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ON THREE-DIMENSIONAL LORENTZIAN α-SASAKIAN MANIFOLDS

(DEDICATED IN OCCASION OF THE 65-YEARS OF PROFESSOR R.K. RAINA)

AHMET YILDIZ, MINE TURAN, AND BILAL EFTAL ACET

ABSTRACT. The object of the present paper is to study three-dimensional Lorentzian α-Sasakian manifolds which are Ricci-semisymmetry, locally ϕ-symmetric and have η-parallel Ricci tensor. An example of a three-dimensional Lorentzian α-Sasakian manifold is given which verifies all the Theorems.

1. INTRODUCTION

The product of an almost contact manifold M and the real line $\mathbb{R}$ carries a natural almost complex structure. However if one takes M to be an almost contact metric manifold and supposes that the product metric G on $M \times \mathbb{R}$ is Kaehlerian, then the structure on M is cosymplectic [9] and not Sasakian. On the other hand Oubina [13] pointed out that if the conformally related metric $e^{2t}G$, t being the coordinate on $\mathbb{R}$, is Kaehlerian, then M is Sasakian and conversely.

In [15], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes:

(i) homogeneous normal contact Riemannian manifolds with $c > 0$,

(ii) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$,

(iii) a warped product space if $c < 0$.

It is known that the manifolds of class (i) are characterized by admitting a Sasakian structure.

In the Gray-Hervella classification of almost Hermitian manifolds [7], there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [4]. An almost contact metric structure on a manifold M

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is called a trans-Sasakian structure [13], [1] if the product manifold $M \times \mathbb{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$ [12] coincides with the class of the trans-Sasakian structures of type (α, β). In fact, in [12], local nature of the two subclasses, namely, C_5 and C_6 structures, of trans-Sasakian structures are characterized completely.

We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [1], β-Kenmotsu [8] and α-Sasakian [8], respectively. In [17] it is proved that trans-Sasakian structures are generalized quasi-Sasakian [8]. Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures. Then, in [18], Yıldız and Murathan introduced Lorentzian α-Sasakian manifolds.

Also, three-dimensional trans-Sasakian manifolds have been studied by De and Tripathi [6], De and Sarkar [5] and many others. Also three-dimensional Lorentzian Para-Sasakian manifolds have been studied by Shaikh and De [14].

An almost contact metric structure (ϕ, ξ, η, g) on M is called a *trans-Sasakian structure* [13] if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4 [7], where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, f d/dt) = (\phi X - f\xi, \eta(X) d/dt),$$

for all vector fields X on M and smooth functions f on $M \times \mathbb{R}$, and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [2]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{1.1}$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β). From the formula (1.1) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{1.2}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{1.3}$$

More generally one has the notion of an α-Sasakian structure [8] which may be defined by

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X), \tag{1.4}$$

where α is a non-zero constant. From the condition one may readily deduce that

$$\nabla_X \xi = -\alpha\phi X, \tag{1.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y). \tag{1.6}$$

Thus $\beta = 0$ and therefore a trans-Sasakian structure of type (α, β) with α a non-zero constant is always α-Sasakian [8]. If $\alpha = 1$, then α-Sasakian manifold is a Sasakian manifold.

Let (x, y, z) be Cartesian coordinates in $\mathbb{R}^3$, then (ϕ, ξ, η, g) given by

$$\xi = \partial/\partial z, \qquad \eta = dz - y dx,$$

$$\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix}$$

is a trans-Sasakian structure of type $(-1/(2e^z),\ 1/2)$ in $\mathbb{R}^3$ [2]. In general, in a three-dimensional K-contact manifold with structure tensors (ϕ, ξ, η, g) for a non-constant function f, if we define $g' = fg + (1 - f)\eta \otimes \eta$; then (ϕ, ξ, η, g') is a trans-Sasakian structure of type $(1/f,\ (1/2)\xi(\ln f))$ ([3], [8], [11]).

The relation between trans-Sasakian, α-Sasakian and β-Kenmotsu structures was discussed by Marrero [11].

Proposition 1.1. [11] *A trans-Sasakian manifold of dimension ≥ 5 is either α-Sasakian, β-Kenmotsu or cosymplectic.*

The paper is organized as follows: After intoroduction in section 2, we introduce the notion of Lorentzian α-Sasakian manifolds. In section 3, we study three-dimensional Lorentzian α-Sasakian manifolds. In the next section we prove that a three-dimensional Ricci -semisymmetric Lorentzian α-Sasakian manifold is a manifold of constant curvature and in section 5, it is shown that such a manifold is locally ϕ-symmetric. In section 6, we prove that three-dimensional Lorentzian α-Sasakian manifold with η-parallel Ricci tensor is also locally ϕ-symmetric. In the last section, we give an example of a locally ϕ-symmetric three-dimensional Lorentzian α-Sasakian manifold.

2. Lorentzian α-Sasakian manifolds

A differentiable manifold of dimension $(2n+1)$ is called a Lorentzian α-Sasakian manifold if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and the Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.5}$$

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi + \eta(Y)X), \tag{2.6}$$

for all $X, Y \in TM$.

Also a Lorentzian α-Sasakian manifold M satisfies

$$\nabla_X \xi = \alpha\phi X, \tag{2.7}$$

$$(\nabla_X \eta)Y = \alpha g(X, \phi Y), \tag{2.8}$$

where ∇ denotes the operator of covariant differentation with respect to the Lorentzian metric g and α is constant.

On the other hand, on a Lorentzian α-Sasakian manifold M the following relations hold [18]:

$$R(\xi, X)Y = \alpha^2(g(X, Y)\xi - \eta(Y)X), \tag{2.9}$$

$$R(X, Y)\xi = \alpha^2(\eta(Y)X - \eta(X)Y), \tag{2.10}$$

$$R(\xi, X)\xi = \alpha^2(\eta(X)\xi + X), \tag{2.11}$$

$$S(X, \xi) = 2n\alpha^2\eta(X), \tag{2.12}$$

$$Q\xi = 2n\alpha^2\xi, \tag{2.13}$$

$$S(\xi,\xi) = -2n\alpha^2, \tag{2.14}$$

for any vector fields X, Y, Z, where S is the Ricci curvature and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

A Lorentzian α-Sasakian manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for any vector fields X, Y, where a, b are functions on M^n.

3. Three-dimensional Lorentzian α-Sasakian manifolds

In a three-dimensional Lorentzian α-Sasakian manifold the curvature tensor satifies

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{\tau}{2}\left[g(Y,Z)X - g(X,Z)Y\right], \end{aligned} \tag{3.1}$$

where τ is the scalar curvature.

Then putting $Z = \xi$ in (3.1) and using (2.4) and (2.12), we have

$$R(X,Y)\xi = \eta(Y)QX - \eta(X)QY - [\frac{\tau}{2} - 2\alpha^2][\eta(Y)X - \eta(X)Y]. \tag{3.2}$$

Using (2.10) in (3.2), we get

$$\eta(Y)QX - \eta(X)QY = [\frac{\tau}{2} - \alpha^2][\eta(Y)X - \eta(X)Y]. \tag{3.3}$$

Putting $Y = \xi$ in (3.3), we obtain

$$QX = [\frac{\tau}{2} - \alpha^2]X + [\frac{\tau}{2} - 3\alpha^2]\eta(X)\xi. \tag{3.4}$$

Then from (3.4), we get

$$S(X,Y) = [\frac{\tau}{2} - \alpha^2]g(X,Y) + [\frac{\tau}{2} - 3\alpha^2]\eta(X)\eta(Y). \tag{3.5}$$

From (3.5), it follows that a Lorentzian α-Sasakian manifold is an η-Einstein manifold.

Lemma 3.1. *A three-dimensional Lorentzian α-Sasakian manifold is a manifold of constant curvature if and only if the scalar curvature is $6\alpha^2$.*

Proof. Using (3.4) and (3.5) in (3.1), we get

$$\begin{aligned} R(X,Y)Z &= [\frac{\tau}{2} - 2\alpha^2][g(Y,Z)X - g(X,Z)Y] \\ &\quad +[\frac{\tau}{2} - 3\alpha^2][g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{3.6}$$

□

From (3.6) the Lemma 3.1 is obvious.

4. Three-dimensional Ricci-Semisymmetric Lorentzian α-Sasakian manifolds

Definition 4.1. *A Lorentzian α-Sasakian manifold is said to be Ricci-semisymmetric if the Ricci tensor S satisfies*

$$R(X,Y)\cdot S=0, \tag{4.1}$$

where $R(X,Y)$ denotes the derivation of the tensor algebra at each point of the manifold.

Let us consider a three-dimensional Lorentzian α-Sasakian manifold which satisfies the condition (4.1). Hence, we can write

$$\begin{aligned}(R(X,Y)\cdot S)(U,V) &= R(X,Y)S(U,V) \\ &\quad -S(R(X,Y)U,V)-S(U,R(X,Y)V)=0.\end{aligned} \tag{4.2}$$

Then from (4.2), we have

$$S(R(X,Y)U,V)+S(U,R(X,Y)V)=0. \tag{4.3}$$

Putting $X=\xi$ in (4.3) and using (2.9) and (2.12), we get

$$S(R(\xi,Y)U,V)=2\alpha^4 g(Y,U)\eta(V)-\alpha^2 S(Y,V)\eta(U), \tag{4.4}$$

and

$$S(U,R(\xi,Y)V)=2\alpha^4 g(Y,V)\eta(U)-\alpha^2 S(Y,U)\eta(V). \tag{4.5}$$

Using (4.4) and (4.5) in (4.3),we get

$$\begin{aligned}&2\alpha^2 g(Y,U)\eta(V)-S(Y,V)\eta(U)\\ &+2\alpha^2 g(Y,V)\eta(U)-S(Y,U)\eta(V)=0, \qquad \alpha\neq 0\end{aligned} \tag{4.6}$$

Let $\{e_1,e_2,\xi\}$ be an orthonormal basis of the tangent space at each point of the three-dimensional Lorentzian α-Sasakian manifold. Then we can write

$$\left\{\begin{array}{c} g(e_i,e_j)=\delta_{ij}, \quad i,j=1,2 \\ g(\xi,\xi)=\eta(\xi)=-1, \\ \eta(e_i)=0, \quad i=1,2 \end{array}\right. . \tag{4.7}$$

Putting $Y=U=e_i$ in (4.6) and using (4.7), we obtain

$$\eta(V)[2\alpha^2 g(e_i,e_i)-S(e_i,e_i)]=0,$$

where since $S(e_i,e_i)=[\frac{\tau}{2}-\alpha^2]g(e_i,e_i)$, we get

$$[3\alpha^2-\frac{\tau}{2}]g(e_i,e_i)=0.$$

This gives

$$\tau=6\alpha^2, \quad \text{since } g(e_i,e_i)\neq 0$$

which implies by Lemma 3.1 that the manifold is of constant curvature.

Hence we can state the following:

Theorem 4.2. *A three-dimensional Ricci-semisymmetric Lorentzian α-Sasakian manifold is a manifold of constant curvature.*

5. Locally ϕ-symmetric three-dimensional Lorentzian α-Sasakian manifolds

Definition 5.1. *A Lorentzian α-Sasakian manifold is said to be locally ϕ-symmetric if*

$$\phi^2(\nabla_W R)(X,Y)Z = 0, \tag{5.1}$$

for all vector fields W, X, Y, Z orthogonal to ξ.

This notion was introduced by Takahashi for Sasakian manifolds [16].

Let us consider a three-dimensional Lorentzian α-Sasakian manifold. Firstly, differentiating (3.6) covariantly with respect to W, we get

$$\begin{aligned}(\nabla_W R)(X,Y)Z \quad = \quad & \frac{d\tau(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ & + \frac{d\tau(W)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ & + [\frac{\tau}{2} - 3\alpha^2][g(Y,Z)(\nabla_W\eta)(X)\xi - g(X,Z)(\nabla_W\eta)(Y)\xi \\ & + g(Y,Z)\eta(X)\nabla_W\xi - g(X,Z)\eta(Y)\nabla_W\xi + (\nabla_W\eta)(Y)\eta(Z)X \\ & + \eta(Y)(\nabla_W\eta)(Z)X - (\nabla_W\eta)X\eta(Z)Y - \eta(X)(\nabla_W\eta)(Z)Y)].\end{aligned}$$

Taking X, Y, Z, W orthogonal to ξ in the above equation, we have

$$\begin{aligned}(\nabla_W R)(X,Y)Z \quad = \quad & \frac{d\tau(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ & + [\frac{\tau}{2} - 3\alpha^2][g(Y,Z)(\nabla_W\eta)(X)\xi - g(X,Z)(\nabla_W\eta)(Y)\xi].\end{aligned} \tag{5.2}$$

Using the equation (2.8) in (5.2), we obtain

$$\begin{aligned}(\nabla_W R)(X,Y)Z \quad = \quad & \frac{d\tau(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ & + [\frac{\tau}{2} - 3\alpha^2][g(Y,Z)g(W,X)\xi + g(X,Z)g(W,Y)\xi].\end{aligned} \tag{5.3}$$

From (5.3), it follows that

$$\phi^2(\nabla_W R)(X,Y)Z = \frac{d\tau(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{5.4}$$

Thus, we obtain the following:

Theorem 5.2. *A three-dimensional Lorentzian α-Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature τ is constant.*

Again if the manifold is Ricci-semisymmetric, then we have seen that $\tau = 6\alpha^2$, i.e., τ =constant and hence from Theorem 5.2, we can state the following:

Theorem 5.3. *A three-dimensional Ricci-semisymmetric Lorentzian α-Sasakian manifold is locally ϕ-symmetric.*

6. Three-dimensional Lorentzian α-Sasakian manifolds with η-parallel Ricci Tensor

Definition 6.1. *The Ricci tensor S of a Lorentzian α-Sasakian manifold M is called η-parallel if it is satisfies*

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{6.1}$$

for all vector fields X, Y and Z.

The notion of Ricci η-parallelity for Sasakian manifolds was introduced by Kon [10].

Now let us consider three-dimensional Lorentzian α-Sasakian manifold with η-parallel Ricci tensor. Then from (3.5), we have

$$S(\phi X, \phi Y) = [\frac{\tau}{2} - \alpha^2] g(\phi X, \phi Y), \tag{6.2}$$

where, using (2.3), we get

$$S(\phi X, \phi Y) = [\frac{\tau}{2} - \alpha^2](g(X, Y) + \eta(X)\eta(Y)). \tag{6.3}$$

Differentiating (6.3) covariantly along Z, we obtain

$$\begin{aligned}(\nabla_Z S)(\phi X, \phi Y) &= \frac{d\tau(Z)}{2}(g(X, Y) + \eta(X)\eta(Y)) \\ &\quad +(\frac{\tau}{2} - \alpha^2)(\eta(Y)(\nabla_Z \eta)(X) + \eta(X)(\nabla_Z \eta)(Y)).\end{aligned} \tag{6.4}$$

Using (6.1) in (6.4), yields

$$\begin{aligned}&\frac{1}{2}[d\tau(Z)(g(X, Y) + \eta(X)\eta(Y)) \\ &+(\tau - 2\alpha^2)(\eta(Y)(\nabla_Z \eta)(X) + \eta(X)(\nabla_Z \eta)(Y))] = 0.\end{aligned} \tag{6.5}$$

Taking a frame field, we get from (6.5), $d\tau(Z) = 0$, for all Z.

Proposition 6.2. *If a three-dimensional Lorentzian α-Sasakian manifold has η-parallel Ricci tensor, then the scalar curvature τ is constant.*

From Theorem 5.2 and Proposition 6.2 we have the following:

Theorem 6.3. *A three-dimensional Lorentzian α-Sasakian manifold with η-parallel Ricci tensor is locally ϕ-symmetric.*

7. Example

We consider the three-dimensional manifold $M = \{(x_1, x_2, x_3) \colon x_i \in \mathbb{R}^3\}$, where (x_1, x_2, x_3) are the standard coordinates of $\mathbb{R}^3$. The vector fields

$$e_1 = e^{x_3}\frac{\partial}{\partial x_2}, \quad e_2 = e^{x_3}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \quad e_3 = \alpha\frac{\partial}{\partial x_3},$$

are linearly indepent at each point of M, where α is constant. Let g be the Lorentzian metric defined by

$$\begin{aligned}g(e_1, e_3) &= g(e_1, e_2) = g(e_2, e_3) = 0 \\ g(e_1, e_1) &= g(e_2, e_2) = 1, \; g(e_3, e_3) = -1,\end{aligned}$$

that is, the form of the metric becomes

$$g = \frac{1}{(e^{x_3})^2}(dx^2)^2 - \frac{1}{\alpha^2}(dx^3)^2,$$

which is a Lorentzian metric.

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_1, \quad \phi e_2 = -e_2, \quad \phi e_3 = 0.$$

Then using the linearity of ϕ and g, we have

$$\begin{aligned} \eta(e_3) &= -1, \\ \phi^2 Z &= Z + \eta(Z)e_3 \text{ and} \\ g(\phi Z, \phi W) &= g(Z, W) + \eta(Z)\eta(W), \end{aligned}$$

for any $Z, W \in \chi(M)$. Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorantzian paracontact structure on M. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\alpha e_1, \quad [e_2, e_3] = -\alpha e_2.$$

Koszul's formula is defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{7.1}$$

Using (7.1) for the Lorentzian metric g, we can easily calculate that

$$\begin{aligned} \nabla_{e_1} e_1 &= -\alpha e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\alpha e_1, \\ \nabla_{e_2} e_1 &= 0, \quad \nabla_{e_2} e_2 = -\alpha e_3, \quad \nabla_{e_2} e_3 = -\alpha e_2, \\ \nabla_{e_3} e_1 &= 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \end{aligned}$$

Hence the structure (ϕ, ξ, η, g) is a Lorantzian α-Sasakian manifold. Now using the above results, we obtain

$$\begin{aligned} R(e_1, e_2)e_3 &= 0, \quad R(e_2, e_3)e_3 = -\alpha^2 e_2, \\ R(e_1, e_3)e_3 &= -\alpha^2 e_1, \quad R(e_1, e_2)e_2 = \alpha^2 e_1, \\ R(e_2, e_3)e_2 &= -\alpha^2 e_3, \quad R(e_1, e_2)e_1 = -\alpha^2 e_2, \\ R(e_3, e_1)e_1 &= \alpha^2 e_3, \quad R(e_2, e_1)e_1 = \alpha^2 e_2, \\ R(e_3, e_2)e_2 &= \alpha^2 e_3. \end{aligned} \tag{7.2}$$

From which it follows that

$$\phi^2(\nabla_W R)(X, Y)Z = 0.$$

Hence, the three-dimensional Lorentzian α-Sasakian manifold is locally ϕ-symmetric.

Also from the above expressions of the curvature tensor we obtain

$$S(e_1, e_1) = S(e_2, e_2) = 0 \text{ and } S(e_3, e_3) = -2\alpha^2. \tag{7.3}$$

Hence

$$\tau = -2\alpha^2,$$

which is a constant. Thus Theorem 5.3 is verified.

Next from the expresions of the Ricci tensor we find that the manifold is Ricci-semisymmetric. Also from (7.2) we see that the manifold is a manifold of constant curvature α^2. Hence Theorem 4.2 is verified.

Finally from (7.3) it follows that

$$(\nabla_Z S)(\phi X, \phi Y) = 0,$$

for all X, Y, Z. Therefore Theorem 6.3 is also verified.

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Art and Science Faculty, Department of Mathematics, Dumlupinar University, Kütahya, TURKEY

E-mail address: ahmetyildiz@dumlupinar.edu.tr

Art and Science Faculty, Department of Mathematics, Dumlupinar University, Kütahya, TURKEY

E-mail address: mineturan@dumlupinar.edu.tr

Art and Science Faculty, Department of Mathematics, Adiyaman University, Adiyaman, TURKEY

E-mail address: eacet@adiyaman.edu.tr