Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 10 Issue 1(2018), Pages 38-54.

EXISTENCE RESULTS FOR SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES

GHAUS UR RAHMAN

Abstract. The paper explores sufficient conditions for the existence of weak solution of the following fractional order differential equations in abstract spaces,

$$\left(D_w^{\alpha_m} - \sum_{i=1}^{m-1} a_i D^{\alpha_i}\right) u(t) = f(t, u(t)) \text{ for t} \in [0,1], \text{ u}(0) = 0, \qquad (1)$$

where $D_w^{\alpha_m} u(\cdot)$ and $D_w^{\alpha_i} u(\cdot)$ are weak fractional Caputo derivatives of the function $u(\cdot) : [0,1] \to E$ of order α_m and α_i, $i = 1, 2, \ldots, m-1,$ respectively. The function $f(t, \cdot) : [0,1] \times E \to E$ is weakly-weakly sequentially continuous for every $t \in [0,1]$ and $f(\cdot, y(\cdot))$ is Pettis integrable for every weakly absolutely continuous function $y(\cdot) : [0,1] \to E$ and E is a non-reflexive Banach space, $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m < 1$ and $a_1, a_2 \ldots a_{m-1}$ are real numbers such that $a := \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} < 1$.

1. Introduction

In the past few years fractional differential equations have fascinated scientists by virtue of its various applications in many branches of sciences e.g physics, chemistry, mathematical biology, fluid dynamics etc. The readers who are engrossed in details of the subject, we refer to, [19, 31, 33]. The distinguishing features of fractional differential equations in that it out line memory and transmitted properties of numerous mathematical models. As a fact, that fractional order models are more realistic and practical than the classical integer order models [21, 27, 28, 31, 32, 33].

The fractional differential equation bearing more than one differential operators is known as multi-term fractional differential equation. Many researchers have intensively studied these types of equations and discussed existence and uniqueness of the solutions of multi-term fractional differential equations [1, 2, 8, 15, 17, 22, 23]. These equations have attracted the attentions of scientists, as they can model a range of physical phenomena which cannot be governed by single term fractional differential equations. Such types of equations can model propagation of mechanical

---

2010 *Mathematics Subject Classification.* 26A33, 47H40, 60H25.

Key words and phrases. weak solution; Riemann-Pettis integral; fractional differential equations; weak measure of non-compactness; fixed point theorem.

Submitted August 31, 2016. Published December 20, 2017.

G. Rahman was supported by the grant No:21-190/SRGP/R&D/HEC/2014 from HEC, Pakistan.

Communicated by I. Stravroulakis.

waves in visco-elastic media [28], a non-markovian diffusion process with memory [29], charge transport in amorphous semiconductors [39]. Presently Gejji in [16], studied some aspects of multi-term fractional diffusion-wave equation.
The study of fractional differential equations in Banach spaces has been taken in two different ways. The first one is to impose compactness conditions that only guarantee the existence of solution. The other approach is to consider dissipative type conditions which ensures the existence as well as uniqueness.
Different researchers have adopted various types of methods for the existence of solution of multi-term fractional differential equations. In [8, 15, 17], the authors used Kransnoselskii's fixed point theorem and the technique associated with the measure of non-compactness. Using Riemann-Liouville fractional derivative the author in [22], exhibited the existence of monotonic solution for the multi-term fractional differential equations in Banach spaces. Moreover, no compactness conditions is assumed on the non-linearity of the input function. In [25], the author studied the existence of weak solution of the Cauchy problem in reflexive Banach spaces equipped with weak topology, the author imposed weak-weak continuity assumption on f. Similarly in [23], the author exhibited the existence of global monotonic solution for the Cauchy problem, and assumed the function f to be Caratheodory which has linear growth. Recently Agarwal et al. in [2], exhibited the existence of solution of multi term fractional differential equations in non-reflexive Banach spaces. Furthermore, the authors used the techniques associated with weak measure of non-compactness and fractional Pettis and fractional Pseudo derivatives. Knowing the nature of the integral used the authors found that the solution is fractional Pseudo differentiable function.

Many people have worked on the existence of weak solution of ordinary differential equations e.g see [10]. The weak measure of non-compactness was introduced by De Blasi [18], and it was used by Cramer, Lakshmitantham and Mitchell [12] and obtained an existence result for weak solutions of Cauchy problem in non-reflexive Banach spaces. The authors imposed weak compactness type conditions in term of the measure of weak non-compactness. Moreover, for existence and uniqueness of solution, the authors imposed weak dissipative type condition. Using these existence results and partial ordering induced by cones, existence of extremal solutions and comparison results are also proved by the authors. Considering the weak measure of non-compactness, some authors have imposed and generalized previous results. We refer to [10, 37]. Some researchers have exhibited the solutions of fractional differential equations in Banach spaces using weak topology(see [3, 9]).
Inspired by the aforementioned work, the goal of the present research is to exhibit weak solution, of multi-term fractional differential equation in non-reflexive Banach spaces under the hypothesis that the given input function is weakly-weakly continuous. The main tools we use are Riemann-Pettis integral, fractional Caputo derivative and weak measure of non-compactness. The solution exhibited in this paper is fractional Caputo weakly differentiable function [13]. At the end of the paper we will provide the conclusion of the results and particular cases of our work.
This paper is organized as follows: In Section 2, we provide preliminaries result to facilitate the readers. In Section 3, we present main results about the existence of solution and related results.

2. Preliminaries

In order to make the manuscript a comprehensive note here we provide some significant and main characteristics of Riemann-Pettis integrable, weakly-weakly continuous functions, weak and strong derivatives and will point out some properties of non-compactness measure. Furthermore, we present the notations, definitions preliminary results about fractional calculus in the abstract spaces, using weak derivative and Riemann Pettis integral. For more detail about this section, the readers may see the monographs([37, 38]), we assume E to be Banach spaces with E^* be its topological dual and $||.||$ be the norm on E. Addition $\langle x, x*\rangle$ we mean the action of the functional $x*$ on the vector $x \in E$. Furthermore, we will denote by E_w, the space E with the weak topology $\sigma(E, E*)$. Assume that the interval $T = [0, 1]$ endowed with the Lebesgue $\sigma-$ algebra $\mathcal{L}(T)$ and the Lebesgue measure λ. According to the custom we will represent by, $L^1(T)$, the space of all measurable and Lebesgue integrable real functions defined on T, and by $L^\infty(T)$ the space of all measurable and essential bounded real functions defined on T.

Definition 2.1. *Let $f : I \times A \subset R \times E \to E$. Then the function $f(t, x)$ is said to be weakly-weakly continuous at (t_0, x_0) if given $\epsilon > 0$, $x^* \in E^*$, there exists a $\delta = \delta(x^*, \epsilon)$ and a weakly open set $\Omega = \Omega(x^*, \epsilon)$ containing x_0 such that*

$$|\langle x^*, f(t, x) - f(t_0, x_0)\rangle| < \epsilon$$

whenever, $|t - t_0| < \delta$ and $x \in \Omega$.

Definition 2.2. *A linear space X is weakly sequentially complete if every weakly Cauchy sequence is weakly convergent in X.*

Definition 2.3. *A function $x : T \to E$ is said to be absolutely continuous on T (AC) if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\| \sum_{i=1}^{n}[x(b_k) - x(a_k)]\| < \epsilon$ for every finite disjoint family $\{(a_k, b_k) : 1 \leq k \leq n\}$ of sub intervals of T such that $\sum_{i=1}^{n}[b_k - a_k] < \epsilon$.*

Definition 2.4. *A function $x(\cdot) : T \to E$ is said to be weakly absolutely continuous (wAC) on T if for every $x^* \in E^*$ the real valued function $t \to \langle x^*, x(t)\rangle$ is AC on T.*

Proposition 2.5. *([34, Theorem 7.3.3]) If E is a weakly sequentially complete space and $x(\cdot) : T \to E$ is a function such that for every $x^* \in E^*$, the real function $t \mapsto \langle x^*, x(t)\rangle$ is differentiable on T, then $x(\cdot)$ is weakly differentiable on T.*

Definition 2.6. *Let $x(\cdot) : T \to E$ be a given function and $\alpha > 0$. The* fractional Riemann-Liouville *integral of order $\alpha > 0$ of $x(\cdot)$ is defined by*

$$I^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \quad t \in T, \tag{2.1}$$

provided that the right side is point-wise defined on T. Also, the fractional Caputo derivative *of order $\alpha \in (0, 1]$ of $x(\cdot)$ is defined by*

$$D^\alpha x(t) := I^{1-\alpha}x'(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x'(s)ds, \quad t \in T, \tag{2.2}$$

provided that the right side is point-wise defined on T.

Definition 2.7. *A vector-valued function $x(\cdot) : T \to E$ is said to be* Riemann integrable *(or R-*integrable*, for short) on T if for any partition $\{t_0, ..., t_n\}$ of T and any choice of points $\xi_i \in [t_{i-1}, t_i]$, $i = 1, ..., n$, the sums*

$$\sum_{i=1}^{n}(t_i - t_{i-1})x(\xi_i) \tag{2.3}$$

converge strongly to some $x_T \in E$ provided $\max\limits_{1\le i\le n} |t_i - t_{i-1}| \to 0$ as $n \to \infty$. The element x_T is called the Riemann-Graves integral of $x(\cdot)$ and it will be denoted by $(R)\int_0^b x(t)dt$.

Definition 2.8. *A function $x(\cdot) : T \to E$ is said to be* scalarly Riemann integrable *if for every $x^* \in E^*$ the real function $t \mapsto \langle x^*, x(t)\rangle$ is Riemann integrable on T.*

Definition 2.9. *A function $x(\cdot) : T \to E$ is said to be Riemann-Pettis integrable (RP-integrable) on T if $x(\cdot)$ is scalarly Riemann integrable and for every interval I $I \subseteq T$, there exists an element $\tau \in E$ such that*

$$\langle x^*,\ \tau\rangle = \int_I \langle x^*,\ x(s)\rangle\, ds$$

for every $x^* \in E^*$. The element τ is represented by $(w)\int_I x(s)ds$ and it is called the weak Riemann integral of $x(\cdot)$ on I.

Definition 2.10. *Let $x(\cdot) : T \to E$ be a weakly differentiable function such that $x(t)$ is Riemann-Pettis integrable function and $\alpha \in (0,\ 1)$. The* fractional Bochner *integral of order α of $x(\cdot)$ is defined by*

$$I_w^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds,\ \ t \in T, \tag{2.4}$$

provided that the right side is point-wise defined on T.

Proposition 2.11. *A weakly measurable function $x(\cdot) : T \to E$ is Pettis integrable on T and $\langle x^*,\ x(\cdot)\rangle \in L^\infty(T)$ for every $x^* \in E^*$, if and only if the function $t \to \phi(t)x(t)$ is Pettis integrable on T for every $\phi \in L^1(T)$.*

Proposition 2.12. [1] *Let us denote by $P^\infty(T, E)$ the space of all weakly measurable and Pettis integrable functions $x(\cdot)$ with the property that for every $x^* \in E^*$ we have $\langle x*,\ x(t)\rangle \in L^\infty(T\,, E)$. Since for each $t \in T$ the real valued function $s \to (t-s)^{\alpha-1}$ is Lebesgue integrable on [0, t] for every $\alpha > 0$ then, by Proposition 2.11, the fractional Pettis integral*

$$I^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds,\ \ \ t \in T$$

exists for every function $x(\cdot) \in P^\infty(T, E)$ as a function from T into E.

Definition 2.13. *A function $f : X \to Y$ defined on a Banach space X into Y is called weakly-weakly sequentially continuous function if the image of each weakly convergent sequence in X is weakly convergent in Y.*

Definition 2.14. *Assume a Banach space X and X^* be the topological dual space of X. If $C_\omega(I, X)$ is space of all continuous mappings, $f : I \to (X, \omega)$, where*

ω represents weak topology. Then the topology of weak uniform convergence is determine by the basis,

$V_n\,(g_1^*,\, g_2^*,\, g_3^* \ldots,g_n^*,\,\epsilon) = \{g \in C_\omega(I,X) : \sup|\langle g_k^*,\, g(t)-u(t)\rangle| < \epsilon,\ \text{ for k} = 1,2,3,\ldots \text{n}\}$

here $u \in C_\omega(I,X)$, $\epsilon > 0$, $n \in N$ and $g_1^*,\, g_2^*,\, g_3^* \ldots, g_n^* \in X^*$.

In the succeeding discussion we shall recall some properties of the Riemann integral. First, let us denote by $P^\infty(T,E)$ the space of all weakly measurable and Pettis integrable functions $x(\cdot) : T \to E$ with the property that $\langle x^*, x(\cdot)\rangle \in L^\infty(T)$ for every $x^* \in E^*$. Also, let $C(T,E)$ denote the space of all strongly continuous functions $x(\cdot) : T \to E$, endowed with the sup norm $||x(\cdot)||_c = sup_{t\in T}\|x(t)\|$. Also, let $C_w(T,E)$ be the space of all weakly continuous functions from T into E_w endowed with the topology of weak uniform convergence.

Proposition 2.15. *If $x(\cdot) : T \to E$ is R-integrable on T, then*

$$I^\alpha x(t) = (P)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \quad t \in T, \tag{2.5}$$

that is, $I^\alpha x(t)$ exists on T as a fractional Pettis integral.

Remark. *If $x(\cdot) : T \to E$ is strongly measurable and R-integrable on T, then the fractional integral $I^\alpha x(t)$ exists a.e. on T as a fractional Bochner integral. This result is a direct consequence of Theorem 15 of* [20] *and Theorem 2.4 from* [24].

Proposition 2.16. *Let $\alpha \in (0,1)$ and let $x(\cdot) : T \to E$ be a strongly differentiable function on T. If the strong derivative $x'(\cdot)$ of $x(\cdot)$ is R-integrable on T , then*

$$D^\alpha x(t) = (B)\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x'(s)ds, \quad t \in T, \tag{2.6}$$

that is, $D^\alpha x(t)$exists a.e. on T as a fractional Bochner integral.

In the forthcoming discussion we will concentrate on the study of Riemann-Pettis integrability and its applications to fractional calculus and fractional differential equations.

Definition 2.17. *A function $x(\cdot) : T \to E$ is said to be* Riemann-Pettis integrable *(*RP-integrable*) on T if $x(\cdot)$ is scalarly Riemann integrable and, for each interval $I \subset T$, there exists an element $z_I \in E$ such that*

$$\langle x^*, z_I\rangle = \int_I \langle x^*, x(s)\rangle\, ds \tag{2.7}$$

for every $x^* \in E^*$. The element z_I will be denoted by $(w)\int_I x(s)ds$ and it is called the weak Riemann integral *of $x(\cdot)$ on I.*

Also, a RP-integrable function is sometime called a *weakly Riemann integrable* function. In fact, RP-integrability on T is equivalent to the weak convergence of the Riemann sums (2.3). It is not difficult to show that every R-integrable function is RP-integrable, and every RP-integrable function is Pettis integrable (see [20]). Alexiewicz and Orlicz [5] give an example which shows that neither RP-integrability nor weakly continuity imply R-integrability. We shall denote by $RP(T,E)$ the set of all RP-integrable functions from T into E.

Proposition 2.18. *(Krein [26], Alexiewicz and Orlicz [5]). Every weakly continuous function from T into E is RP-integrable on T, that is, $C_w(T,E) \subset RP(T,E)$.*

The following properties are the consequences of definitions and properties of weak differentiability [4].“

Proposition 2.19. *If $x(\cdot): T \to E$ is weakly continuous on T, then the function $y(\cdot): T \to E$, given by*

$$y(t) = (w)\int_0^t x(s)ds, \quad t \in T, \tag{2.8}$$

is weakly differentiable on T and $y'_w(t) = x(t)$ for every $t \in T$.

Proposition 2.20. *If $x(\cdot): T \to E$ is weakly differentiable on T and $x'_w(\cdot)$ is weakly continuous on T, then*

$$x(t) = x(0) + (w)\int_0^t x'_w(s)ds, \quad t \in T. \tag{2.9}$$

Proposition 2.21. *If $x(\cdot): T \to E$ is RP-integrable on T, then*

$$I_w^\alpha x(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \quad t \in T, \tag{2.10}$$

exists on T as a fractional Pettis integral. Moreover, I_w^α is a linear operator from $RP(T,E)$ into $P^\infty(T,E)$, and for $\alpha > 0$, $\beta > 0$ we have

$$I_w^\alpha I_w^\beta x(t) = I_w^{\alpha+\beta} x(t), \; t \in T. \tag{2.11}$$

Proposition 2.22. *Let $\alpha \in (0,1)$ and let $x(\cdot): T \to E$ be a weakly differentiable function on T. If the weak derivative $x'_w(\cdot)$ of $x(\cdot)$ is RP-integrable on T, then*

$$D_w^\alpha x(t) := I_w^{1-\alpha} x'(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x'_w(s)ds, \; t \in T, \tag{2.12}$$

exists a.e. on T as a fractional Bochner integral (and so as a fractional Pettis integral).

Clearly, if $x(\cdot): T \to E$ is R-integrable on T, then $D_w^\alpha x(t)$ exists on T as a fractional Bochner integral and $D_w^\alpha x(t) = D^\alpha x(t)$ for $t \in T$.

Definition 2.23. *Two weakly measurable functions $x(\cdot): T \to E$ and $y(\cdot): T \to E$ are said to be weakly equivalent if for every $x^* \in E^*$ we have that $\langle x^*, x(t)\rangle = \langle x^*, y(t)\rangle$ for a.e $t \in T$. If two weakly measurable functions $x(\cdot): T \to E$ and $y(\cdot): T \to E$ are weakly equivalent on T, then we will write $x(\cdot) \simeq y(\cdot)$ for $t \in T$.*

Lemma 2.24. *If $x(\cdot): T \to E$ is weakly differentiable a.e. on T and $x'_w(\cdot)$ is RP-integrable on T, then the function*

$$x_{1-\alpha}(t) := \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds, \; t \in T,$$

is wAC and weakly differentiable a.e. on T. Moreover, $(x_{1-\alpha})'_w(\cdot)$ is RP-integrable and

$$(x_{1-\alpha})'_w(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0) + I_w^{1-\alpha} x'_w(t) \quad a.e. \text{ on } T. \tag{2.13}$$

Remark. *Relation (2.13) can be written as*

$$D_w^\alpha x(t) = (x_{1-\alpha})'_w(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0) \quad a.e.\ on\ T. \tag{2.14}$$

In this case $(x_{1-\alpha})'_w(t)$ *will be denoted by* ${}^{RL}D_w^\alpha x(t)$ *and it is called the* weak Riemann-Liouville derivative *of* $x(\cdot)$. *The formula (2.14) suggests us that we can extend the definition of the weak Caputo fractional derivative for RP-integrable functions. Therefore, if* $x(\cdot) \in RP(T,E)$, *then its weak Caputo derivative is defined by (2.14). It follows that the weak Caputo fractional derivatives* $D_w^\alpha x(t)$ *are also defined for functions* $x(\cdot)$ *for which the weak Riemann-Liouville fractional derivatives* ${}^{RL}D_w^\alpha x(t)$ *exist.*

Remark. *If* $x(\cdot)$ *is not weakly differentiable, then* $x_{1-\alpha}(\cdot)$ *will not be weakly differentiable.*

Remark. *If* $x(\cdot) : T \to E$ *is weakly differentiable a.e. on* T *and* $x'_w(\cdot)$ *is RP-integrable, then*

$$x_\alpha(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds,\ t \in T,$$

is wAC and weakly differentiable a.e. on T *and*

$$(x_\alpha)'_w(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x(0) + I_w^\alpha x'_w(t) \ a.e.\ on\ T.$$

Also, if E *is a weakly sequentially complete space and* $x(\cdot) : T \to E$ *is weakly differentiable, then the function* $x_\alpha(\cdot)$ *is AC on* T.

Corollary 2.25. *Let* E *be a weakly sequentially complete space. If* $y(\cdot) : T \to E$ *is weakly differentiable, then the function* $y_{1-\alpha}(\cdot)$ *is AC on* T.

Proposition 2.26. *If* $x(\cdot) : T \to E$ *is weakly differentiable a.e. on* T *and* $x'_w(\cdot)$ *is RP-integrable on* T *and* $\alpha, \beta \in (0,1)$, *then*

(a) $I_w^\alpha D_w^\alpha x(t) = x(t) - x(0)$ *on* T;

(b) $D_w^\alpha I_w^\alpha x(t) = x(t)$ *on* T.

Theorem 2.27. [4] *If* $y(\cdot) \in RP(T,E)$, *then the Abel integral equation*

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds = y(t),\ t \in T = T, \tag{2.15}$$

has a solution in $x(\cdot) \in RP(T,E)$ *if and only if the function* $y_{1-\alpha}(\cdot)$ *has the following properties:*

(a) $y_{1-\alpha}(\cdot)$ *is wAC on* T,

(b)$y_{1-\alpha}(\cdot)$ *is weakly differentiable a.e. on* T *and*

$$x(t) = (y_{1-\alpha})'_w(t),\ for\ a.e.\ t \in T, \tag{2.16}$$

(c) $y_{1-\alpha}(0) = 0$.

Lemma 2.28. *Let* $y(\cdot) : T \to E$ *be a weakly differentiable function such that* y'_w *is Riemann-Pettis integrable and* $0 \le \alpha \le \beta \le 1$. *Then,*

(a) $I^\alpha D_w^\beta y(t) = D^{\beta-\alpha} y(t)$ *on* T,

(b) If $y(0) = 0$, *then* $D_w^\beta I^\alpha y(t) = D^{\beta-\alpha} y(t)$ *and* $I^\beta D_w^\alpha y(t) = I^{\beta-\alpha} y(t)$ *on* T".

Theorem 2.29. [30] *Let E be a metrizable locally convex topological vector space and let K be a closed convex subset of E, let Q be a weakly sequentially continuous map of K into itself. If for some $y \in K$ the implication*

$$\overline{V} = \overline{connv}(Q(V) \cup y) \Rightarrow \text{ is relatively weakly compact,}$$

holds for every subset V of K, then Q has a fixed point.

Let us denote by $P_{wk}(E)$ the set of all weakly compact subsets of E. The weak measure of non-compactness is the set function $\beta : P_{wk}(E) \to \mathbb{R}^+$ defined by

$$\beta(A) = \inf\{r > 0;\ \textit{there exist } K \in P_{wk}(E) \textit{ such } A \subset K + rB_1\},$$

where B_1 is the closed unite ball in E. The properties of weak non-compactness measure are analogous to the properties of measure of non-compactness. If A, $B \in P_{wk}(E)$,
(N_1) $\beta(B) \geq \beta(A)$; whenever $B \supseteq A$
(N_2) $\beta(A) = \beta(cl_w(A))$, where $cl_w(A)$ denotes the weak closure of A;
(N_3) $\beta(A) = 0$ if and only if cl_w is weakly compact;
(N_4) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;
(N_5) $\beta(A) = \beta(conv(A))$;
(N_6)) $\beta(A) + \beta(B) \geq \beta(A + B)$;
(N_7) $\beta(x + A) = \beta(A)$, for all $x \in E$;
(N_8) $\beta(\kappa A) = |\kappa|\beta(A)$, for all $\kappa \in \mathbb{R}$;
(N_9) $\beta(\cup_{0\leq r\leq r_0} rA) = r_0\beta(A)$
(N_{10}) $2diam(A) \geq \beta(A)$

Lemma 2.30. [7, 12] *Let $Y \subset C(T, E)$ be bounded and equi-continuous, then*

$$(i)\, \textit{the} \text{ function t} \to \beta(\text{Y(t)}) \text{ is continuous on T},$$

$$(ii)\, \beta_c(Y) = \sup_{t\in T} \beta(Y(t)) = \beta(Y(t)),$$

where $\beta_c(\cdot)$ denotes the weak measure of non-compactness in C(T,E) and $Y(t) = \{u(t);\ u \in Y\}$, $t \in T$.

3. Main Results

Assume the following multi-term fractional differential equation

$$\left(D_w^{\alpha_m} - \sum_{i=1}^{m-1} a_i D^{\alpha_i}\right) u(t) = f(t, u(t)) \text{ for t} \in [0,1],\ \text{u}(0) = 0, \tag{3.1}$$

where $D_w^{\alpha_m} u(\cdot)$ and $D_w^{\alpha_i} u(\cdot)$ are fractional Caputo weak derivatives of the function $u(\cdot) : [0,1] \to E$ of order α_m and α_i, $i = 1, 2, \ldots, m-1$, respectively, E is a non-reflexive Banach space, $0 < \alpha_1 < \alpha_2 < \alpha_3 \ldots < \alpha_m < 1$ and $a_1, a_2, a_3, \ldots a_{m-1}$ are real numbers such that $a := \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} < 1$. Along with Cauchy problem (3.1), consider the following integral equation

$$u(t) = \sum_{i=1}^{m-1} (P) \int_0^t \frac{a_i (t-s)^{(\alpha_m - \alpha_i - 1)}}{\Gamma(\alpha_m - \alpha_i)} u(s) ds + (P) \int_0^t \frac{(t-s)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f(s, u(s)) ds, \quad t \in T = [0,1], \tag{3.2}$$

where the integral is in the sense of Pettis.

Definition 3.1. *A continuous function $y(\cdot) : T \to E$ is said to be weak solution of (3.1) if,*
(i) $y(\cdot)$ is weakly differentiable, $y'_w(\cdot)$ is RP-integrable,
(ii) $y(\cdot)$ has Caputo fractional weak-derivative of order $\alpha_i \in (0,1), i = 1,2,3,\dots,m$,
(iii) $(D^{\alpha_m} - \sum_{i=1}^{m-1})a_i D^{\alpha_i}) \simeq f(t, y(t))$, for all t $\in$ T,
(iv) $y(0) = 0$.

Lemma 3.2. *Assume $f(\cdot,\cdot) : T \times E \to E$ such that $f(\cdot, u(\cdot))$ is weakly continuous for every continuous function $u : T \to E$. Then $u(\cdot)$ is weak solution of (3.1), if and only if it satisfy the integral equation (3.2).*

Proof. If a continuous function $y(\cdot) : T \to E$ is a weak solution of (3.1), then from lemma (2.28) it follows that $y(t) = \sum_{i=1}^{m-1} (I_w^{\alpha-\alpha_i} u(t)) + I_w^{\alpha} f(t, y(t))$on T; that is, $y(\cdot)$ satisfies the integral equation (3.2). Conversely suppose that a continuous function $y(\cdot) : T \to E$ satisfies the integral equation (3.2). The weakly continuous function $z(\cdot) := f(\cdot, y(\cdot))$ satisfies the Abel equation,

$$\int_0^t \frac{(t-s)^{\alpha_m - 1}}{\Gamma(\alpha_m)} z(s)ds = v(t), \;\; \text{t} \in \text{T}$$

, from (2.27) it follows that $v_{1-\alpha_m}(\cdot)$ is weakly differentiable a.e on T and

$$z(t) \simeq \frac{d_p}{dt} v_{1-\alpha_m}(t), \;\; \text{for } \text{t} \in \text{T}.$$

Since $y(\cdot)$ is continuous on T and $f(\cdot, y(\cdot))$ is weakly continuous also satisfies (h3), we have

$$\lim_{t \to 0^+} I^{\alpha} y(t) = \lim_{t \to 0^+} I^{\alpha} f(t, y(t)) = 0 \;\; \text{for} \;\; \alpha \in (0,1)$$

now taking limit $t \to 0^+$ of (3.2), we obtain $y(0) = 0$ and consequently $v(0) = 0$. Since $v(0) = 0$, by remark (2) we have

$$z(t) \simeq \frac{d_p}{dt} v_{1-\alpha_m} = D_w^{\alpha_m} v(t); \quad t \in T.$$

Using Lemma (2.28) we have

$$\begin{aligned} D_w^{\alpha_m} v(t) \quad &= D_w^{\alpha_m} y(t) - \sum_{i=1}^{m-1} a_i D_w^{\alpha_m} I^{\alpha_m - \alpha_i} y(t) \\ &= \;\; D_w^{\alpha_m} y(t) - \sum_{i=1}^{m-1} a_i D_w^{\alpha_i} y(t) \end{aligned}$$

we obtain

$$\left(D_w^{\alpha_m} - \sum_{i=1}^{m-1} a_i D^{\alpha_i} \right) y(t) \simeq f(t, \, y(t)) \;\; \text{for t} \in [0,1].$$

Hence the continuous function $y(\cdot)$ satisfy the conditions of definition (3.1) and thus $y(\cdot)$ is a solution of (3.1). $\square$

Theorem 3.3. *Let $r > 0$. Assume $f(\cdot,\cdot) : T \times E \to E$ be a function such that:*
(h1) $f(t,\cdot)$ is weakly-weakly continuous for every $t \in T$;
(h2) $g : [0,\infty) \to [0,\infty)$ is a non-decreasing continuous function such that $g(0) = 0$ and $g(t) < t$ for all $t > 0$;
(h3) $||f(t,u)|| \leq M$ for all $(t,u) \in T \times E$
(h4) for all $A \subseteq E$ we have

$$\beta(f(T \times A)) \leq g(\beta(A)),$$

where $g(\cdot)$ is a Gripenberg function. Then (3.2) admits a solution $u(\cdot)$ on an interval $[0,a_0]$ with

$$a_0 < \min\left\{1, \left[\frac{r(1-a)\Gamma(\alpha_m+1)}{M}\right]^{1/\alpha_m}\right\}.$$

Proof: Suppose the nonlinear operator, $Q\,(\cdot) : C_w(T_0,E) \to C_w(T_0,E)$ defined as

$$(Qu)(t) = \sum_{i=1}^{m-1} (P)\int_0^t \frac{a_i(t-s)^{\alpha_m-\alpha_i-1}}{\Gamma(\alpha_m-\alpha_i)} u(s)ds + (P)\int_0^t \frac{(t-s)^{\alpha_m-1}}{\Gamma(\alpha_m)} f(s,u(s))ds, \qquad ,$$

for all $t \in [0,a_0] = T_0$. If $y(\cdot) \in C_w(T_0,E)$, then by remark 2 the operator Q makes sense. To show that Q is well defined, let $t, s \in T_0$ with $t > s$. Without loss of generality, assume that $(Qy)(t) - (Qy)(s) \neq 0$. Then by the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and $\|(Qy)(t) - (Qy)(s)\| = |\langle y^*, (Qy)(t) - (Qy)(s)\rangle|$. Then

$$\begin{aligned}
&\|(Qy)(t) - (Qy)(s)\| = |\langle y^*, (Qy)(t) - (Qy)(s)\rangle| \leq \\
\leq\ & \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m-\alpha_i)} \int_0^s \left[(s-\tau)^{\alpha_m-\alpha_i-1} - (t-\tau)^{\alpha_m-\alpha_i-1}\right] |\langle y^*, u(\tau)\rangle|\, d\tau \\
& + \int_s^t (t-\tau)^{\alpha_m-\alpha_i-1} |\langle y^*, f(\tau,u(\tau))\rangle|\, d\tau + \\
& + \frac{1}{\Gamma(\alpha_m)} \int_0^s \left[(s-\tau)^{\alpha_m-1} - (t-\tau)^{\alpha_m-1}\right] |\langle y^*, u(\tau)\rangle|\, d\tau \qquad (3.3)\\
& + \int_s^t (t-\tau)^{\alpha_m-1} |\langle y^*, f(\tau,u(\tau))\rangle|\, d\tau \\
\leq\ & 2\left[\sum_{i=1}^{m-1} \frac{r|a_i|}{\Gamma(\alpha_m-\alpha_i+1)} + \frac{M}{\Gamma(\alpha_m+1)}\right](t-s)^{\alpha_m}.
\end{aligned}$$

so Q maps $C_w(T_0,E)$ into itself. Let $r > 0$ and let $\widetilde{B}$ be the convex, closed and equi-continues set defined by

$$\begin{aligned}
\widetilde{B} &= \{y(\cdot) \in C_w(T_0,E); \|y(\cdot)\|_c \leq r,\ \|y(t) - y(s)\| \leq \\
&\leq 2\left[\sum_{i=1}^{m-1} \frac{r|a_i|}{\Gamma(\alpha_m-\alpha_i+1)} + \frac{M}{\Gamma(\alpha_m+1)}\right](t-s)^{\alpha_m} \text{ for all } t,s \in T_0\}.
\end{aligned}$$

We will show that Q maps $\widetilde{B}$ into itself and Q restricted to the set $\widetilde{B}$ is continuous. To show that $Q : \widetilde{B} \to \widetilde{B}$, let $y(\cdot) \in \widetilde{B}$ and $t \in T_0$. Again, without loss of generality, assume that $(Qy)(t) \neq 0$. By the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and $\|(Qy)(t)\| = |\langle y^*, (Qy)(t)\rangle|$. Then by (h3), we have

$$
\begin{aligned}
& \|(Qu)(t)\| = |\langle y^*, (Qy)(t)\rangle| \leq \\
\leq \;& \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i)} \int_0^t (t-s)^{\alpha_m - \alpha_i - 1} |\langle y^*, u(\tau)\rangle| \, ds + \\
& \frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1} |\langle y^*, f(\tau, u(\tau))\rangle| \, ds \leq \\
\leq \;& \sum_{i=1}^{m-1} \frac{r|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{Mt^{\alpha_m}}{\Gamma(\alpha_m + 1)} \leq ra + (1-a)r = r
\end{aligned}
$$

and using (3.3) it follows that Q maps $\widetilde{B}$ into $\widetilde{B}$. Next, we show that Q restricted to $\widetilde{B}$ is a continuous operator. Let fix $y(\cdot) \in \widetilde{B}$, $\epsilon > 0$. Furthermore, we fix $y^* \in E^*$ such that $||y^*|| \leq 1$. We will choose a_0 such that, $a_0 < \frac{\epsilon(\alpha_m - \alpha_i)}{2r + (\alpha_m - \alpha_i)}$. As $f(\cdot, \cdot)$ is weakly-weakly continuous so we have by Krasnoselskii type lemma (see [35]), that there exists a weak neighborhood W of 0 in E such that $|\langle y^*, f(s, y(s) - f(s), z(s))| < \Gamma(1 + \alpha_m)$ for $s \in T_0$ and $z(\cdot) \in \widetilde{B}$ with $y(s) - z(s) \in W$. Hence we have,

$$
\begin{aligned}
& |\langle y^*, (Qy)(t) - (Qz(t))\rangle| \leq \\
& \leq \left| \left\langle y^*, \sum_{i=1}^{m-1} \int_0^t \frac{a_i (t-s)^{\alpha_m - \alpha_i - 1}}{\Gamma(\alpha_m - \alpha_i)} [y(s) - z(s)] ds \right\rangle \right| + \\
& + \left| \left\langle y^*, \sum_{i=1}^{m-1} \int_0^t \frac{a_i (t-s)^{\alpha_m - \alpha_i - 1}}{\Gamma(\alpha_m - \alpha_i)} [f(s, y(s)) - f(s, z(s))] ds \right\rangle \right| \\
& \leq \left(\frac{2r + (\alpha_m - \alpha_i)}{(\alpha_m - \alpha_i)} \right) a_0^{\alpha_m} \\
& \leq \varepsilon.
\end{aligned} \tag{3.4}
$$

Hence we proved that Q restricted to $\widetilde{B}$ is a continuous operator . Now Suppose that $V \subset \widetilde{B}$ such that $V = \overline{co}(Q(V) \cup \{y(\cdot)\})$ for some $y(\cdot) \in \widetilde{B}$. Let $t \in T_0$ and $\varepsilon > 0$. If we choose $\eta > 0$ such that $\eta < \left(\frac{\varepsilon \Gamma(\alpha_m + 1)}{M + r\Gamma(\alpha_m + 1)} \right)^{1/\alpha_m}$ and

$$
\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^t (t-s)^{\alpha_m - \alpha_i - 1} u(s) ds + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^t (t-s)^{\alpha_m - 1} f(s, u(s)) ds \neq 0
$$

then, by the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and

$$\begin{aligned}
&\left\|\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-\alpha_i-1} u(s)ds + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-1} f(s,u(s))ds\right\| \\
= & \left|\left\langle y^*, \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-\alpha_i-1} u(s)ds + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-1} f(s,u(s))ds \right\rangle\right| \\
\leq & \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m-\alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-\alpha_i-1} \left|\langle y^*, u(\tau)\rangle\right| ds + \\
& + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-1} \left|\langle y^*, f(\tau,u(\tau))\rangle\right| ds \\
\leq & \sum_{i=1}^{m-1} \frac{r|a_i|\eta^{\alpha_m-\alpha_i}}{\Gamma(\alpha_m-\alpha_i+1)} + \frac{M\eta^{\alpha_m}}{\Gamma(\alpha_m+1)} \leq r\eta^{\alpha_m} + \frac{M\eta^{\alpha_m}}{\Gamma(\alpha_m+1)} \\
\leq & \frac{M + r\Gamma(\alpha_m+1)}{\Gamma(\alpha_m+1)} \eta^{\alpha_m} < \varepsilon.
\end{aligned}$$

and thus using property (x) measure of the non-compactness we infer

$$\begin{aligned}
&\beta\left(\left\{\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-\alpha_i-1} u(s)ds + \right.\right. \\
&\left.\left. \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-1} f(s,u(s))ds,\, u \in V\right\}\right) \leq 2\varepsilon.
\end{aligned} \tag{3.5}$$

Since by Lemma 2.30 the function $t \to v(t) := \beta(V(t))$ is continuous on $[0, t-\eta]$ it follows that $s \to (t-s)^{\alpha_m-1} g(v(s))$ is continuous on $[0, t-\eta]$. Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left\|(t-\tau)^{\alpha_m-1} g(v(\tau)) - (t-s)^{\alpha_m-1} g(v(s))\right\| < \frac{\varepsilon}{2}$$

and

$$\|g(v(\xi)) - g(v(\tau))\| < \frac{\varepsilon}{2\eta^{\alpha_m-1}}.$$

If $|\tau - s| < \delta$ and $|\tau - \xi| < \delta$ with $\tau,\, s,\, \xi \in [0, t-\eta]$, then it follows that

$$\begin{aligned}
|(t-\tau)^{\alpha_m-1} g(v(\xi)) - (t-s)^{\alpha_m-1} g(v(s))| &\leq |(t-\tau)^{\alpha_m-1} g(v(\tau)) - (t-s)^{\alpha_m-1} g(v(s))| \\
&+ (t-\tau)^{\alpha_m-1} |g(v(\xi)) - g(v(\tau))| \\
&< \varepsilon,
\end{aligned}$$

that is

$$|(t-\tau)^{\alpha_m-1} g(v(\xi)) - (t-s)^{\alpha_m-1} g(v(s))| < \varepsilon, \tag{3.6}$$

for all τ, s, $\xi \in [0, t-\eta]$ with $|\tau - s| < \delta$ and $|\tau - \xi| < \delta$. Consider the following partition of the interval $[0, t-\eta]$ into n parts $0 = t_0 < t_1 \ldots < t_n = t - \eta$ such that $t_i - t_{i-1} < \delta$ $(i = 1, 2, \ldots, n)$. By Lemma 2.30 for each i there exists $s_i \in [t_{i-1}, t_i]$ such that $\beta(V([t_{i-1}, t_i])) = v(s_i)$, $i = 1, 2, \ldots, n$. Then we have by([36], Theorem 2.2 and [6, 11])

$$\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m-\alpha_i-1} V(s)ds + \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m-1} f(s, V(s))ds \subset$$

$$\subset \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-\alpha_i-1} V(s)ds + \sum_{j=1}^{n} \frac{1}{\Gamma(\alpha_m)} \int_{t_{i-1}}^{t_i} (t-s)^{\alpha_m-1} f(s,V(s))ds \subset$$

$$\subset \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-\alpha_i-1} V(s)ds +$$

$$+\frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i-t_{i-1}) \overline{conv}\{(t-s)^{\alpha_m-1} f(s,u(s)); s\in[t_{i-1},t_i],\ u\in V\},$$

and so

$$\beta\left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m-\alpha_i-1} V(s)ds + \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m-1} f(s,V(s))ds\right) \le$$

$$\le \beta\left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m-\alpha_i-1} V(s)ds\right) +$$

$$+\frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i-t_{i-1}) \beta\left(\overline{conv}\{(t-s)^{\alpha_m-1} f(s,u(s)); s\in[t_{i-1},t_i],\ u\in V\}\right)$$

$$= \beta\left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m-\alpha_i-1} V(s)ds\right) +$$

$$+\frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i-t_{i-1}) \beta\left(\{(t-s)^{\alpha_m-1} f(s,u(s)); s\in[t_{i-1},t_i],\ u\in V\}\right) \le$$

$$\le \beta\left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m-\alpha_i-1} V(s)ds\right) +$$

$$+\frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i-t_{i-1})(t-t_i)^{\alpha_m-1} \beta\left(f([0,a_0]\times V[t_{i-1},t_i])\right)$$

$$\le \beta\left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m-\alpha_i-1} V(s)ds\right) +$$

$$+\frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i-t_{i-1})(t-t_i)^{\alpha_m-1} g(V[t_{i-1},t_i]) \le$$

$$\le \beta\left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m-\alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m-\alpha_i-1} V(s)ds\right) +$$

$$+\frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i-t_{i-1})(t-t_i)^{\alpha_m-1} g(v(s_i)).$$

Using (3.6) we have that

$$|(t-t_i)^{\alpha_m-1} g(v(s_i)) - (t-s)^{\alpha_m-1} g(v(s))| < \varepsilon.$$

This implies that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha_m)}\sum_{j=1}^{n}(t_i-t_{i-1})(t-t_i)^{\alpha_m-1}g(v(s_i)) \qquad (3.7)\\ \leq\ &\frac{1}{\Gamma(\alpha_m)}\int_0^{t-\eta}(t-s)^{\alpha_m-1}g(v(s))ds+\varepsilon(t-\eta)/\Gamma(\alpha_m).\end{aligned}$$

By using (3.5) we claim that

$$\beta\left(\sum_{i=1}^{m-1}\frac{a_i}{\Gamma(\alpha_m-\alpha_i)}\int_{t-\eta}^{t}(t-s)^{\alpha_m-\alpha_i-1}V(s)ds\right)\leq 2\varepsilon. \qquad (3.8)$$

Because if we let that

$$A(t)=\frac{1}{\Gamma(\alpha_m)}\int_{t-\eta}^{t}(t-s)^{\alpha_m-1}f(s,V(s))ds,$$

$$B(t)=\sum_{i=1}^{m-1}\frac{a_i}{\Gamma(\alpha_m-\alpha_i)}\int_{t-\eta}^{t}(t-s)^{\alpha_m-\alpha_i-1}V(s)ds,$$

then $a+B(t)\subset A(t)+B(t)$ for $a\in A(t)$, implies that $\beta(B(t))\leq\beta(A(t)+B(t))<2\varepsilon$. From relations (3.7) and (3.8) we obtain

$$\beta\left(\sum_{i=1}^{m-1}\frac{a_i}{\Gamma(\alpha_m-\alpha_i)}\int_0^{t-\eta}(t-s)^{\alpha_m-\alpha_i-1}V(s)ds+\frac{1}{\Gamma(\alpha_m)}\int_0^{t-\eta}(t-s)^{\alpha_m-1}f(s,V(s))ds\right)\leq \qquad (3.9)$$

$$\leq 2\varepsilon+\frac{1}{\Gamma(\alpha_m)}\int_0^{t-\eta}(t-s)^{\alpha_m-1}g(V(s))ds+\varepsilon(t-\eta)/\Gamma(\alpha_m).$$

Since

$$\begin{aligned}(QV)(t)\subset&\sum_{i=1}^{m-1}\frac{a_i}{\Gamma(\alpha_m-\alpha_i)}\int_0^{t-\eta}(t-s)^{\alpha_m-\alpha_i-1}V(s)ds+\\ &\sum_{i=1}^{m-1}\frac{a_i}{\Gamma(\alpha_m-\alpha_i)}\int_{t-\eta}^{t}(t-s)^{\alpha_m-\alpha_i-1}V(s)ds+\\ &+\frac{1}{\Gamma(\alpha_m)}\int_0^{t-\eta}(t-s)^{\alpha_m-1}f(s,V(s))ds+\frac{1}{\Gamma(\alpha_m)}\int_{t-\eta}^{t}(t-s)^{\alpha_m-1}f(s,V(s))ds,\end{aligned}$$

then by virtue of (3.5) and (3.9) we have

$$\begin{aligned}\beta((QV)(t))\ &\leq\ \frac{1}{\Gamma(\alpha_m)}\int_0^{t-\eta}(t-s)^{\alpha_m-1}g(v(s))ds+\varepsilon(t-\eta)/\Gamma(\alpha_m)+4\varepsilon\\ &\leq\ \frac{1}{\Gamma(\alpha_m)}\int_0^{t}(t-s)^{\alpha_m-1}g(v(s))ds+\varepsilon((t+4)/\Gamma(\alpha_m)).\end{aligned}$$

As the last inequality is true for every $\varepsilon>0$, we infer

$$\beta((QV)(t))\leq\frac{1}{\Gamma(\alpha_m)}\int_0^{t}(t-s)^{\alpha_m-1}g(v(s))ds,\ t\in[0,a_0],$$

Because $V=\overline{co}(Q(V)\cup\{y(\cdot)\})$ then

$$\beta(V(t))=\beta\left(\overline{co}(Q(V(t))\cup\{y(t)\})\right)\leq\beta((QV(t)))$$

and thus

$$v(t) \leq \frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1} g(v(s))ds \ for\ t \in T_0.$$

Since $g(\cdot)$ is a Gripenberg function, it follows that $v(t) = 0$ for $t \in T_0$. Since V as a subset of $\widetilde{B}$ is equicontinuous, by Lemma 2.30 we infer

$$\beta_c(V(T_0)) = \sup_{t \in T_0} \beta(V(t)) = 0.$$

Thus, by Arzelá-Ascoli's theorem we obtain that V is weakly relatively compact in $C(T_0, E)$. Using Theorem 2.29 there exists a fixed point of the operator Q which is a solution of (3.2). Therefore, by Lemma 3.2 we have a solution of (3.1).

Concluding Remarks:

Assuming the Cauchy problem (3.1) we exhibited the existence of weak solution of multi-term fractional differential equations in a non-reflexive Banach space equipped with the weak topology. Which turn out to be mostly parallel to those of the single term prototype. Furthermore, we used fractional Riemann-Pettis integral, weak fractional Caputo derivative and weak measure of non-compactness.

Remark. *If* $\alpha_1 = \alpha_2 \ldots \alpha_{m-1} = 0$ *and* $\alpha_m = 1$ *and put* $Dy(\cdot) = y'$*, then from theorem 2.27 we obtain the following generalization of some known results (*[12, 14]*). Moreover, if* $\alpha_1 = \alpha_2 \ldots \alpha_{m-1} = 0$ *and* $\alpha_m = \alpha$*, where* $0 < \alpha < 1$ *one can get the special case* [4].

Acknowledgments. The author would like to thank the anonymous referee for his/her comments that helped us improve this article.

This paper is a part of the project No: 21-190/SRGP/R&D/HEC/2014 granted by Higher Education Commission, Islamabad Pakistan. The author is thankful to HEC for approving the project. Moreover, the author is thankful to Professor Dr. Vasile Lupulescu, University of Constantin Brncui, Trgu-Jiu, Romnia, for his support and help.

References

[1] R. P. Agarwal, V. Lupulescu, D. O'Regan, Ghaus ur Rahman, *Fractional calculus and fractional differential equations in nonreflexive Banach spaces*, Commun. Nonlinear Sci. Numer. Simul**20** (2015)59–73.

[2] R. P. Agarwal, V. Lupulescu, D. O'Regon, Ghaus ur Rahman, *Multi-term Fractional Differential Equation in nonreflexive Banach spaces*, Adv. Difference Equ. 2013, 2013:302.

[3] R. P. Agarwal, M. Belmekki, M. Benchohra, *A survey on semilinear differential equations and inclusions involving Riemann Liouville fractional derivative*, Adv. Difference Equ. (2009) 47. Art. ID 981728.

[4] R. P. Agarwal, V. Lupulescu, D. O'Regon, Ghaus ur Rahman, *Weak solution of fractional differential equations in non-reflexive Banach spaces via Riemann Pettis integral*, Math. Nachar **289** **4**(2016) 395–409.

[5] A. Alexiewicz, W. Orlicz, *Remarks on Riemann-integration of vector valued functions*, Studia Math. **12**(1951) 125–132.

[6] A. Ben Amar, *Some fixed point theorems and existence of weak solutions of Volterra integral equation under Henstock-Kurzweil-Pettis integrability*, Comment. Math. Univ. Carolin **52**(2011) 177–190.

[7] A. Ambrosetti, *Un teorema di esistenza per le equazioni differenziali negli spazi di Banach*, Rend. Sem. Mat. Univ. Padova **39**(1967) 349–369.

[8] A. Babakhani, V. G-Gejji, *Existence of positive solutions of nonlinear fractional differential equations*, J.Math. Anal. App. **278**(2003)434–442

[9] M. Benchohra, F. Mostefai, *Weak solutions for nonlinear fractional differential equations with integral boundary conditions in Banach spaces*, Opuscula Math. **32** **1**(2012) 31–40.

[10] M. Cichoń, *Weak solutions of ordinary differential equations in Banach spaces*, Discuss. Math. Differ. Inc. Conrol Optimal **15**(1995) 5–14.
[11] M. Cichoń, On bounded weak solutions of a nonlinear differential equation in Banach spaces, Funct. Approx. Comment. Math. **21**(1992) 27–35.
[12] E. Cramer, V. Lakshmikantham, A. R. Mitchell*On the existence of weak solutions of differential equations in nonreflexive Banach spaces*, Nonlinear Anal. **2**(1978) 259–262.
[13] M. Cichoń *On solutions of differential equations in Banach spaces*, Nonlinear Anal. **60**(2005) 651–667.
[14] I. Kubiaczyk, I.Kubiaczyk, *Kneser,s theorem for weak solutions of ordinary differential equations in Banach spaces*, Publ.Inst.Math.(Beograd)(N.S) **32 46** (1982) 99–103.
[15] V.G-Gejji, *Positive solution of a system of non-autonomous fractional differential equation*, J.Math.Anal.Appl. **302**(2005)56–64.
[16] V.G-Gejji, *Boundary value problems for multi-term fractional differential equations*, J.Math.Anal.Appl. **345**(2008)754–765.
[17] V.G-Gejji, Hossein Jafri, *Solving a multi-order fractional differential equation using Adomain decomposition*, App. Math.Comput. **189**(2007) 541–548.
[18] *F. De Blasi, On a property of the unit sphere in a Banach space*, Bull. Math. Soc. Sci. Math. R. S. Romania **21**(1977) 259–262.
[19] K. Diethelem, *The Analysis of Fractional Differential Equations*, Springer Heidelberg Dordrecht London New York, 2004.
[20] R. Gordon, *Riemann integration in Banach spaces*, Rocky Mountain J. Math. **21 3**(1991) 923–949.
[21] Y. Hu , JH. He, *On Fractal Space-Time And Fractional Calculus*, Thermal Science, **20 3**(2016)773–777 .
[22] H.A.H. Salem,*Monotonic solutions of multi-term fractional differential equations*, Comment. Math. **47 1** (2007) 1.
[23] H.A.H. Salem,*Global monotonic solutions of multi-term fractional differential equations*, Applied Mathematics and Computation **217** (2011)6597–6603.
[24] H. A. H. Salem, A. M. A. El-Sayed, O.L. Moustafa, *A note on the fractional Calculus in Banach spaces*, Studia Sci. Math. Hungar. **42 2** (2005) 115–130.
[25] H. A. H. Salem, A. M. A. El-Sayed, *Weak solution for fractional order integral equations in reflexive Banach spaces*, Math. Slovaca, 55(2005)(2),169–181.
[26] M. Kerner, Gewöhnliche *Differentialeichungen der allgemeinen Analysis, Prace Matematyczno - Fizyezne* **40**(1932) 47–87.
[27] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.
[28] F. Mainardi,*Fractional diffusive waves in visco elastic solids*, J.L. Wegner, F.R. Norwood(Eds.), Nonlinear Waves in Solids, Fairfield, 1995,pp.9397.
[29] R. Metzler, J.Klafter, *Boundary value problems for fractional diffusion equations*, Phys.A **278** (2000)107–125.
[30] I. Kubiaczyk, S. Szufla, *Kenser's theorem for weak solutions of ordinary differential equations in Banach spaces*, Publ. Inst. Math. (Beograd) (N.S.) **32 46** (1982) 99–103.
[31] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley and Sons, New York, 1993.
[32] M. Monsrefi-Torbati and J.K. Hammond, *Physical and geometrical interpretation of fractional operators*, J. Franklin Inst. **335 6** (1998)1077–1086.
[33] S. Samko, A. Kilbas and O. Marichev,*Fraction Integrals and Derivatives*, Gordon and Breach Science Publisher, 1993.
[34] S. Schwabik, Y. Guoju, *Topics in Banach space integration*, World Scientific, Singapore, 2005.
[35] S. Szufla, *Sets of fixed points of nonlinear mappings in functions spaces*, Funkcial. Ekvac. 22(1979) 121–126.
[36] R.F. Geitz, *Geometry and the Pettis integral*, Trans. Amer. Math. Soc. **169 2**(1982) 535-548.
[37] A. M. Gomaa,*Weak and strong solutions for differential equations in Banach spaces*, Chaos, Solitons and Fractionals **18**(2003) 687–692.
[38] D. O'Regan, *Weak solutions of ordinary differential equations in Banach spaces*, Appl. Math. Lett. **12**(1999) 101–105.

[39] *H. Scher, E. Montroll, Anomalous transite-time dispersion in amorphous solids*, Phy.Rev.B **12**(1975)24–55.

Ghaus ur Rahman
Department of Mathematics and Statistics University of Swat, Khyber Pakhtunkhwa, Pakistan
E-mail address: ghaus957@yahoo.com