k} A_j,$$

for some L_n and each fixed $n \in \mathbb{N}$. Since $A_j \triangle B_j$ is a finite set for $j \in \{1, 2, \dots, k\}$, there exists $m_0 \in \mathbb{N}$ such that

$$\Big(\bigcup_{j=1}^{k} B_j\Big) \cap \{m : m \geq m_0\} = \Big(\bigcup_{j=1}^{k} A_j\Big) \cap \{m : m \geq m_0\}.$$

If $m \geq m_0$ and $m \notin B$ then

$$m \notin \bigcup_{j=1}^{k} B_j \text{ and so } m \notin \bigcup_{j=1}^{k} A_j.$$

Thus, we have $\|x_{mn} - L_n\|_0^+ < \frac{1}{k} < \delta$, for some L_n and each fixed $n \in \mathbb{N}$. This implies that

$$FN - \lim_{\substack{m\to\infty \\ m\in M}} x_{mn} = L_n.$$

Hence, we have

$$F\mathcal{I}^* - \lim_{m\to\infty} x_{mn} = L_n$$

for some L_n and each fixed $n \in \mathbb{N}$.

Similarly, for the set $C(\varepsilon) = \{n \in \mathbb{N} : \|x_{mn} - K_m\|_0^+ \geq \varepsilon\} \in \mathcal{I}$, we have

$$F\mathcal{I}^* - \lim_{n\to\infty} x_{mn} = K_m$$

for some K_m and each fixed $m \in \mathbb{N}$. Hence, a double sequence (x_{mn}) is $Fr(\mathcal{I}_2^*, \mathcal{I}^*)$-convergent. □

Now, we give the definitions of $Fr(\mathcal{I}_2, \mathcal{I})$-Cauchy sequence and $Fr(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy sequence.

Definition 2.4. A double sequence (x_{mn}) in X is said to be regularly $(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence with respect to fuzzy norm on X ($Fr(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence), if it is $\mathcal{I}_2$-Cauchy double sequence with respect to fuzzy norm on X and for each $\varepsilon > 0$ there exist $k_n = k_n(\varepsilon) \in \mathbb{N}$ and $l_m = l_m(\varepsilon) \in \mathbb{N}$ such that the following statements hold:

$$\begin{aligned} A_1(\varepsilon) &= \{m \in \mathbb{N} : \|x_{mn} - x_{k_n n}\|_0^+ \geq \varepsilon\} \in \mathcal{I}, \ \ (n \in \mathbb{N}), \\ A_2(\varepsilon) &= \{n \in \mathbb{N} : \|x_{mn} - x_{m l_m}\|_0^+ \geq \varepsilon\} \in \mathcal{I}, \ \ (m \in \mathbb{N}). \end{aligned}$$

Theorem 2.4. *If a double sequence (x_{mn}) in X is $Fr(\mathcal{I}_2, \mathcal{I})$-convergent, then (x_{mn}) is $Fr(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence.*

Proof. Let (x_{mn}) be a $Fr(\mathcal{I}_2, \mathcal{I})$-convergent double sequence in X. Then, (x_{mn}) is $\mathcal{I}_2$-convergent and by Lemma 1.5, it is $\mathcal{I}_2$-Cauchy double sequence. Also for each $\varepsilon > 0$, we have

$$A_1\Big(\frac{\varepsilon}{2}\Big) = \Big\{m \in \mathbb{N} : \|x_{mn} - L_n\|_0^+ \geq \frac{\varepsilon}{2}\Big\} \in \mathcal{I}$$

for some L_n and each fixed $n \in \mathbb{N}$ and also

$$A_2\Big(\frac{\varepsilon}{2}\Big) = \Big\{n \in \mathbb{N} : \|x_{mn} - K_m\|_0^+ \geq \frac{\varepsilon}{2}\Big\} \in \mathcal{I}$$

for some K_m and each fixed $m \in \mathbb{N}$. Since $\mathcal{I}$ is an admissible ideal, the sets

$$A_1^c\Big(\frac{\varepsilon}{2}\Big) = \Big\{m \in \mathbb{N} : \|x_{mn} - L_n\|_0^+ < \frac{\varepsilon}{2}\Big\}, \ (n \in \mathbb{N})$$

for some L_n and

$$A_2^c\Big(\frac{\varepsilon}{2}\Big) = \Big\{n \in \mathbb{N} : \|x_{mn} - K_m\|_0^+ < \frac{\varepsilon}{2}\Big\},\ (m \in \mathbb{N})$$

for some K_m, are nonempty and belong to $\mathcal{F}(\mathcal{I})$. For $k_n \in A_1^c(\frac{\varepsilon}{2})$, ($n \in \mathbb{N}$ and $k_n > 0$) we have

$$\|x_{k_n n} - L_n\|_0^+ < \frac{\varepsilon}{2},$$

for some L_n. Now, for each $\varepsilon > 0$, we define the set

$$B_1(\varepsilon) = \{m \in \mathbb{N} : \|x_{mn} - x_{k_n n}\|_0^+ \geq \varepsilon\},\ \ (n \in \mathbb{N}),$$

where $k_n = k_n(\varepsilon) \in \mathbb{N}$. Let $m \in B_1(\varepsilon)$. Since $\|.\|_0^+$ is a norm in the usual sense, then for $k_n \in A_1^c(\frac{\varepsilon}{2})$, ($n \in \mathbb{N}$ and $k_n > 0$) we have

$$\begin{aligned}\varepsilon \leq \|x_{mn} - x_{k_n n}\|_0^+ &\leq \|x_{mn} - L_n\|_0^+ + \|x_{k_n n} - L_n\|_0^+ \\ &< \|x_{mn} - L_n\|_0^+ + \frac{\varepsilon}{2},\end{aligned}$$

for some L_n. This shows that

$$\frac{\varepsilon}{2} < \|x_{mn} - L_n\|_0^+ \text{ and so } m \in A_1\left(\frac{\varepsilon}{2}\right).$$

Hence, we have $B_1(\varepsilon) \subset A_1(\frac{\varepsilon}{2})$.

Similarly, for each $\varepsilon > 0$ and for $l_m \in A_2^c(\frac{\varepsilon}{2})$ ($m \in \mathbb{N}$ and $l_m > 0$) we have

$$\|x_{ml_m} - K_m\|_0^+ < \frac{\varepsilon}{2},\ (m \in \mathbb{N})$$

for some K_m. Therefore, it can be seen that

$$B_2(\varepsilon) = \{m \in \mathbb{N} : \|x_{ml_m} - K_m\|_0^+ \geq \varepsilon\} \subset A_2(\frac{\varepsilon}{2}).$$

Hence, we have $B_1(\varepsilon), B_2(\varepsilon) \in \mathcal{I}$. This shows that (x_{mn}) is $Fr(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence. □

Definition 2.5. A double sequence (x_{mn}) is said to be regularly $(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy double sequence with respect to fuzzy norm on X ($Fr(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy double sequence), if there exist the sets $M \in \mathcal{F}(\mathcal{I}_2)$, $M_1 \in \mathcal{F}(\mathcal{I})$ and $M_2 \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N}\times\mathbb{N} \setminus M \in \mathcal{I}_2$, $\mathbb{N} \setminus M_1 \in \mathcal{I}$ and $\mathbb{N} \setminus M_2 \in \mathcal{I}$), for each $\varepsilon > 0$ there exist $N = N(\varepsilon)$, $s = s(\varepsilon)$, $t = t(\varepsilon)$, $(s,t) \in M$, $k_n = k_n(\varepsilon)$, $l_m = l_m(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned}&\|x_{mn} - x_{st}\|_0^+ < \varepsilon,\ \text{ for } (m,n),(s,t) \in M,\\ &\|x_{mn} - x_{k_n n}\|_0^+ < \varepsilon,\ \text{ for each } m \in M_1 \text{ and each fixed } n \in \mathbb{N},\\ &\|x_{mn} - x_{ml_m}\|_0^+ < \varepsilon,\ \text{ for each } n \in M_2 \text{and each fixed } m \in \mathbb{N},\end{aligned}$$

whenever $m, n, s, t, k_n, l_m \geq N$.

Theorem 2.5. *If a double sequence (x_{mn}) in X is $Fr(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy double sequence, then it is $Fr(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence.*

Proof. Since a double sequence (x_{mn}) in X is $Fr(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy double sequence, it is $\mathcal{I}_2^*$-Cauchy double sequence. We know that $\mathcal{I}_2^*$-Cauchy double sequence implies $\mathcal{I}_2$-Cauchy double sequence by Lemma 1.4. Also, there exist the sets $M_1, M_2 \in \mathcal{F}(\mathcal{I})$ and for each $\varepsilon > 0$ there exist $k_n = k_n(\varepsilon) \in \mathbb{N}$ and $l_m = l_m(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned}\|x_{mn} - x_{k_n n}\|_0^+ &< \varepsilon,\ \text{ for each } m \in M_1 \text{ and each fixed } n \in \mathbb{N},\\ \|x_{mn} - x_{ml_m}\|_0^+ &< \varepsilon,\ \text{ for each } n \in M_2 \text{ and each fixed } m \in \mathbb{N},\end{aligned}$$

for $N = N(\varepsilon) \in \mathbb{N}$ and $m, n, k_n, l_m \geq N$.

Therefore, for $H_1 = \mathbb{N}\backslash M_1 \in \mathcal{I}$ and $H_2 = \mathbb{N}\backslash M_2 \in \mathcal{I}$ we have

$$A_1(\varepsilon) = \{m \in \mathbb{N} : \|x_{mn} - x_{k_n n}\|_0^+ \geq \varepsilon\} \subset H_1 \cup \{1, 2, \ldots, N-1\}, \ (n \in \mathbb{N})$$

for $m \in M_1$ and

$$A_2(\varepsilon) = \{n \in \mathbb{N} : \|x_{mn} - x_{ml_m}\|_0^+ \geq \varepsilon\} \subset H_2 \cup \{1, 2, \ldots, N-1\}, \ (m \in \mathbb{N})$$

for $n \in M_2$. Since $\mathcal{I}$ is an admissible ideal,

$$H_1 \cup \{1, 2, \ldots, N-1\} \in \mathcal{I} \text{ and } H_2 \cup \{1, 2, \ldots, N-1\} \in \mathcal{I}.$$

Hence, we have $A_1(\varepsilon), A_2(\varepsilon) \in \mathcal{I}$ and (x_{mn}) is $Fr(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence. □

Theorem 2.6. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with property (AP2), $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal with property (AP). If a double sequence (x_{mn}) in X is $Fr(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence, then it is $Fr(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy double sequence.*

Proof. Since $Fr(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence, it is $\mathcal{I}_2$-Cauchy double sequence. We know that $\mathcal{I}_2$-Cauchy double sequence implies $\mathcal{I}_2^*$-Cauchy double sequence by Lemma 1.7. Also, for every $\varepsilon > 0$ there exist $k_n = k_n(\varepsilon) \in \mathbb{N}$ and $l_m = l_m(\varepsilon) \in \mathbb{N}$ such that the following statements hold:

$$\begin{aligned} A_1(\varepsilon) &= \{m \in \mathbb{N} : \|x_{mn} - x_{k_n n}\|_0^+ \geq \varepsilon\} \in \mathcal{I}, \ (n \in \mathbb{N}), \\ A_2(\varepsilon) &= \{n \in \mathbb{N} : \|x_{mn} - x_{ml_m}\|_0^+ \geq \varepsilon\} \in \mathcal{I}, \ (m \in \mathbb{N}). \end{aligned}$$

Let

$$P_i = \left\{ m \in \mathbb{N} : \|x_{mn} - x_{k_{n_i} n}\|_0^+ < \frac{1}{i} \right\}; \ (i = 1, 2, \ldots)$$

and

$$R_i = \left\{ n \in \mathbb{N} : \|x_{mn} - x_{ml_{m_i}}\|_0^+ < \frac{1}{i} \right\}; \ (i = 1, 2, \ldots),$$

where $k_{n_i} = k_n(1\backslash i)$ and $l_{m_i} = l_m(1\backslash i)$. It is clear that $P_i, R_i \in \mathcal{F}(\mathcal{I})$, $(i = 1, 2, \ldots)$. Since $\mathcal{I}$ has the property (AP), then by Lemma 1.6 there exist the sets $P, R \subset \mathbb{N}$ such that $P, R \in \mathcal{F}(\mathcal{I})$ and $P\backslash P_i$ and $R\backslash R_i$ are finite for all i. Now, firstly we show that for every $\varepsilon > 0$,

$$\|x_{mn} - x_{k_n n}\|_0^+ < \varepsilon, \text{ for each } m \in P \text{ and each fixed } n \in \mathbb{N}.$$

To prove this, let $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $j > 2/\varepsilon$. If $m \in P$ then $P\backslash P_i$ is a finite set, so there exists $k = k(j)$ such that $m \in P_j$ for all $m, k_n > k(j)$. Therefore,

$$\|x_{mn} - x_{k_{n_i} n}\|_0^+ < \frac{1}{j} \ \textit{and} \ \|x_{k_n n} - x_{k_{n_i} n}\|_0^+ < \frac{1}{j}$$

for all $m, n, k_n > k(j)$. Since $\|.\|_0^+$ is a norm in the usual sense, then it follows that

$$\begin{aligned} \|x_{mn} - x_{k_n n}\|_0^+ &\leq \|x_{mn} - x_{k_{n_i} n}\|_0^+ + \|x_{k_n n} - x_{k_{n_i} n}\|_0^+ \\ &< \frac{1}{j} + \frac{1}{j} = \frac{2}{j} < \varepsilon \end{aligned}$$

for all $m, n, k_n > k(j)$. Thus, for any $\varepsilon > 0$ there exists $k = k(\varepsilon)$ such that for $m, n, k_n > k(\varepsilon)$

$$\|x_{mn} - x_{k_n n}\|_0^+ < \varepsilon, \text{ for each } m \in P \text{ and each fixed } n \in \mathbb{N}.$$

Similarly, we can show that for any $\varepsilon > 0$ there exists $l = l(\varepsilon)$ such that for $m, n, l_m > l(\varepsilon)$

$$\|x_{mn} - x_{ml_m}\|_0^+ < \varepsilon, \quad \text{for each} \quad n \in R \quad \text{and each fixed } m \in \mathbb{N}.$$

This shows that the sequence (x_{mn}) is an $\mathcal{I}_2^*$-Cauchy double sequence. □

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