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SPACES OF D_{L^p}-TYPE AND A CONVOLUTION PRODUCT ASSOCIATED WITH THE RIEMANN-LIOUVILLE OPERATOR

(DEDICATED IN OCCASION OF THE 65-YEARS OF PROFESSOR R.K. RAINA)

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Abstract. We define and study the spaces $\mathscr{M}_p(\mathbb{R}^2)$, $1 \leqslant p \leqslant +\infty$, that are of D_{L^p}-type. Using the harmonic analysis related to the Fourier transform connected with the Riemann-Liouville operator, we give a new characterization of the dual space $\mathscr{M}'_p(\mathbb{R}^2)$ and we describe its bounded subsets. Next, we define a convolution product in $\mathscr{M}'_p(\mathbb{R}^2) \times \mathcal{M}_r(\mathbb{R}^2)$, $1 \leqslant r \leqslant p < +\infty$, where $\mathcal{M}_r(\mathbb{R}^2)$ is the closure of the space $\mathscr{S}_*(\mathbb{R}^2)$ in $\mathscr{M}_r(\mathbb{R}^2)$ and we prove some new results.

1. Introduction

The space $D_{L_p}(\mathbb{R}^n)$; $1 \leq p \leq +\infty$; is formed by the measurable functions on $\mathbb{R}^n$ such that for all $\alpha \in \mathbb{N}^n$; the function $D^\alpha(f)$ belongs to $L^p(\mathbb{R}^n, dx)$ (the space of functions with p^{th} power integrable on $\mathbb{R}^n$ with respect to the Lebesgue measure dx on $\mathbb{R}^n$).

Many aspects of these spaces have been studied [1, 2, 6, 24]. In [8]; the authors have defined some spaces of functions that are of D_{L_p}-type but replacing the usual derivative by the Bessel operator $\frac{1}{r^{2\alpha+1}}\frac{d}{dr}\left(r^{2\alpha+1}\frac{d}{dr}\right)$, and they have established many results for these spaces.

In [3]; we define the so-called Riemann-Liouville operator; defined on $\mathscr{C}_*(\mathbb{R}^2)$ (the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable) by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f\left(rs\sqrt{1-t^2}, x+rt\right) \\ \qquad\qquad \times \left(1-t^2\right)^{\alpha-\frac{1}{2}}\left(1-s^2\right)^{\alpha-1} dtds; & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f\left(r\sqrt{1-t^2}, x+rt\right)\dfrac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha = 0. \end{cases}$$

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The dual operator ${}^t\mathscr{R}_\alpha$ is defined by

$$ {}^t\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{2\alpha}{\pi}\displaystyle\int_r^{+\infty}\int_{-\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} f(u,x+v) \\ \qquad\qquad \times\left(u^2-v^2-r^2\right)^{\alpha-1} u du dv; & \text{if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}} f\left(\sqrt{r^2+(x-y)^2}, y\right) dy; & \text{if } \alpha=0. \end{cases} $$

The operators $\mathscr{R}_\alpha$ and ${}^t\mathscr{R}_\alpha$ generalize the mean operator $\mathscr{R}_0$ and its dual ${}^t\mathscr{R}_0$ ([28]), defined respectively by

$$ \mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f\left(r\sin\theta, x+r\cos\theta\right) d\theta, $$

and

$$ {}^t\mathscr{R}_0(f)(r,x) = \frac{1}{\pi}\int_{\mathbb{R}} f\left(\sqrt{r^2+(x-y)^2}, y\right) dy. $$

The mean operator $\mathscr{R}_0$ and its dual ${}^t\mathscr{R}_0$ play an important role and have many applications, for example, in image processing of so-called synthetic aperture radar (SAR) data [12, 14], or in the linearized inverse scaterring problem in acoustics [11].

The Fourier transform $\mathscr{F}_\alpha$ connected with the Riemann-Liouville operator $\mathscr{R}_\alpha$ is defined by

$$ \mathscr{F}_\alpha(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\varphi_{\mu,\lambda}(r,x) d\nu_\alpha(r,x), $$

where

- $\varphi_{\mu,\lambda}(r,x) = \mathscr{R}_\alpha\left(cos(\mu.)e^{-i\lambda.}\right)(r,x)$.
- $d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}} dr\otimes dx$.

We have constructed the harmonic analysis related to the Fourier transform $\mathscr{F}_\alpha$ (inversion formula, Plancherel formula, Paley-Wiener theorem, Plancherel theorem...)

On the other hand the uncertainty principle play an important role in harmonic analysis, and many aspects of these principle have been studied. In particular, the Heisenberg-Pauli-Weyl inequality [13] has been established for several Fourier transforms [21, 22, 23].

In [17, 18, 19, 20], the author gave many generalizations of this inequality for the usual Fourier transform. In this context, we have established in [4] an $L^p - L^q$ version of Hardy theorem's for the Fourier transform $\mathscr{F}_\alpha$. Also, in [16] the second author with the other have established the Heisenberg-Pauli-Weyl inequality for $\mathscr{F}_\alpha$.

Our investigation in the present work consists to define and study some function spaces denoted by $\mathscr{M}_p(\mathbb{R}^2)$, $1 \leqslant p \leqslant +\infty$, similar to D_{L^p}, but replacing the usual derivatives by the operator

$$ \Delta_\alpha = -\left(\frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right). $$

For this, let $\mathscr{S}_*(\mathbb{R}^2)$ be the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives and even with respect to the first variable and $\mathscr{S}'_*(\mathbb{R}^2)$ its topological dual that is the space of tempered distribution on $\mathbb{R}^2$ even with resect to the first variable. Then, the singular partial differential operator Δ_α is continuous from $\mathscr{S}_*(\mathbb{R}^2)$ into itself. Moreover, for all $T \in \mathscr{S}'_*(\mathbb{R}^2)$ we define $\Delta_\alpha(T)$ by

$$\forall \varphi \in \mathscr{S}_*(\mathbb{R}^2); \quad \langle \Delta_\alpha(T), \varphi\rangle = \langle T, \Delta_\alpha(\varphi)\rangle,$$

then, Δ_α is also continuous from $\mathscr{S}'_*(\mathbb{R}^2)$ into itself.

The space $\mathscr{M}_p(\mathbb{R}^2)$ consists off all measurable functions on $\mathbb{R}^2$ even with respect to the first variable such that for all $k \in \mathbb{N}$; there exists a function denoted by $\Delta_\alpha^k(f) \in L^p(d\nu_\alpha)$ satisfying $\Delta_\alpha^k(T_f) = T_{\Delta_\alpha^k(f)}$; where

. T_f is the tempered distribution, even with respect to the first variable given by the function f.

. $L^p(d\nu_\alpha)$, $1 \leqslant p \leqslant +\infty$, is the space of measurable functions f on $[0,+\infty[\times\mathbb{R}$, such that

$$\|f\|_{p,\nu_\alpha} = \Big(\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|^p \, d\nu_\alpha(r,x)\Big)^{1/p} < +\infty, \quad 1 \leqslant p < +\infty;$$
$$\|f\|_{\infty,\nu_\alpha} = \underset{(r,x)\in[0,+\infty[\times\mathbb{R}}{ess\ sup} |f(r,x)| < \infty, \qquad p = +\infty.$$

Using the convolution product and the Fourier transform $\mathscr{F}_\alpha$ associated with the Riemann-Liouville operator, we establish firstly the following results which give a nice characterization of the elements of the dual space $\mathscr{M}'_p(\mathbb{R}^2)$

• Let $T \in \mathscr{S}'_*(\mathbb{R}^2)$. Then T belongs to $\mathscr{M}'_p(\mathbb{R}^2)$, $1 \leqslant p < +\infty$, if and only if there exist $m \in \mathbb{N}$ and $\{f_0, ..., f_m\} \subset L^{p'}(d\nu_\alpha)$;
$p' = \frac{p}{p-1}$, such that

$$T = \sum_{k=0}^{m} \Delta_\alpha^k(T_{f_k}),$$

where

$$\forall \varphi \in \mathscr{M}_p(\mathbb{R}^2); \quad \langle \Delta_\alpha^k(T_f), \varphi\rangle = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\Delta_\alpha^k(\varphi)(r,x)d\nu_\alpha(r,x);$$

• Let $T \in \mathscr{S}'_*(\mathbb{R}^2)$, $p \in [1,+\infty[$, and $p' = \frac{p}{p-1}$. Then T belongs to $\mathscr{M}'_p(\mathbb{R}^2)$ if, and only if for every $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$, the function $T * \varphi$ belongs to the space $L^{p'}(d\nu_\alpha)$, where

. $T * \varphi$ is the function defined by $T * \varphi(r,x) = \langle T, \tau_{(r,-x)}(\check{\varphi})\rangle$, with $\check{\varphi}(r,x) = \varphi(r,-x)$.

. $\tau_{(r,x)}$ is the translation operator associated with the Riemann-Liouville transform, defined on $L^p(d\nu_\alpha)$ by

$$\tau_{(r,x)}f(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^{\pi} f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}(\theta)d\theta.$$

. $\mathscr{D}_*(\mathbb{R}^2)$ is the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable and with compact support.

Next, by the fact that a subset of $\mathscr{M}'_p(\mathbb{R}^2)$ is bounded if, and only if it is equicontinuous, we show the coming result that is a good description of the bounded subsets of the dual space $\mathscr{M}'_p(\mathbb{R}^2)$

• Let $p \in [1,\infty[$ and let B' be a subset of $\mathscr{M}'_p(\mathbb{R}^2)$. The following assertions are equivalent

(i) B' is weakly (equivalently strongly) bounded in $\mathscr{M}'_p(\mathbb{R}^2)$,

(ii) there exist $C > 0$ and $m \in \mathbb{N}$, such that for every $T \in B'$, it is possible to find $f_0, ..., f_m \in L^{p'}(d\nu_\alpha)$ satisfying

$$T = \sum_{k=0}^{m} \Delta_\alpha^k (T_{f_k}) \text{ with } \max_{0 \leqslant k \leqslant m} \|f_k\|_{p',\nu_\alpha} \leqslant C,$$

(iii) for every $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$, the set $\{T * \varphi,\ T \in B'\}$ is bounded in $L^{p'}(d\nu_\alpha)$.

Finally, we define and study a convolution product on the space $\mathscr{M}'_p(\mathbb{R}^2) \times \mathcal{M}_r(\mathbb{R}^2)$, $1 \leqslant r \leqslant p < +\infty$, where $\mathcal{M}_r(\mathbb{R}^2)$ is the closure of the space $\mathscr{S}_*(\mathbb{R}^2)$ in $\mathscr{M}_r(\mathbb{R}^2)$. More precisely

• Let $p \in [1, +\infty[$. For every $(r, x) \in [0, +\infty[\times\mathbb{R}$, the translation operator $\tau_{(r,x)}$ is continuous from $\mathscr{M}_p(\mathbb{R}^2)$ into itself.

• Let $1 \leqslant r \leqslant p < \infty$ and $q \in [1, +\infty]$, such that

$$\frac{1}{q} = \frac{1}{r} - \frac{1}{p}.$$

Then for every $T \in \mathscr{M}'{}_p(\mathbb{R}^2)$, the mapping

$$\phi \longrightarrow T * \phi$$

is continuous from $\mathcal{M}_r(\mathbb{R}^2)$ into $\mathscr{M}_q(\mathbb{R}^2)$.

2. The Fourier transform associated with the Riemann-Liouville operator

In this section, we recall some harmonic analysis results related to the convolution product and the Fourier transform associated with Riemann-Liouville operator. For more details see [3, 5, 7, 16].

Let D and Ξ be the singular partial differential operators defined by

$$\begin{cases} D = \dfrac{\partial}{\partial x}; \\ \\ \Xi = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}; \qquad (r,x) \in]0, +\infty[\times\mathbb{R}, \quad \alpha \geqslant 0. \end{cases}$$

For all $(\mu, \lambda) \in \mathbb{C}^2$; the system

$$\begin{cases} Du(r,x) = -i\lambda u(r,x); \\ \\ \Xi u(r,x) = -\mu^2 u(r,x); \\ u(0,0) = 1, \qquad \dfrac{\partial u}{\partial r}(0,x) = 0;\ \forall x \in \mathbb{R}. \end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$, given by

$$\forall (r,x) \in [0, +\infty[\times\mathbb{R}; \qquad \varphi_{\mu,\lambda}(r,x) = j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)\exp(-i\lambda x), \tag{2.1}$$

where j_α is the modified Bessel function defined by

$$j_\alpha(s) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(s)}{s^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{s}{2}\right)^{2k},$$

and J_α is the Bessel function of first kind and index α [9, 10, 15, 29]. The modified Bessel function j_α has the integral representation

$$j_\alpha(s) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{\alpha-\frac{1}{2}}\exp\left(-ist\right)dt. \tag{2.2}$$

Consequently, for all $k\in\mathbb{N}$ and $s\in\mathbb{R}$; we have

$$|j_\alpha^{(k)}(s)| \leqslant 1. \tag{2.3}$$

The eigenfunction function $\varphi_{\mu,\lambda}$ satisfies the following properties

$$\bullet \sup_{(r,x)\in\mathbb{R}^2}|\varphi_{\mu,\lambda}(r,x)| = 1 \qquad \text{if, and only if } (\mu,\lambda)\in\Gamma, \tag{2.4}$$

where Γ is the set defined by

$$\Gamma = \mathbb{R}^2 \cup \left\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}^2,\ |\mu|\leqslant|\lambda|\right\}. \tag{2.5}$$

• The function $\varphi_{\mu,\lambda}$ has the following Mehler integral representation

$$\varphi_{\mu,\lambda}(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1}\cos\left(\mu rs\sqrt{1-t^2}\right)\exp\left(-i\lambda(x+rt)\right) \\ \qquad\qquad \times\left(1-t^2\right)^{\alpha-\frac{1}{2}}\left(1-s^2\right)^{\alpha-1}dtds; & \text{if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\cos\left(r\mu\sqrt{1-t^2}\right)\exp(-i\lambda(x+rt)) \\ \qquad\qquad \times\dfrac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha=0. \end{cases}$$

The precedent integral representation allows us to define the Riemann-Liouville transform $\mathscr{R}_\alpha$ associated with the operators Δ_1 ans Δ_2 by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f\left(rs\sqrt{1-t^2},x+rt\right) \\ \qquad\qquad \times\left(1-t^2\right)^{\alpha-\frac{1}{2}}\left(1-s^2\right)^{\alpha-1}dtds; & \text{if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f\left(r\sqrt{1-t^2},x+rt\right)\dfrac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha=0. \end{cases}$$

where f is any continuous functions on $\mathbb{R}^2$, even with respect to the first variable.

• From the precedent integral representation of the eigenfunction $\varphi_{\mu,\lambda}$, we have

$$\forall (r,x)\in[0,+\infty[\times\mathbb{R},\quad \varphi_{\mu,\lambda}(r,x) = \mathscr{R}_\alpha(\cos(\mu.)e^{-\lambda.})(r,x).$$

In the following, we will define the convolution product and the Fourier transform associated with the Riemann-Liouville transform. For this, we use the product formula for the function $\varphi_{\mu,\lambda}$ given by

$\forall (r,x),(s,y)\in[0,+\infty[\times\mathbb{R}$,

$$\varphi_{\mu,\lambda}(r,x)\ \varphi_{\mu,\lambda}(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi \varphi_{\mu,\lambda}\left(\sqrt{r^2+s^2+2rs\cos\theta},x+y\right) \times\sin^{2\alpha}(\theta)d\theta. \tag{2.6}$$

Definition 2.1.

(i) The translation operator associated with Riemann-Liouville transform is defined on $L^1(d\nu_\alpha)$, by for all $(r,x),(s,y)\in[0,+\infty[\times\mathbb{R}$,

$$\tau_{(r,x)}f(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta},x+y\right)\sin^{2\alpha}(\theta)d\theta.$$

(ii) The convolution product of $f, g \in L^1(d\nu_\alpha)$ is defined for all $(r, x) \in [0, +\infty[\times\mathbb{R}$, by

$$f * g(r, x) = \int_0^{+\infty}\int_{\mathbb{R}} \tau_{(r,-x)}(\check{f})(s, y)g(s, y)d\nu_\alpha(s, y),$$

where $\check{f}(s, y) = f(s, -y)$.

We have the following properties

- The product formula (2.6), can be written
$$\tau_{(r,x)}(\varphi_{\mu,\lambda})(s, y) = \varphi_{\mu,\lambda}(r, x)\varphi_{\mu,\lambda}(s, y).$$
- For all $f \in L^p(d\nu_\alpha), 1 \leqslant p \leqslant +\infty$, and for all $(r, x) \in [0, +\infty[\times\mathbb{R}$, the function $\tau_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$ and we have
$$\left\|\tau_{(r,x)}(f)\right\|_{p,\nu_\alpha} \leqslant \|f\|_{p,\nu_\alpha}. \tag{2.7}$$
- For $f, g \in L^1(d\nu_\alpha)$, the function $f * g$ belongs to $L^1(d\nu_\alpha)$; the convolution product is commutative, associative and we have
$$\|f * g\|_{1,\nu_\alpha} \leqslant \|f\|_{1,\nu_\alpha}\|g\||_{1,\nu_\alpha}.$$
Moreover, if $1 \leqslant p, q, r \leqslant +\infty$ are such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and if $f \in L^p(d\nu_\alpha),\ g \in L^q(d\nu_\alpha)$, then the function $f * g$ belongs to $L^r(d\nu_\alpha)$, and we have
$$\|f * g\|_{r,\nu_\alpha} \leqslant \|f\|_{p,\nu_\alpha}\|g\|_{q,\nu_\alpha}. \tag{2.8}$$

In the sequel, we use the following notations

- Γ_+ is the subset of Γ given by
$$\Gamma_+ = \mathbb{R}_+ \times \mathbb{R} \cup \big\{(it, x);\ (t, x) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant |x|\big\}.$$
- $\mathscr{B}_{\Gamma_+}$ is the σ-algebra defined on Γ_+ by
$$\mathscr{B}_{\Gamma_+} = \big\{\theta^{-1}(B)\ ,\ B \in \mathscr{B}_{Bor}([0, +\infty[\times\mathbb{R})\big\},$$
where θ is the bijective function defined on the set Γ_+ by
$$\theta(\mu, \lambda) = (\sqrt{\mu^2 + \lambda^2}, \lambda). \tag{2.9}$$
- $d\gamma_\alpha$ is the measure defined on $\mathscr{B}_{\Gamma_+}$ by
$$\forall\ A \in \mathscr{B}_{\Gamma_+};\ \gamma_\alpha(A) = \nu_\alpha(\theta(A)).$$
- $L^p(d\gamma_\alpha),\ p \in [1, +\infty[$, is the space of measurable functions f on Γ_+, such that
$$\|f\|_{p,\gamma_\alpha} = \Big(\int\int_{\Gamma_+} |f(\mu, \lambda)|^p d\gamma_\alpha(\mu, \lambda)\Big)^{\frac{1}{p}} < \infty, \quad \text{if } p \in [1, +\infty[,$$
$$\|f\|_{\infty,\gamma_\alpha} = \operatorname*{ess\,sup}_{(r,x)\in[0,+\infty[\times\mathbb{R}} |f(r, x)| < +\infty, \qquad \text{if } p = +\infty.$$

Proposition 2.2.

i) *For all non negative measurable function g on Γ_+, we have*

$$\int\int_{\Gamma_+} g(\mu, \lambda)d\gamma_\alpha(\mu, \lambda) = \frac{1}{2^\alpha\Gamma(\alpha + 1)\sqrt{2\pi}}\Big(\int_0^{+\infty}\int_{\mathbb{R}} g(\mu, \lambda)(\mu^2 + \lambda^2)^\alpha \mu d\mu d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} g(i\mu, \lambda)(\lambda^2 - \mu^2)^\alpha \mu d\mu d\lambda\Big).$$

ii) *For all nonnegative measurable function f on $[0,+\infty[\times\mathbb{R}$ (respectively integrable on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $d\nu_\alpha$) $fo\theta$ is a nonnegative measurable function on Γ_+ (respectively integrable on Γ_+ with respect to the measure $d\gamma_\alpha$) and we have*

$$\int\int_{\Gamma_+}(f\circ\theta)(\mu,\lambda)d\gamma_\alpha(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)d\nu_\alpha(r,x).$$

Definition 2.3. The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$, by

$$\forall(\mu,\lambda)\in\Gamma,\qquad \mathscr{F}_\alpha(f)(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x),$$

where $\varphi_{\mu,\lambda}$ is the eigenfunction given by the relation (2.1) and Γ is the set defined by the relation (2.5).

We have the following properties

- From the relation (2.4), we deduce that for $f\in L^1(d\nu_\alpha)$ the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^\infty(d\gamma_\alpha)$ and we have
$$\big\|\mathscr{F}_\alpha(f)\big\|_{\infty,\gamma_\alpha}\leqslant\|f\|_{1,\nu_\alpha}.\tag{2.10}$$
- For $f\in L^1(d\nu_\alpha)$, we have
$$\forall(\mu,\lambda)\in\Gamma,\qquad \mathscr{F}_\alpha(f)(\mu,\lambda)=\widetilde{\mathscr{F}_\alpha}(f)\circ\theta(\mu,\lambda),\tag{2.11}$$
where
$$\forall(\mu,\lambda)\in\mathbb{R}^2,\quad \widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)j_\alpha(r\mu)\exp(-i\lambda x)d\nu_\alpha(r,x),\tag{2.12}$$
and θ is the function defined by (2.9).
- Let $f\in L^1(d\nu_\alpha)$ such that the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^1(d\gamma_\alpha)$, then we have the following inversion formula for $\mathscr{F}_\alpha$, for almost every $(r,x)\in[0,+\infty[\times\mathbb{R}$,
$$f(r,x)=\int\int_{\Gamma_+}\mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda).$$
- Let $f\in L^1(d\nu_\alpha)$. For all $(s,y)\in[0,+\infty[\times\mathbb{R}$, we have
$$\forall(\mu,\lambda)\in\Gamma,\qquad \mathscr{F}_\alpha\left(\tau_{(s,y)}(f)\right)(\mu,\lambda)=\overline{\varphi_{\mu,\lambda}(s,y)}\mathscr{F}_\alpha(f)(\mu,\lambda).$$
- For $f,g\in L^1(d\nu_\alpha)$, we have
$$\forall(\mu,\lambda)\in\Gamma,\qquad \mathscr{F}_\alpha(f\ *\ g)(\mu,\lambda)=\mathscr{F}_\alpha(f)(\mu,\lambda)\mathscr{F}_\alpha(g)(\mu,\lambda).$$
- Let $p\in[1,+\infty]$. the function f belongs to $L^p(d\nu_\alpha)$ if, and only if the function $f\circ\theta$ belongs to the space $L^p(d\gamma_\alpha)$ and we have
$$\big\|f\circ\theta\big\|_{p,\gamma_\alpha}=\|f\|_{p,\nu_\alpha}.\tag{2.13}$$

  Since the mapping $\widetilde{\mathscr{F}_\alpha}$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself, then the relations (2.11) and (2.13) show that the Fourier transform $\mathscr{F}_\alpha$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ into $L^2(d\gamma_\alpha)$, namely, for every $f\in L^2(d\nu_\alpha)$, the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^2(d\gamma_\alpha$ and we have
$$\|\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}=\|f\|_{2,\nu_\alpha}.\tag{2.14}$$

Proposition 2.4. *For all f in $L^p(d\nu_\alpha)$, $p \in [1,2]$; the function $\mathscr{F}_\alpha(f)$ lies in $L^{p'}(d\gamma_\alpha)$, $p' = \frac{p}{p-1}$, and we have*

$$\left\|\mathscr{F}_\alpha(f)\right\|_{p',\gamma_\alpha} \leqslant \|f\|_{p,\nu_\alpha}.$$

Proof. The result follows from the relations (2.10), (2.14) and the Riesz-Thorin theorem's [25, 26]. □

We denote by

- $\mathscr{S}_*(\Gamma)$ (see [3, 28]) the space of functions $f : \Gamma \longrightarrow \mathbb{C}$ infinitely differentiable, even with respect to the first variable and rapidly decreasing together with all their derivatives, that means for all $k_1, k_2, k_3 \in \mathbb{N}$,

$$\sup_{(\mu,\lambda)\in\Gamma} \left(1+\mu^2+2\lambda^2\right)^{k_1} \left|\left(\frac{\partial}{\partial\mu}\right)^{k_2}\left(\frac{\partial}{\partial\lambda}\right)^{k_3} f(\mu,\lambda)\right| < +\infty,$$

  where

$$\frac{\partial f}{\partial\mu}(\mu,\lambda) = \begin{cases} \dfrac{\partial}{\partial r}\left(f(r,\lambda)\right), & \text{if } \mu = r \in \mathbb{R}; \\ \dfrac{1}{i}\dfrac{\partial}{\partial t}\left(f(it,\lambda)\right), & \text{if } \mu = it, |t| \leqslant |\lambda|. \end{cases}$$

- $\mathscr{S}'_*\left(\mathbb{R}^2\right)$ and $\mathscr{S}'_*(\Gamma)$ are respectively the dual spaces of $\mathscr{S}_*\left(\mathbb{R}^2\right)$ and $\mathscr{S}_*(\Gamma)$.

Each of these spaces is equipped with its usual topology.

Remark 2.5. (See [3]) The Fourier transform $\mathscr{F}_\alpha$ is a topological isomorphism from $\mathscr{S}_*(\mathbb{R}^2)$ onto $\mathscr{S}_*(\Gamma)$. The inverse mapping is given by for all $(r,x) \in \mathbb{R}^2$,

$$\mathscr{F}_\alpha{}^{-1}(f)(r,x) = \int\int_{\Gamma_+} f(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda).$$

Definition 2.6. The Fourier transform $\mathscr{F}_\alpha$ is defined for all $T \in \mathscr{S}'_*(\mathbb{R}^2)$ by

$$\langle \mathscr{F}_\alpha(T), \varphi\rangle = \langle T, \mathscr{F}_\alpha^{-1}(\varphi)\rangle, \qquad \varphi \in \mathscr{S}_*(\Gamma).$$

Since the Fourier transform $\mathscr{F}_\alpha$ is an isomorphism from $\mathscr{S}_*(\mathbb{R}^2)$ into $\mathscr{S}_*(\Gamma)$, we deduce that $\mathscr{F}_\alpha$ is also an isomorphism from $\mathscr{S}'_*(\mathbb{R}^2)$ into $\mathscr{S}'_*(\Gamma)$.

3. The space $\mathscr{M}_p(\mathbb{R}^2)$

We denote by

- Δ_α the partial differential operator defined by

$$\Delta_\alpha = -\left(\frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right).$$

- For all $f \in L^p(d\nu_\alpha), p \in [1,+\infty]$, T_f is the element of $\mathscr{S}'_*(\mathbb{R}^2)$ defined by

$$\forall\varphi \in \mathscr{S}_*(\mathbb{R}^2), \qquad \langle T_f, \varphi\rangle = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\varphi(r,x)d\nu_\alpha(r,x).$$

- For all $g \in L^p(d\gamma_\alpha), p \in [1,+\infty]$, T_g is the element of $\mathscr{S}'_*(\Gamma)$ defined by

$$\forall\psi \in \mathscr{S}_*(\Gamma), \qquad \langle T_g, \psi\rangle = \int\int_{\Gamma_+} g(\mu,\lambda)\psi(\mu,\lambda)d\gamma_\alpha(\mu,\lambda),$$

From proposition 2.4 and remark 2.5, we deduce that for all $f \in L^p(d\nu_\alpha),\ 1 \leqslant p \leqslant 2$, the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^{p'}(d\gamma_\alpha)$ and we have

$$\mathscr{F}_\alpha(T_f) = T_{\mathscr{F}_\alpha(\check{f})}, \tag{3.1}$$

with $p' = \dfrac{p}{p-1}$.

On the other hand, since the operator Δ_α is continuous from $\mathscr{S}_*(\mathbb{R}^2)$ into itself; we define Δ_α on $S'_*(\mathbb{R}^2)$ by setting

$$\forall \varphi \in \mathscr{S}_*(\mathbb{R}^2),\ \forall T \in S'_*(\mathbb{R}^2); \quad \langle \Delta_\alpha(T), \varphi\rangle = \langle T, \Delta_\alpha(\varphi)\rangle.$$

Then Δ_α becomes a continuous operator from $\mathscr{S}'_*(\mathbb{R}^2)$ into itself; moreover for all $f \in \mathscr{S}_*(\mathbb{R}^2)$ and for all integer k we have

$$\Delta_\alpha^k(T_f) = T_{\Delta_\alpha^k(f)}.$$

Definition 3.1. Let $p \in [1, +\infty]$. We define $\mathscr{M}_p(\mathbb{R}^2)$ to be the space of measurable functions f on $\mathbb{R}^2$, even with respect to the first variable, and such that for all $k \in \mathbb{N}$ there exists a function $\Delta_\alpha^k(f) \in L^p(d\nu_\alpha)$, satisfying

$$\Delta_\alpha^k(T_f) = T_{\Delta_\alpha^k(f)}, \quad \text{in } \mathscr{S}'_*(\mathbb{R}^2).$$

The space $\mathscr{M}_p(\mathbb{R}^2)$ is equipped with the topology generated by the family of norms

$$\gamma_{m,p}(f) = \max_{0 \leqslant k \leqslant m} \left\|\Delta_\alpha^k(f)\right\|_{p,\nu_\alpha}, \qquad m \in \mathbb{N},$$

Also, we define a distance d_p, on $\mathscr{M}_p(\mathbb{R}^2)$ by

$$\forall (f,g) \in \mathscr{M}_p(\mathbb{R}^2), \qquad d_p(f,g) = \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{\gamma_{m,p}(f-g)}{1+\gamma_{m,p}(f-g)}.$$

Then, a sequence $(f_k)_{k\in\mathbb{N}}$ converges to 0 in $\big(\mathscr{M}_p(\mathbb{R}^2), d_p\big)$ if,and only if

$$\forall m \in \mathbb{N}; \qquad \gamma_{m,p}(f_k) \underset{k\to\infty}{\longrightarrow} 0 .$$

In the following, we will give some properties of the space $\mathscr{M}_p(\mathbb{R}^2)$.

Proposition 3.2. $\big(\mathscr{M}_p(\mathbb{R}^2), d_p\big)$ *is a Frchet space.*

Proof. Let $(f_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $(\mathscr{M}_p(\mathbb{R}^2), d_p)$ and $(\Delta_\alpha^k(f_m))_{m\in\mathbb{N}} \subset L^p(d\nu_\alpha)$, such that

$$\Delta_\alpha^k(T_{f_m}) = T_{\Delta_\alpha^k(f_m)}, \qquad k \in \mathbb{N}.$$

Then, for all $k \in \mathbb{N}$; $(\Delta_\alpha^k(f_m))_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^p(d\nu_\alpha)$. We put

$$g_k = \lim_{m\to+\infty} \Delta_\alpha^k(f_m), \qquad k \in \mathbb{N}, \tag{3.2}$$

in $L^p(d\nu_\alpha)$. Thus,

$$\forall\ k \in \mathbb{N}; \qquad \lim_{m\to+\infty} \Delta_\alpha^k(T_{f_m}) = \lim_{m\to+\infty} T_{\Delta_\alpha^k(f_m)} = T_{g_k},\ \text{ in } \mathscr{S}'_*(\mathbb{R}^2).$$

Since the operator Δ_α is continuous from $\mathscr{S}'_*(\mathbb{R}^2)$ into itself and using the relation (3.2) we deduce that for all $k \in \mathbb{N}$

$$\Delta_\alpha^k(T_{g_0}) = T_{g_k}$$

This equality shows that the function g_0 belongs to the space $\mathscr{M}_p(\mathbb{R}^2)$ and that for all $k \in \mathbb{N}$, $\Delta_\alpha^k(g_0) = g_k$.

Now, the relation (3.2) implies that the sequence $(f_m)_m$ converges to g_0 in $(\mathscr{M}_p(\mathbb{R}^2), d_p)$ □

Remark 3.3. The operator Δ_α is continuous from $\mathscr{M}_p(\mathbb{R}^2)$ into itself. Moreover, for all $m \in \mathbb{N}$; we have

$$\forall f \in \mathscr{M}_p(\mathbb{R}^2); \quad \gamma_{m,p}(\Delta_\alpha(f)) \leqslant \gamma_{m+1,p}(f).$$

We denote by

• $\mathscr{C}_*(\mathbb{R}^2)$ the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable.

• $\mathscr{E}_*(\mathbb{R}^2)$ the subspace of $\mathscr{C}_*(\mathbb{R}^2)$consisting of infinitely differentiable functions on $\mathbb{R}^2$.

Proposition 3.4. *Let $p \in [1, 2]$ and $f \in \mathscr{M}_p(\mathbb{R}^2)$ then*

(i) *For all $k \in \mathbb{N}$, the function*

$$(\mu, \lambda) \longmapsto (1 + \mu^2 + 2\lambda^2)^k \mathscr{F}_\alpha(f)(\mu, \lambda)$$

belongs to the space $L^{p'}(d\gamma_\alpha)$, with $p' = \frac{p}{p-1}$.

(ii) $\mathscr{M}_p(\mathbb{R}^2) \cap \mathscr{C}_*(\mathbb{R}^2) \subset \mathscr{E}_*(\mathbb{R}^2)$.

Proof. (i) Let $f \in \mathscr{M}_p(\mathbb{R}^2),\ 1 \leqslant p \leqslant 2$. From the relation (3.1), we have

$$\mathscr{F}_\alpha(\Delta_\alpha^k(T_f)) = \mathscr{F}_\alpha(T_{\Delta_\alpha^k(f)}) = T_{\mathscr{F}_\alpha(\Delta_\alpha^k(\check{f}))}.$$

On the other hand,

$$\begin{aligned}\mathscr{F}_\alpha(\Delta_\alpha^k(T_f)) &= (\mu^2 + 2\lambda^2)^k \mathscr{F}_\alpha(T_f),\\ &= T_{(\mu^2+2\lambda^2)^k \mathscr{F}_\alpha(\check{f})},\end{aligned}$$

hence,

$$(\mu^2 + 2\lambda^2)^k \mathscr{F}_\alpha(f) = \mathscr{F}_\alpha(\Delta_\alpha^k(f)),$$

this equality, together with the fact that the function $\mathscr{F}_\alpha(\Delta_\alpha^k(f))$ belongs to the space $L^{p'}(d\nu_\alpha)$ implies (i).

(ii) Let $f \in \mathscr{M}_p(\mathbb{R}^2) \cap \mathscr{C}_*(\mathbb{R}^2)$. From the assertion (i) and the relations (2.11) and (2.13), we deduce that for all $k \in \mathbb{N}$, the function

$$(\mu, \lambda) \longmapsto (\mu^2 + \lambda^2)^k \widetilde{\mathscr{F}_\alpha}(f)(\mu, \lambda),$$

belongs to the space $L^{p'}(d\nu_\alpha)$. Hence, by using Hölder's inequality, we deduce that the function $\widetilde{\mathscr{F}_\alpha}(f)$ belongs to the space $L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$.

On the other hand, the transform $\widetilde{\mathscr{F}_\alpha}$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself, then from the inversion formula for $\widetilde{\mathscr{F}_\alpha}$, and using the continuity of the function f, we have for all $(r, x) \in \mathbb{R}^2$,

$$f(r, x) = \int_0^{+\infty} \int_{\mathbb{R}} \widetilde{\mathscr{F}_\alpha}(f)(\mu, \lambda) j_\alpha(r\mu) \exp(i\lambda x) d\nu_\alpha(\mu, \lambda). \tag{3.3}$$

Then, the result follows from the derivative theorem, the relations (2.3) and (3.3). □

Proposition 3.5. *Let $p \in [1, 2]$. Then, for all $\eta \in [2, +\infty]$,*

$$\mathscr{M}_p(\mathbb{R}^2) \cap \mathscr{C}_*(\mathbb{R}^2) \subset \mathscr{M}_\eta(\mathbb{R}^2).$$

Proof. Let $f \in \mathscr{M}_p(\mathbb{R}^2) \cap \mathscr{C}_*(\mathbb{R}^2)$, $p \in [1,2]$, $\eta \geqslant 2$ and $\eta' = \eta/(\eta-1)$. From proposition 3.4, we deduce that $f \in \mathscr{E}_*(\mathbb{R}^2)$ and that for all $k \in \mathbb{N}$, the function

$$(\mu,\lambda) \longmapsto (\mu^2+\lambda^2)^k \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda),$$

belongs to the space $L^{p'}(d\nu_\alpha)$. By applying Hölder's inequality it follows that this last function belongs to the space $L^{\eta'}(d\nu_\alpha)$.

On the other hand, for all $(r,x) \in \mathbb{R}^2$,

$$\begin{aligned}\Delta_\alpha^k(f)(r,x) &= \int_0^{+\infty}\int_{\mathbb{R}} \left(\mu^2+\lambda^2\right)^k \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda) j_\alpha(r\mu)\exp(i\lambda x) d\nu_\alpha(\mu,\lambda),\\ &= \widetilde{\mathscr{F}}_\alpha\left((\mu^2+\lambda^2)^k \widetilde{\mathscr{F}}_\alpha(\check{f})\right)(r,x).\end{aligned}$$

From proposition 2.4 and the fact that for all $g \in L^{\eta'}(d\nu_\alpha)$,

$$\|\mathscr{F}_\alpha(g)\|_{\eta,\gamma} = \left\|\widetilde{\mathscr{F}}_\alpha(g)\right\|_{\eta,\nu_\alpha},$$

we deduce that for all $k \in \mathbb{N}$, the function $\Delta_\alpha^k(f)$ belongs to the space $L^\eta(d\nu_\alpha)$. □

4. The dual space $\mathscr{M}'_p(\mathbb{R}^2)$

In this section, we will give a new characterization of the dual space $\mathscr{M}'_p(\mathbb{R}^2)$ of $\mathscr{M}_p(\mathbb{R}^2)$.

It is well known that for every $f \in \mathscr{M}_p(\mathbb{R}^2)$, the family $\{\mathfrak{N}_{m,p,\varepsilon}(f), m \in \mathbb{N}, \varepsilon > 0\}$, defined by

$$\mathfrak{N}_{m,p,\varepsilon}(f) = \left\{g \in \mathscr{M}_p(\mathbb{R}^2), \gamma_{m,p}(f-g) < \varepsilon\right\}$$

is a basis of neighborhoods of f in $\left(\mathscr{M}_p(\mathbb{R}^2), d_p\right)$.

Hence, $T \in \mathscr{M}'_p(\mathbb{R}^2)$ if, and only if there exist $m \in \mathbb{N}$ and $C > 0$, such that

$$\forall\ f \in \mathscr{M}_p(\mathbb{R}^2); \qquad |\langle T, f\rangle| \leqslant C\gamma_{m,p}(f). \tag{4.1}$$

For $f \in L^{p'}(d\nu_\alpha)$ and $\varphi \in \mathscr{M}_p(\mathbb{R}^2)$, we put

$$\langle \Delta_\alpha^k(T_f), \varphi\rangle = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\Delta_\alpha^k(\varphi)(r,x) d\nu_\alpha(r,x); \tag{4.2}$$

with $\Delta_\alpha^k(T_\varphi) = T_{\Delta_\alpha^k(\varphi)}$. Then

$$\begin{aligned}|\langle \Delta_\alpha^k(T_f), \varphi\rangle| &\leqslant \|f\|_{p',\nu_\alpha} \|\Delta_\alpha^k(\varphi)\|_{p,\nu_\alpha}\\ &\leqslant \|f\|_{p',\nu_\alpha}\gamma_{k,p}(\varphi),\end{aligned}$$

this proves that for all $f \in L^{p'}(d\nu_\alpha)$ and $k \in \mathbb{N}$, the functional $\Delta_\alpha^k(T_f)$ defined by the relation (4.2), belongs to the space $\mathscr{M}'_p\left(\mathbb{R}^2\right)$.

In the following, we will prove that every element of $\mathscr{M}'_p(\mathbb{R}^2)$ is also of this type.

Theorem 4.1. *Let $T \in \mathscr{S}'_*(\mathbb{R}^2)$. Then T belongs to $\mathscr{M}'_p(\mathbb{R}^2)$, $1 \leqslant p < +\infty$, if and only if there exist $m \in \mathbb{N}$ and $\{f_0, ..., f_m\} \subset L^{p'}(d\nu_\alpha)$, such that*

$$T = \sum_{k=0}^{m} \Delta_\alpha^k(T_{f_k}), \tag{4.3}$$

where $\Delta_\alpha^k(T_{f_k})$ is given by the relation (4.2).

Proof. It is clear that if

$$T=\sum_{k=0}^{m}\Delta_\alpha^k(T_{f_k}),\qquad \{f_0,...,f_m\}\subset L^{p'}(d\nu_\alpha),$$

then T belongs to the space $\mathscr{M}'_p(\mathbb{R}^2)$. Conversely, suppose that $T\in\mathscr{M}'_p(\mathbb{R}^2)$. From the relation (4.1), there exist $m\in\mathbb{N}$ and $C>0$, such that

$$\forall\ \varphi\in\mathscr{M}_p(\mathbb{R}^2),\qquad |\langle T,\varphi\rangle|\leqslant C\gamma_{m,p}(\varphi).$$

Let $\Big(L^p(d\nu_\alpha)\Big)^{m+1}=\Big\{(f_0,...,f_m),\ f_k\in L^p(d\nu_\alpha),\ 0\leqslant k\leqslant m\Big\}$, equipped with the norm

$$\Big\|(f_0,...,f_m)\Big\|_{(L^p(d\nu_\alpha))^{m+1}}=\ \underset{0\leqslant k\leqslant m}{max}\ \|f_k\|_{p,\nu_\alpha}.$$

We consider the mappings

$$\begin{aligned}\mathcal{A}:\mathscr{M}_p(\mathbb{R}^2)&\longrightarrow\Big(L^p(d\nu_\alpha)\Big)^{m+1}\\ \varphi&\longmapsto(\varphi,\Delta_\alpha(\varphi),...,\Delta_\alpha^m(\varphi)),\end{aligned}$$

and

$$\mathfrak{B}:\mathcal{A}(\mathscr{M}_p(\mathbb{R}^2))\longrightarrow\mathbb{C};\ \mathfrak{B}(\mathcal{A}\varphi)=\langle T,\varphi\rangle.$$

From the relation (4.1), we deduce that

$$\begin{aligned}\Big|\mathfrak{B}\big(\mathcal{A}(\varphi)\big)\Big|&=|\langle T,\varphi\rangle|\\ &\leqslant C\big\|\mathcal{A}(\varphi)\big\|_{(L^p(d\nu_\alpha))^{m+1}},\end{aligned}$$

this means that $\mathfrak{B}$ is a continuous functional on the subspace $\mathcal{A}(\mathscr{M}_p(\mathbb{R}^2))$ of the space $\Big(L^p(d\nu_\alpha)\Big)^{m+1}$. From Hahn Banach theorem, there exists a continuous extension of $\mathfrak{B}$ to $(L^p(d\nu_\alpha))^{m+1}$, denoted again by $\mathfrak{B}$.

By Riesz theorem, there exist $\big(f_0,...,f_m\big)\in\Big(L^{p'}(d\nu_\alpha)\Big)^{m+1}$, such that for all $(\varphi_0,...,\varphi_m)\in\Big(L^p(d\nu_\alpha)\Big)^{m+1}$,

$$\mathfrak{B}(\varphi_0,...,\varphi_m)=\sum_{k=0}^{m}\int_0^{+\infty}\int_{\mathbb{R}}f_k(r,x)\varphi_k(r,x)d\nu_\alpha(r,x).$$

By means of the relation (4.2), we deduce that for $\varphi\in\mathscr{M}_p(\mathbb{R}^2)$, we have

$$\langle T,\varphi\rangle=\sum_{k=0}^{m}\int_0^{+\infty}\int_{\mathbb{R}}f_k(r,x)\Delta_\alpha^k(\varphi)(r,x)d\nu_\alpha(r,x)=\sum_{k=0}^{m}\langle\Delta_\alpha^k(T_{f_k}),\varphi\rangle.$$

□

Proposition 4.2. *Let $p\geqslant 2$. Then for all $T\in\mathscr{M}'_p(\mathbb{R}^2)$, there exist $m\in\mathbb{N}$ and $F\in L^p(d\gamma_\alpha)$; such that*

$$\mathscr{F}_\alpha(T)=T_{(1+\mu^2+2\lambda^2)^mF}.$$

Proof. Let $T \in \mathscr{M}'_p(\mathbb{R}^2)$. From Theorem 4.1, there exist $m \in \mathbb{N}$ and $(f_0, ..., f_m) \subset \left(L^{p'}(d\nu_\alpha)\right)^{m+1}$, $p' = \frac{p}{p-1}$, such that

$$T = \sum_{k=0}^{m} \Delta_\alpha^k (T_{f_k}).$$

Consequently,

$$\mathscr{F}_\alpha(T) = \sum_{k=0}^{m} \mathscr{F}_\alpha(\Delta_\alpha^k(T_{f_k})) = \sum_{k=0}^{m} (\mu^2 + 2\lambda^2)^k \mathscr{F}_\alpha(T_{f_k}).$$

From the relation (3.1), we get

$$\mathscr{F}_\alpha(T) = T_{(1+\mu^2+2\lambda^2)^m F},$$

where

$$F = \sum_{k=0}^{m} \frac{(\mu^2 + 2\lambda^2)^k}{(1 + \mu^2 + 2\lambda^2)^m} \mathscr{F}_\alpha(\check{f}_k).$$

□

Proposition 4.3. *Let $T \in \mathscr{S}'_*(\mathbb{R}^2)$, then $T \in \mathscr{M}'_2(\mathbb{R}^2)$ if, and only if there exist $m \in \mathbb{N}$ and $F \in L^2(d\gamma_\alpha)$, such that*

$$\mathscr{F}_\alpha(T) = T_{(1+\mu^2+2\lambda^2)^m F}. \tag{4.4}$$

Proof. From Proposition 4.2, we deduce that if $T \in \mathscr{M}'_2(\mathbb{R}^2)$, then there exist $m \in \mathbb{N}$ and $F \in L^2(d\gamma_\alpha)$ verifying (4.4).

Conversely, suppose that (4.4) holds with $F \in L^2(d\gamma_\alpha)$. Since $\mathscr{F}_\alpha$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ into $L^2(d\gamma_\alpha)$, then there exists $G \in L^2(d\nu_\alpha)$, such that $\mathscr{F}_\alpha(G) = F$ and from the relation (3.1), we have

$$\mathscr{F}_\alpha(T_{\check{G}}) = T_F.$$

Consequently,

$$\mathscr{F}_\alpha(T) = \mathscr{F}_\alpha((I + \Delta_\alpha)^m (T_{\check{G}})),$$

thus,

$$T = \sum_{k=0}^{m} C_m^k \Delta_\alpha^k (T_{\check{G}}),$$

and Theorem 4.1 implies that T belongs to $\mathscr{M}'_2(\mathbb{R}^2)$. □

We denote by

- $\mathscr{D}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable and with compact support, equipped with its usual topology.
- For $a > 0$, $\mathscr{D}_{*,a}(\mathbb{R}^2)$ the subspace of $\mathscr{D}_*(\mathbb{R}^2)$, consisting of function f, such that $supp f \subset \mathcal{B}(0, a) = \left\{(r, x) \in \mathbb{R}^2,\ r^2 + x^2 \leqslant a^2\right\}$.
- For $a > 0$, $\mathscr{D}'_{*,a}(\mathbb{R}^2)$ the dual space of $\mathscr{D}_{*,a}(\mathbb{R}^2)$.
- For $a > 0$ and $m \in \mathbb{N}$, $\mathcal{W}_a^m(\mathbb{R}^2)$ the space of function $f : \mathbb{R}^2 \longmapsto \mathbb{C}$; of class C^{2m} on $\mathbb{R}^2$, even with respect to the first variable and with support in $\mathcal{B}(0, a)$, normed by

$$\mathcal{N}_{\infty,m}(f) = \max_{0 \leq k \leq m} \left\|\Delta_\alpha^k(f)\right\|_{\infty,\nu_\alpha}.$$

Lemma 4.4. *For all $m \in \mathbb{N}$, there exists $\beta \in \mathbb{N}$ sufficiently large, such that the function*

$$(\mu,\lambda) \longmapsto \mathfrak{g}_\beta(\mu,\lambda) = \widetilde{\mathscr{F}}_\alpha\Big(\frac{1}{(1+r^2+x^2)^\beta}\Big)(\mu,\lambda), \tag{4.5}$$

is of class C^{2m} on $\mathbb{R}^2$, even with respect to the first variable, and infinitely differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Proof. From the relation (2.12), we have for all $(\mu,\lambda) \in \mathbb{R}^2$,

$$\mathfrak{g}_\beta(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} \frac{j_\alpha(r\mu)\exp(-i\lambda x)}{(1+r^2+x^2)^\beta} d\nu_\alpha(r,x).$$

Using the relation (2.3) and the derivative theorem, we can choose β sufficiently large, such that the function $\mathfrak{g}_\beta$ is of class C^{2m} on $\mathbb{R}^2$.

On the other hand, from the integral representation (2.2) of the function j_α and applying the Fubini's theorem, we get

$$\begin{aligned}\mathfrak{g}_\beta(\mu,\lambda) &= \frac{1}{\pi 2^{\alpha-\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}\\ &\times \int_{\mathbb{R}}\Big(\int_0^{+\infty}\frac{r}{(1+r^2+x^2)^\beta}\Big(\int_0^r (r^2-t^2)^{\alpha-\frac{1}{2}}\cos(\mu t)dt\Big)dr\Big)\exp(-i\lambda x)dx\\ &= \frac{1}{\pi 2^{\alpha-\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}\int_{\mathbb{R}}\Big(\int_0^{+\infty}\cos(\mu t)\Big(\int_t^{+\infty}\frac{(r^2-t^2)^{\alpha-\frac{1}{2}}}{(1+r^2+x^2)^\beta}rdr\Big)dt\Big)\exp(-i\lambda x)dx,\end{aligned}$$

using the change of variables $s = \frac{r^2-t^2}{1+r^2+x^2}$, we obtain

$$\begin{aligned}\mathfrak{g}_\beta(\mu,\lambda) &= \frac{\Gamma(\beta-\alpha-\frac{1}{2})}{\pi 2^{\alpha+1/2}\Gamma(\beta)}\int_{\mathbb{R}}\int_0^{+\infty}\frac{\cos(\mu t)\exp(-i\lambda x)}{(1+t^2+x^2)^{\beta-\alpha-1/2}}dtdx,\\ &= \frac{\Gamma(\beta-\alpha-1/2)}{2^{\alpha+1/2}\Gamma(\beta)}\int_0^{+\infty}\frac{j_0(t\sqrt{\mu^2+\lambda^2})}{(1+t^2)^{\beta-\alpha-\frac{1}{2}}}tdt.\end{aligned}$$

again, from the relation (2.2), we deduce that for all $x \in \mathbb{R}$,

$$\begin{aligned}\int_0^{+\infty}\frac{j_0(tx)}{(1+t^2)^{\beta-\alpha-\frac{1}{2}}}tdt &= \int_0^{+\infty}\cos(sx)\Big(\int_s^{+\infty}\frac{(t^2-s^2)^{-1/2}}{(1+t^2)^{\beta-\alpha-\frac{1}{2}}}tdt\Big)ds,\\ &= \frac{\sqrt{\pi}}{2}\frac{\Gamma(\beta-\alpha-1)}{\Gamma(\beta-\alpha-\frac{1}{2})}\int_0^{+\infty}\frac{\cos(sx)}{(1+s^2)^{\beta-\alpha-1}}ds,\end{aligned}$$

and therefore for all $(\mu,\lambda) \in \mathbb{R}^2$,

$$\mathfrak{g}_\beta(\mu,\lambda) = \frac{\sqrt{\pi}}{2^{\alpha+\frac{3}{2}}}\frac{\Gamma(\beta-\alpha-1)}{\Gamma(\beta)}\int_0^{+\infty}\frac{\cos(s\sqrt{\mu^2+\lambda^2})}{(1+s^2)^{\beta-\alpha-1}}ds.$$

Now, from [10, 29] it follows that for all $(\mu,\lambda) \in \mathbb{R}^2 \setminus \{(0,0)\}$,

$$\mathfrak{g}_\beta(\mu,\lambda) = \frac{1}{2^{\beta+1}\Gamma(\beta)}(\sqrt{\mu^2+\lambda^2})^{\beta-\alpha-\frac{3}{2}}\mathrm{K}_{\beta-\alpha-\frac{3}{2}}(\sqrt{\mu^2+\lambda^2})$$

where $\mathrm{K}_{\beta-\alpha-\frac{3}{2}}$ is the Bessel function of second kind and index $\beta-\alpha-\frac{3}{2}$, called also the Mac-Donald function.

This shows that the function $\mathfrak{g}_\beta$ is infinitely differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$, even with respect to the first variable. □

Proposition 4.5. *Let $a > 0$ and $m \in \mathbb{N}$. Then there exists $n_o \in \mathbb{N}$, such that for every $n \in \mathbb{N}$, $n \geqslant n_o$, it is possible to find $\varphi_n \in \mathscr{D}_{*,a}(\mathbb{R}^2)$ and $\psi_n \in \mathcal{W}_a^m(\mathbb{R}^2)$ satisfying*

$$\delta = (I + \Delta_\alpha)^n T_{\varphi_n} + T_{\psi_n}$$

in $\mathscr{S}'_*(\mathbb{R}^2)$. Where δ is the Dirac distribution.

Proof. Let $\kappa \in \mathscr{D}_{*,a}(\mathbb{R}^2)$, such that

$$\forall\ (r,x) \in \mathbb{R}^2, \quad r^2 + x^2 \leqslant \frac{a^2}{4}, \qquad \kappa(r,x) = 1.$$

From Lemma 4.4; there exists $n_0 \in \mathbb{N}$, such that for all $n \geqslant n_0$, the function $\mathfrak{g}_n$ is of class C^{2m} on $\mathbb{R}^2$, even with respect to the first variable and infinitely differentiable on $\mathbb{R}^2 \backslash \{(0,0)\}$. Since the transform $\widetilde{\mathscr{F}_\alpha}$ defined by relation (2.12), is an isomorphism from $\mathscr{S}_*(\mathbb{R}^2)$ onto itself, and for all $\varphi \in \mathscr{S}_*(\mathbb{R}^2)$, $(r,x) \in \mathbb{R}^2$, we have

$$\varphi(r,x) = \int_0^{+\infty}\int_{\mathbb{R}} \widetilde{\mathscr{F}_\alpha}(\varphi)(s,y) j_\alpha(rs)\exp(ixy) d\nu_\alpha(s,y). \tag{4.6}$$

Then, from the relations (4.5) and (4.6), we deduce that for all $\varphi \in \mathscr{S}_*(\mathbb{R}^2)$, we have

$$\begin{aligned}
\langle (I + \Delta_\alpha)^n T_{\mathfrak{g}_n}, \varphi \rangle &= \langle T_{\mathfrak{g}_n}, (I + \Delta_\alpha)^n \varphi \rangle, \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \mathfrak{g}_n(r,x)(I + \Delta_\alpha)^n \varphi(r,x) d\nu_\alpha(r,x) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \frac{1}{(1 + r^2 + x^2)^n} \widetilde{\mathscr{F}_\alpha}\big((I + \Delta_\alpha)^n\big)(\varphi)(r,x) d\nu_\alpha(r,x), \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \widetilde{\mathscr{F}_\alpha}(\varphi)(r,x) d\nu_\alpha(r,x), \\
&= \varphi(0,0).
\end{aligned}$$

This means that for all $n \geqslant n_0$; $(I + \Delta_\alpha)^n T_{\mathfrak{g}_n} = \delta$. Then

$$\kappa(I + \Delta_\alpha)^n T_{\mathfrak{g}_n} = (I + \Delta_\alpha)^n T_{\mathfrak{g}_n} = \delta. \tag{4.7}$$

Using the fact that the function $\mathfrak{g}_n$ is infinitely differentiable on $\mathbb{R}^2 \backslash \{(0,0)\}$, even with respect to the first variable, we deduce that the function

$$\varphi_n(r,x) = (\kappa - 1)(I + \Delta_\alpha)^n \mathfrak{g}_n + (I + \Delta_\alpha)^n((1 - \kappa)\mathfrak{g}_n), \tag{4.8}$$

belongs to the space $\mathscr{D}_{*,a}(\mathbb{R}^2)$. From the relation (4.7), we have

$$T_{(\kappa-1)(I+\Delta_\alpha)^n \mathfrak{g}_n} = (\kappa - 1)(I + \Delta_\alpha)^n T_{\mathfrak{g}_n} = 0,$$

and this implies, by using the relation (4.8) that

$$T_{\varphi_n} = T_{(I+\Delta_\alpha)^n((1-\kappa)\mathfrak{g}_n)} = (I + \Delta_\alpha)^n (T_{(1-\kappa)\mathfrak{g}_n}).$$

Hence,

$$T_{\varphi_n} + (I + \Delta_\alpha)^n T_{\kappa \mathfrak{g}_n} = (I + \Delta_\alpha)^n T_{\mathfrak{g}_n} = \delta,$$

and this completes the proof of the proposition if we pick $\psi_n = \kappa \mathfrak{g}_n$. □

In the following, we will prove that the elements of all bounded subset $\mathcal{B}' \subset \mathscr{D}'_{*,a}(\mathbb{R}^2)$, can be continuously extended to the space $\mathcal{W}_a^m(\mathbb{R}^2)$. For this we define some new families of norms on the space $\mathscr{D}_{*,a}(\mathbb{R}^2)$.

For $f \in \mathscr{D}_{*,a}(\mathbb{R}^2)$, $a > 0$, we denote by

- $\mathcal{P}_m(f) = \max_{k_1+k_2\leqslant m} \left\|(\frac{\partial}{\partial r})^{k_1}(\frac{\partial}{\partial x})^{k_2} f\right\|_{\infty,\nu_\alpha}$,
- $\widetilde{\mathcal{P}}_m(f) = \max_{k_1+k_2\leqslant m} \left\|\ell_\alpha^{k_1}(\frac{\partial}{\partial x})^{k_2} f\right\|_{\infty,\nu_\alpha}$,
- $\mathcal{N}_{p,m}(f) = \max_{0\leq k\leq m} \|\Delta_\alpha^k(f)\|_{p,\nu_\alpha},\ \ p \in [1, +\infty]$.

where ℓ_α is the Bessel operator defined by

$$\ell_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}.$$

Lemma 4.6. *(i)For all $m \in \mathbb{N}$, there exists $C_1 > 0$, such that*

$$\forall\ \varphi \in \mathscr{D}_{*,a}(\mathbb{R}^2), \qquad \mathcal{P}_m(\varphi) \leqslant C_1\widetilde{\mathcal{P}}_m(\varphi).$$

(ii) For all $m \in \mathbb{N}$, there exist $C_2 > 0$ and $m' \in \mathbb{N}$, such that

$$\forall\ \varphi \in \mathscr{D}_{*,a}(\mathbb{R}^2), \qquad \widetilde{\mathcal{P}}_m(\varphi) \leqslant C_2\mathcal{N}_{p,m'}(\varphi).$$

Proof. (i) Let $\varphi \in \mathscr{D}_{*,a}(\mathbb{R}^2)$. By induction on k_1, we have

$$(\frac{\partial}{\partial r})^{k_1}(\frac{\partial}{\partial x})^{k_2}\varphi(r,x) = \sum_{n=0}^{k_1} P_n(r)(\frac{\partial}{\partial r^2})^n(\frac{\partial}{\partial x})^{k_2}\varphi(r,x), \tag{4.9}$$

where $\frac{\partial}{\partial r^2} = \frac{1}{r}\frac{\partial}{\partial r}$ and P_n is a real polynomial. Also, by induction, for all $n \geqslant 1$, we get

$$\begin{aligned}&(\frac{\partial}{\partial r^2})^n(\frac{\partial}{\partial x})^{k_2}\varphi(r,x) = \\ &\int_0^1 \cdots \int_0^1 \ell_\alpha^n(\frac{\partial}{\partial x})^{k_2}\varphi(rt_1\dots t_n, x)t_1^{2\alpha+1+2(n-1)}\dots t_n^{2\alpha+1}dt_1\dots dt_n,\end{aligned} \tag{4.10}$$

from the relations (4.9) and (4.10), it follows that for all $m \in \mathbb{N}$,

$$\mathcal{P}_m(\varphi) \leqslant C_1\widetilde{\mathcal{P}}_m(\varphi).$$

(ii) Let $p \in [1, +\infty]$, $m \in \mathbb{N}$ and $m_1 \in \mathbb{N}$, such that

$$\left\|\frac{1}{\left(1+r^2+x^2\right)^{m_1}}\right\|_{1,\nu_\alpha} < +\infty,$$

then, for all $\varphi \in \mathscr{D}_{*,a}$ and $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 \leqslant m$, we have

$$\begin{aligned}
\left\|\ell_\alpha^{k_1}\big(\frac{\partial}{\partial x}\big)^{k_2}(\varphi)\right\|_{\infty,\nu_\alpha} &= \left\|\widetilde{\mathscr{F}}_\alpha^{-1}\Big(\widetilde{\mathscr{F}}_\alpha\big(\ell_\alpha^{k_1}(\frac{\partial}{\partial x})^{k_2}(\varphi)\big)\Big)\right\|_{\infty,\nu_\alpha} \\
&\leqslant \left\|\widetilde{\mathscr{F}}_\alpha(\ell_\alpha^{k_1}\big(\frac{\partial}{\partial x}\big)^{k_2}\varphi)\right\|_{1,\nu_\alpha} \\
&= \left\|\mu^{2k_1}\lambda^{k_2}\widetilde{\mathscr{F}}_\alpha(\varphi)\right\|_{1,\nu_\alpha} \\
&\leqslant \left\|(1+\mu^2+\lambda^2)^m\widetilde{\mathscr{F}}_\alpha(\varphi)\right\|_{1,\nu_\alpha} \\
&= \left\|\frac{1}{(1+\mu^2+\lambda^2)^{m_1}}\widetilde{\mathscr{F}}_\alpha\Big((I+\Delta_\alpha)^{m+m_1}\varphi\Big)\right\|_{1,\nu_\alpha} \\
&\leqslant \left\|\frac{1}{(1+\mu^2+\lambda^2)^{m_1}}\right\|_{1,\nu_\alpha} \left\|\widetilde{\mathscr{F}}_\alpha\Big((I+\Delta_\alpha)^{m+m_1}\varphi\Big)\right\|_{\infty,\nu_\alpha} \\
&\leqslant \left\|\frac{1}{(1+\mu^2+\lambda^2)^{m_1}}\right\|_{1,\nu_\alpha} \left\|(I+\Delta_\alpha)^{m+m_1}\varphi\right\|_{1,\nu_\alpha},
\end{aligned}$$

and by Hölder's inequality, we get

$$\begin{aligned}
\left\|\ell_\alpha^{k_1}\big(\frac{\partial}{\partial x}\big)^{k_2}\varphi\right\|_{\infty,\nu_\alpha} &\leqslant \left\|\frac{1}{(1+\mu^2+\lambda^2)^{m_1}}\right\|_{1,\nu_\alpha}\Big(\nu(\mathcal{B}(0,a))\Big)^{\frac{1}{p'}}\left\|(I+\Delta_\alpha)^{m+m_1}\varphi\right\|_{p,\nu_\alpha}, \\
&\leqslant \left\|\frac{1}{(1+\mu^2+\lambda^2)^{m_1}}\right\|_{1,\nu_\alpha}\Big(\nu(\mathcal{B}(0,a))\Big)^{\frac{1}{p'}}2^{m+m_1}\mathcal{N}_{p,m+m_1}(\varphi).
\end{aligned}$$

which implies that

$$\widetilde{\mathcal{P}}_m(\varphi) \leqslant 2^{m+m_1}\Big(\nu(\mathcal{B}(0,a))\Big)^{\frac{1}{p'}}\|\frac{1}{(1+\mu^2+\lambda^2)^{m_1}}\|_{1,\nu_\alpha}\mathcal{N}_{p,m+m_1}(\varphi).$$

□

Theorem 4.7. *Let $a > 0$ and let B' be a weakly bounded set of $\mathscr{D}'_{*,a}(\mathbb{R}^2)$. Then, there exists $m \in \mathbb{N}$, such that the elements of B' can be continuously extended to $\mathcal{W}_a^m(\mathbb{R}^2)$. Moreover, the family of these extensions is equicontinuous.*

Proof. Let $p \in [1, +\infty[$. Since B' is weakly bounded in $\mathscr{D}'_{*,a}(\mathbb{R}^2)$, then from [27] and Lemma 4.6 there exist a positive constant C and $m \in \mathbb{N}$, such that

$$\forall T \ \in B',\ \forall\varphi \in \mathscr{D}_{*,a}(\mathbb{R}^2), \qquad |\langle T, \varphi\rangle| \leqslant C\mathcal{N}_{p,m}(\varphi). \tag{4.11}$$

We consider the mappings

$$\begin{aligned}
\mathcal{A} : \mathcal{W}_a^m(\mathbb{R}^2) &\longrightarrow \Big(L^p(d\nu_\alpha)\Big)^{m+1}, \\
\varphi &\longmapsto (\Delta_\alpha^k(\varphi))_{0\leqslant k\leqslant m},
\end{aligned}$$

and for all $T \in B'$,

$$\mathfrak{L}_T : \mathcal{A}(\mathscr{D}_{*,a}(\mathbb{R}^2)) \longrightarrow \mathbb{C}; \qquad \langle \mathfrak{L}_T, \mathcal{A}\varphi\rangle = \langle T, \varphi\rangle.$$

From the relation (4.11), we deduce that

$$\forall\,\varphi \in \mathscr{D}_{*,a}(\mathbb{R}^2); \qquad |\langle \mathfrak{L}_T, \mathcal{A}\varphi\rangle| \leqslant C\left\|\mathcal{A}\varphi\right\|_{(L^p(d\nu_\alpha))^{m+1}},$$

this means that $\mathfrak{L}_T$ is a continuous functional on the subspace $\mathcal{A}\left(\mathscr{D}_{*,a}(\mathbb{R}^2)\right)$ of the space $\Big(L^p(d\nu_\alpha)\Big)^{m+1}$, and that for all $T \in B'$,

$$\|\mathfrak{L}_T\|_{\mathcal{A}(\mathscr{D}_{*,a}(\mathbb{R}^2))} = \sup_{\|\mathcal{A}\varphi\|_{(L^p(d\nu_\alpha))^{m+1}} \leqslant 1} |\langle \mathfrak{L}_T, \mathcal{A}\varphi\rangle| \leqslant C$$

From the Hahn Banach theorem, $\mathfrak{L}_T$ can be continuously extended to the space $\Big(L^p(d\nu_\alpha)\Big)^{m+1}$, denoted again by $\mathfrak{L}_T$. Furthermore, for all $T \in B'$

$$\begin{aligned} \|\mathfrak{L}_T\|_{(L^p(d\nu_\alpha))^{m+1}} &= \sup_{\|\psi\|_{(L^p(d\nu_\alpha))^{m+1}} \leqslant 1} \big|\langle \mathfrak{L}_T, \psi\rangle\big| \\ &= \|\mathfrak{L}_T\|_{\mathcal{A}(\mathscr{D}_{*,a}(\mathbb{R}^2))} \leqslant C. \end{aligned} \tag{4.12}$$

Now, from the Riesz theorem, for all $T \in B'$, there exists $(f_{k,T})_{0 \leqslant k \leqslant m} \subset L^{p'}(d\nu_\alpha)$, such that for all $\psi = (\psi_0, ..., \psi_m) \in \Big(L^p(d\nu_\alpha)\Big)^{m+1}$,

$$\langle \mathfrak{L}_T, \psi\rangle = \sum_{k=0}^{m} \int_0^{+\infty} \int_{\mathbb{R}} f_{k,T}(r,x)\psi_k(r,x) d\nu_\alpha(r,x);$$

with

$$\|\mathfrak{L}_T\|_{(L^p(d\nu_\alpha))^{m+1}} = \max_{0 \leqslant k \leq m} \|f_{k,T}\|_{p',\nu_\alpha}.$$

Thus, from the relation (4.12) it follows that

$$\forall T \in B',\ \forall k \in \mathbb{N},\ 0 \leqslant k \leqslant m; \qquad \|f_{k,T}\|_{p',\nu_\alpha} \leqslant C. \tag{4.13}$$

In particular, for $\varphi \in \mathcal{W}_a^m(\mathbb{R}^2)$, we have

$$\langle \mathfrak{L}_T, \mathcal{A}\varphi\rangle = \sum_{k=0}^{m} \int_0^{+\infty} \int_{\mathbb{R}} f_{k,T}(r,x) \Delta_\alpha^k(\varphi)(r,x) d\nu_\alpha(r,x).$$

Using Hölder's inequality and the relation (4.13), we get for all $T \in B'$ and $\varphi \in \mathcal{W}_a^m(\mathbb{R}^2)$,

$$\big|\langle \mathfrak{L}_T, \mathcal{A}\varphi\rangle\big| \leqslant C(m+1)\Big(\nu_\alpha(\mathcal{B}(0,a))\Big)^{1/p} \mathcal{N}_{\infty,m}(\varphi),$$

this shows that the mapping $\mathfrak{L}_T o \mathcal{A}$ is a continuous extension of T on $\mathcal{W}_a^m(\mathbb{R}^2)$, and that the family $\{\mathfrak{L}_T o \mathcal{A}\}_{T \in B'}$ is equicontinuous, when applied to $\mathcal{W}_a^m(\mathbb{R}^2)$. □

In the following, we will give a new characterization of the space $\mathscr{M}_p'(\mathbb{R}^2)$.

Theorem 4.8. *Let $T \in \mathscr{S}_*'(\mathbb{R}^2)$, $p \in [1, +\infty[$, and $p' = \frac{p}{p-1}$. Then T belongs to the space $\mathscr{M}_p'(\mathbb{R}^2)$ if, and only if for every $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$, the function $T * \varphi$ belongs to the space $L^{p'}(d\nu_\alpha)$, where*

$$T * \varphi(r,x) = \langle T, \tau_{(r,-x)}(\check{\varphi})\rangle.$$

Proof. • Let $T \in \mathscr{M}_p'(\mathbb{R}^2)$. From Theorem 4.1, there exist $m \in \mathbb{N}$ and $f_0, ..., f_m \in L^{p'}(d\nu_\alpha)$, such that

$$T = \sum_{k=0}^{m} \Delta_\alpha^k(T_{f_k}),$$

in $\mathscr{M}'_p(\mathbb{R}^2)$. Thus, for every $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$;

$$T * \varphi = \sum_{k=0}^{m} T_{f_k} * \Delta_\alpha^k(\varphi) = \sum_{k=0}^{m} f_k * \Delta_\alpha^k(\varphi).$$

Since, for all $k \in \mathbb{N},\ 0 \leqslant k \leqslant m,\ f_k \in L^{p'}(d\nu_\alpha)$ and $\Delta_\alpha^k(\varphi) \in L^1(d\nu_\alpha)$; then from the inequality (2.8), we deduce that $f_k * \Delta_\alpha^k(\varphi) \in L^{p'}(d\nu_\alpha)$. This implies that the function $T * \varphi$ belongs to the space $L^{p'}(d\nu_\alpha)$.

• Conversely, let $T \in \mathscr{S}'_*(\mathbb{R}^2)$ such that for every $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$) the function $T * \varphi$ belongs to the space $L^{p'}(d\nu_\alpha)$. For $\varphi,\ \psi$ in $\mathscr{D}_*(\mathbb{R}^2)$, we have

$$\langle T_{T*\varphi}, \psi\rangle = \langle T, \varphi * \check{\psi}\rangle = \langle T, \psi * \check{\varphi}\rangle = \langle T_{T*\psi}, \varphi\rangle.$$

Thus, from Hlder's inequality and using the hypothesis, we obtain

$$|\langle T_{T*\varphi}, \psi\rangle| \leqslant \|T * \psi\|_{p',\nu_\alpha} \|\varphi\|_{p,\nu_\alpha},$$

from which, we deduce that the set

$$B' = \left\{T_{T*\varphi},\ \varphi \in \mathscr{D}_*(\mathbb{R}^2);\ \|\varphi\|_{p,\nu_\alpha} \leqslant 1\right\},$$

is bounded in $\mathscr{D}'_*(\mathbb{R}^2)$. Now, using Theorem 4.7, it follows that for all $a > 0$ there exists $m \in \mathbb{N}$, such that for all $\varphi \in \mathscr{D}_*(\mathbb{R}^2);\ \|\varphi\|_{p,\nu_\alpha} \leqslant 1$, the mapping $T_{T*\varphi}$ can be continuously extended to the space $\mathcal{W}_a^m(\mathbb{R}^2)$ and the family of these extensions is equicontinuous, which means that there exists $C > 0$, such that for all $\varphi \in \mathscr{D}_*(\mathbb{R}^2),\ \|\varphi\|_{p,\nu_\alpha} \leqslant 1$, and $\psi \in \mathcal{W}_a^m(\mathbb{R}^2)$,

$$|\langle T_{T*\varphi}, \psi\rangle| \leqslant C\mathcal{N}_{\infty,m}(\psi).$$

This involves that for all $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$, for all $\psi \in \mathcal{W}_a^m(\mathbb{R}^2)$,

$$|\langle T_{T*\varphi}, \psi\rangle| \leqslant C\mathcal{N}_{\infty,m}(\psi)\|\varphi\|_{p,\nu_\alpha}. \tag{4.14}$$

On the other hand, we have for all $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$, and $\psi \in \mathcal{W}_a^m(\mathbb{R}^2)$,

$$\langle T_{T*\varphi}, \psi\rangle = \langle T * T_\psi, \check{\varphi}\rangle, \tag{4.15}$$

where, for all $\varphi \in \mathscr{S}_*(\mathbb{R}^2)$,

$$\langle T * T_\psi, \varphi\rangle = \langle T, T_\psi * \varphi\rangle = \langle T, \psi * \varphi\rangle.$$

The relations (4.14) and (4.15) lead to, for all $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$,

$$|\langle T * T_\psi, \varphi\rangle| \leqslant C\mathcal{N}_{\infty,m}(\psi)\|\varphi\|_{p,\nu_\alpha}.$$

This last inequality shows that the functional $T * T_\psi$ can be continuously extended to the space $L^p(d\nu_\alpha)$ and from Riesz theorem there exists $g \in L^{p'}(d\nu_\alpha)$, such that

$$T * T_\psi = T_g \tag{4.16}$$

Furthermore, from proposition 4.5, there exist $n \in \mathbb{N},\ \psi_n \in \mathcal{W}_a^m(\mathbb{R}^2)$, and $\varphi_n \in \mathscr{D}_{*,a}(\mathbb{R}^2)$ satisfying

$$\delta = (I + L)^n T_{\psi_n} + T_{\varphi_n},$$

then,

$$T = (I + L)^n (T * T_{\psi_n}) + T * T_{\varphi_n} = (I + L)^n (T * T_{\psi_n}) + T_{T*\varphi_n}.$$

We complete the proof by using the hypothesis, the relation (4.16) and Theorem 4.1. □

In the following, we will give a characterization of the bounded subsets of $\mathscr{M}'_p(\mathbb{R}^2)$.

Theorem 4.9. *Let $p \in [1, \infty[$ and let B' be a subset of $\mathscr{M}'_p(\mathbb{R}^2)$. The following assertions are equivalent*

(i) *the subset B' is weakly bounded in $\mathscr{M}'_p(\mathbb{R}^2)$,*

(ii) *there exist $C > 0$ and $m \in \mathbb{N}$, such that for every $T \in B'$, it is possible to find $f_0, ..., f_m \in L^{p'}(d\nu_\alpha)$ satisfying*

$$T = \sum_{k=0}^{m} \Delta_\alpha^k (T_{f_k}) \text{ with } \max_{0 \leqslant k \leqslant m} \|f_k\|_{p',\nu_\alpha} \leqslant C,$$

(iii) *for every $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$, the set $\{T * \varphi,\ T \in B'\}$ is bounded in $L^{p'}(d\nu_\alpha)$.*

Proof. (1) Suppose that the subset B' is weakly bounded in $\mathscr{M}'_p(\mathbb{R}^2)$, then from [27] B' is equicontinuous. There exist $C > 0$ and $m \in \mathbb{N}$, such that

$$\forall\ T \in B',\ \forall\ f \in \mathscr{M}_p(\mathbb{R}^2), \qquad |\langle T, f\rangle| \leqslant C\gamma_{m,p}(f). \tag{4.17}$$

As in the proof of theorem 4.7, we consider the mappings

$$\mathcal{A} : \mathscr{M}_p(\mathbb{R}^2) \longrightarrow \Big(L^p(d\nu_\alpha)\Big)^{m+1},$$
$$f \longmapsto (f, \Delta_\alpha(f), ..., \Delta_\alpha^m(f)),$$

and for all $T \in B'$,

$$\mathfrak{L}_T : \mathcal{A}(\mathscr{M}_p(\mathbb{R}^2)) \longrightarrow \mathbb{C}; \quad \langle \mathfrak{L}_T, \mathcal{A}(f)\rangle = \langle T, f\rangle.$$

Then, the relation (4.17) implies that for all $\varphi \in \mathscr{M}_p(\mathbb{R}^2)$,

$$\big|\mathfrak{L}_T(\mathcal{A}\varphi)\big| \leqslant C\|\mathcal{A}\varphi\|_{(L^p(d\nu_\alpha))^{m+1}}.$$

Using Hahn Banach theorem and Riesz theorem, we deduce that $\mathfrak{L}_T$ can be continuously extended to $\Big(L^p(d\nu_\alpha)\Big)^{m+1}$, denoted again by $\mathfrak{L}_T$, and that there exists $(f_k)_{0\leqslant k\leqslant m} \subset L^{p'}(d\nu_\alpha)$, $p' = \frac{p}{p-1}$, verifying for all $\psi = (\psi_0, ..., \psi_m) \in \Big(L^p(d\nu_\alpha)\Big)^{m+1}$,

$$\langle \mathfrak{L}_T, \psi\rangle = \sum_{k=0}^{m} \int_0^{+\infty}\int_{\mathbb{R}} f_k(r,x)\psi_k(r,x)d\nu_\alpha(r,x),$$

with $\big\|\mathfrak{L}_T\big\|_{(L^p(d\nu_\alpha))^{m+1}} = \max_{0\leqslant k\leq m} \|f_k\|_{p',\nu_\alpha} \leqslant C$.

In particular, if $\psi = \mathcal{A}(f),\ f \in \mathscr{M}_p(\mathbb{R}^2)$,

$$\langle \mathfrak{L}_T, \mathcal{A}(f)\rangle = \langle T, f\rangle = \sum_{k=0}^{m} \langle \Delta_\alpha^k(T_{f_k}), f\rangle,$$

this proves that (i) implies (ii).

(2) Suppose that there exist $C > 0$ and $m \in \mathbb{N}$, such that for every $T \in B'$ one can find $f_0, ..., f_m \in L^{p'}(d\nu_\alpha)$, satisfying

$$T = \sum_{k=0}^{m} \Delta_\alpha^k(T_{f_k}), \qquad \max_{0\leqslant k\leqslant m} \|f_k\|_{p',\nu_\alpha} \leqslant C,$$

then, for all $f \in \mathscr{M}_p(\mathbb{R}^2)$ and $T \in B'$,

$$\langle T, f\rangle = \sum_{k=0}^{m} \int_0^{+\infty} \int_{\mathbb{R}} f_k(r,x) g_k(r,x) d\nu_\alpha(r,x);$$

consequently,

$$|\langle T, f\rangle| \leqslant C(m+1)\gamma_{m,p}(f),$$

which means that the set B' is weakly bounded in $\mathscr{M}'_p(\mathbb{R}^2)$ and proves that (ii) implies (i).

(3) Suppose that (ii) holds. Let $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$, then from Theorem 4.8 we know that for all $T \in B'$ the function $T * \varphi$ belongs to the space $L^{p'}(d\nu_\alpha)$. But

$$T * \varphi = \sum_{k=0}^{m} T_{f_k} * \Delta_\alpha^k(\varphi)$$

thus, for all $T \in B'$,

$$\big\|T * \varphi\big\|_{p',\nu_\alpha} \leqslant C(m+1)\gamma_{m,p}(\varphi).$$

This shows that the set $\big\{T * \varphi,\ T \in B'\big\}$ is bounded in $L^{p'}(d\nu_\alpha)$ and therefore (ii) involves (iii).

(4) Suppose that (iii) holds and let $T \in B'$. For all $\varphi, \psi \in \mathscr{D}_*(\mathbb{R}^2)$, we have

$$|\langle T_{T*\varphi}, \psi\rangle| = |\langle T_{T*\psi}, \varphi\rangle| \leqslant \|T * \psi\|_{p',\nu_\alpha} \|\varphi\|_{p,\nu_\alpha},$$

from which, we deduce that the set

$$\Big\{T_{T*\varphi},\ T \in B',\ \varphi \in \mathscr{D}_*(\mathbb{R}^2);\ \|\varphi\|_{p,\nu_\alpha} \leqslant 1\Big\},$$

is bounded in $\mathscr{D}'_*(\mathbb{R}^2)$.

Now, using Theorem 4.7, it follows that for all $a > 0$ there exists $m \in \mathbb{N}$, such that for all $\varphi \in \mathscr{D}_*(\mathbb{R}^2); \|\varphi\|_{p,\nu_\alpha} \leqslant 1$, and $T \in B'$, the mapping $T_{T*\varphi}$ can be continuously extended on the space $\mathcal{W}_a^m(\mathbb{R}^2)$ and the family of these extensions is equicontinuous. This means that there exists $C > 0$ such that for all $T \in B'$, $\varphi \in \mathscr{D}_*(\mathbb{R}^2)$, and $\psi \in \mathcal{W}_a^m(\mathbb{R}^2)$, the inequality (4.14) holds. Using the relations (4.14) and (4.15), we deduce that the functional $T * T_\psi$ can be continuously extended on the space $L^p(d\nu_\alpha)$ and from Riesz theorem there exists $g_{T,\psi} \in L^{p'}(d\nu_\alpha)$, such that

$$T * T_\psi = T_{g_{T,\psi}}. \tag{4.18}$$

Applying again the relations (4.14) and (4.15), we deduce that for all $T \in B'$,

$$\|g_{T,\psi}\|_{p',\nu_\alpha} \leqslant C\mathcal{N}_{\infty,m}(\psi). \tag{4.19}$$

Again by Proposition 4.5, it follows that there exist $n \in \mathbb{N}, \psi_n \in \mathcal{W}_a^m(\mathbb{R}^2)$ and $\varphi_n \in \mathscr{D}_{*,a}(\mathbb{R}^2)$ verifying for all $T \in B'$,

$$T = T * \delta = (I + \Delta_\alpha)^n (T * T_{\psi_n}) + T_{T*\varphi_n},$$

and by the relation (4.18), we get

$$T = (I + \Delta_\alpha)^n T_{g_{T,n}} + T_{T*\varphi_n}. \tag{4.20}$$

On the other hand, from the hypothesis there exists $C_1 > 0$, such that

$$\forall T \in B', \quad \|T * \varphi_n\|_{p',\nu_\alpha} \leqslant C_1, \tag{4.21}$$

However, by the relation (4.19), we have

$$\forall\, T \in B', \quad \|g_{T,n}\|_{p',\nu_\alpha} \leqslant C_2 \mathcal{N}_{\infty,m}(\varphi_n). \tag{4.22}$$

The relations (4.2), (4.20), (4.21) and (4.22) show that the set B' is bounded in $\mathscr{M}'_p(\mathbb{R}^2)$. □

5. Convolution product on the space $\mathscr{M}'_p(\mathbb{R}^2) \times \mathcal{M}_r(\mathbb{R}^2)$

In this section, we define and study a convolution product on the space $\mathscr{M}'_p(\mathbb{R}^2) \times \mathcal{M}_r(\mathbb{R}^2)$, $1 \leqslant r \leqslant p < +\infty$, where $\mathcal{M}_r(\mathbb{R}^2)$ is the closure of the space $\mathscr{S}_*(\mathbb{R}^2)$ in $\mathscr{M}_r(\mathbb{R}^2)$.

Proposition 5.1. *Let $p \in [1, +\infty[$. For every $(r, x) \in [0, +\infty[\times\mathbb{R}$, the translation operator $\tau_{(r,x)}$ given by Definition 2.1 (i), is a continuous mapping from $\mathscr{M}_p(\mathbb{R}^2)$ into itself. Moreover, for all $f \in \mathscr{M}_p(\mathbb{R}^2)$ and $k \in \mathbb{N}$, we have*

$$\Delta_\alpha^k(\tau_{(r,x)}(f)) \quad = \quad \tau_{(r,x)}(\Delta_\alpha^k(f)), \tag{5.1}$$

where

$$\Delta_\alpha^k(T_f) = T_{\Delta_\alpha^k(f)}.$$

Proof. Let $f \in \mathscr{M}_p(\mathbb{R}^2)$. Since for all $(r, x) \in [0, +\infty[\times\mathbb{R}$, the translation operator $\tau_{(r,x)}$ is continuous from $L^p(d\nu_\alpha)$ into itself; then the function $\tau_{(r,x)}(f)$ belongs to the space $L^p(d\nu_\alpha)$. Moreover; for all $\varphi \in \mathscr{S}_*(\mathbb{R}^2)$ and $k \in \mathbb{N}$; we have

$$\begin{aligned}
\langle \Delta_\alpha^k(T_{\tau_{(r,x)}(f)}), \varphi\rangle &= \langle T_{\tau_{(r,x)}(f)}, \Delta_\alpha^k(\varphi)\rangle \\
&= \int_0^{+\infty}\int_{\mathbb{R}} f(s,y)\tau_{(r,-x)}(\Delta_\alpha^k(\varphi))(s,y)\,d\nu_\alpha(s,y) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} f(s,y)\Delta_\alpha^k(\tau_{(r,-x)}(\varphi))(s,y)\,d\nu_\alpha(s,y) \\
&= \langle T_f, \Delta_\alpha^k(\tau_{(r,-x)}(\varphi))\rangle \\
&= \langle \Delta_\alpha^k(T_f), \tau_{(r,-x)}(\varphi)\rangle \\
&= \langle T_{\Delta_\alpha^k(f)}, \tau_{(r,-x)}(\varphi)\rangle \\
&= \langle T_{\tau_{(r,x)}(\Delta_\alpha^k(f))}, \varphi\rangle.
\end{aligned}$$

Since the operator $\tau_{(r,x)}$ is continuous from $L^p(d\nu_\alpha)$ into itself, we deduce that for all $f \in \mathscr{M}_p(\mathbb{R}^2)$ and $(r, x) \in [0, +\infty[\times\mathbb{R}$, the function $\tau_{(r,x)}(f)$ belongs to the space $\mathscr{M}_p(\mathbb{R}^2)$ and that for all $k \in \mathbb{N}$, $\Delta_\alpha^k(\tau_{(r,x)}(f)) = \tau_{(r,x)}(\Delta_\alpha^k(f))$. Moreover, from the relations (2.7) and (5.1) , we have

$$\gamma_{m,p}(\tau_{(r,x)}(f)) = \max_{0\leqslant k\leqslant m} \left\|\tau_{(r,x)}(\Delta_\alpha^k(f))\right\|_{p,\nu_\alpha} \leqslant \max_{0\leqslant k\leqslant m} \|\Delta_\alpha^k(f)\|_{p,\nu_\alpha} = \gamma_{m,p}(f),$$

which shows that the operator $\tau_{(r,x)}$ is continuous from $\mathscr{M}_p(\mathbb{R}^2)$ into itself. □

The precedent proposition allows us to define the coming convolution product

Definition 5.2. The convolution product of $T \in \mathscr{M}'_p(\mathbb{R}^2)$ and $f \in \mathscr{M}_p(\mathbb{R}^2)$ is defined by

$$\forall (r,x) \in [0, +\infty[\times\mathbb{R}; \qquad T * f(r,x) = \langle T, \tau_{(r,-x)}(\check{f})\rangle.$$

Let $T \in \mathscr{M}'_p(\mathbb{R}^2)$ and $\varphi \in \mathscr{M}_p(\mathbb{R}^2)$. From Theorem 4.1, there exist $m \in \mathbb{N}$ and $\{f_0, ..., f_m\} \subset L^{p'}(d\nu_\alpha)$, such that

$$T = \sum_{k=0}^{m} \Delta_\alpha^k(T_{f_k}).$$

Thus,

$$\begin{aligned} T * \varphi(r,x) &= \sum_{k=0}^{m} \langle \Delta_\alpha^k(T_{f_k}), \tau_{(r,-x)}(\check{\varphi})\rangle \\ &= \sum_{k=0}^{m} \langle T_{f_k}, \tau_{(r,-x)}(\Delta_\alpha^{\check{k}}(\varphi))\rangle \\ &= \sum_{k=0}^{m} \left(f_k * \Delta_\alpha^k(\varphi)\right)(r,x) \end{aligned}$$

Using the relation (2.8) and the fact that $\varphi \in \mathscr{M}_p(\mathbb{R}^2)$ we deduce that the function $T * \varphi$ belongs to $L^\infty(d\nu_\alpha)$ and

$$||T * \varphi||_{\infty,\nu_\alpha} \leqslant \gamma_{m,p}(\varphi)\left(\sum_{k=0}^{m} ||f_k||_{p',\nu_\alpha}\right) \tag{5.2}$$

Let $T \in \mathscr{M}'{}_p(\mathbb{R}^2)$, $T = \sum_{k=0}^{m} \Delta_\alpha^k(T_{f_k})$ with $\{f_k\}_{0 \leqslant k \leqslant m} \subset L^{p'}(d\nu_\alpha)$ and $\phi \in \mathcal{M}_r(\mathbb{R}^2)$, $1 \leqslant r \leqslant p$. From the inequality (2.8), it follows that for $0 \leqslant k \leqslant m$ the function $f_k * \Delta_\alpha^k(\phi)$ belongs to the space $L^q(d\nu_\alpha)$ with, $1/q = 1/r + 1/p' - 1 = 1/r - 1/p$ and by using the density of $\mathscr{S}_*(\mathbb{R}^2)$ in $\mathcal{M}_r(\mathbb{R}^2)$, we deduce that the expression $\sum_{k=0}^{m} f_k * \Delta_\alpha^k(\phi)$ is independent of the sequence $\{f_k\}_{0 \leqslant k \leqslant m}$. Then, we put

$$T * \phi = \sum_{k=0}^{m} f_k * \Delta_\alpha^k(\phi).$$

Again, from the relation (2.8), we deduce that the function $T * \phi$ belongs to the space $L^q(d\nu_\alpha)$ and

$$\|T * \phi\|_{q,\nu_\alpha} \leqslant \gamma_{m,r}(\phi)(\sum_{k=0}^{m} \|f_k\|_{p',\nu_\alpha}) \tag{5.3}$$

This allows us to say that

$$\mathscr{M}'_p(\mathbb{R}^2) * \mathcal{M}_r(\mathbb{R}^2) \subset L^q(d\nu_\alpha).$$

Lemma 5.3. *Let $1 \leqslant r \leqslant p < \infty$. Then*

i) The operator Δ_α is continuous from $\mathcal{M}_r(\mathbb{R}^2)$ into itself.

ii) For all $T \in \mathscr{M}'{}_p(\mathbb{R}^2)$ and $\phi \in \mathcal{M}_r(\mathbb{R}^2)$, the function $T * \phi$ belongs to the space $\mathscr{M}_q(\mathbb{R}^2)$ and we have

$$\forall k \in \mathbb{N}, \quad \Delta_\alpha^k(T * \phi) = T * \Delta_\alpha^k(\phi).$$

Proof. i) Let $f \in \mathcal{M}_r(\mathbb{R}^2)$. There exists $(f_k)_k \subset \mathscr{S}_*(\mathbb{R}^2)$ such that

$$\forall m \in \mathbb{N}, \quad \lim_{k \to +\infty} \gamma_{m,r}(f_k - f) = 0.$$

However,

$$\gamma_{m,r}(\Delta_\alpha(f_k) - \Delta_\alpha(f)) \leqslant \gamma_{m+1,r}(f_k - f),$$

thus, the sequence $(\Delta_\alpha(f_k))_k$ of $\mathscr{S}_*(\mathbb{R}^2)$ converges to $\Delta_\alpha(f)$ in $\mathcal{M}_r(\mathbb{R}^2)$, which shows that the function $\Delta_\alpha(f)$ belongs to the space $\mathcal{M}_r(\mathbb{R}^2)$.

ii) If $\phi \in \mathscr{S}_*(\mathbb{R}^2)$, then the function $T * \phi$ is infinitely differentiable, and we have

$$\Delta_\alpha^k(T_{T*\phi}) = T_{\Delta_\alpha^k(T*\phi)} = T_{T*\Delta_\alpha^k(\phi)},$$

therefore, the result follows from the density of $\mathscr{S}_*(\mathbb{R}^2)$ in $\mathcal{M}_r(\mathbb{R}^2)$, the relation (5.3) and the fact that the operator Δ_α is continuous from $\mathcal{M}_r(\mathbb{R}^2)$ into itself. □

Proposition 5.4. *Let $1 \leqslant r \leqslant p < \infty$ and $q \in [1, +\infty]$, such that*

$$\frac{1}{q} = \frac{1}{r} - \frac{1}{p}. \tag{5.4}$$

Then for every $T \in \mathscr{M}'_p(\mathbb{R}^2)$, the mapping

$$\phi \longrightarrow T * \phi$$

is continuous from $\mathcal{M}_r(\mathbb{R}^2)$ into $\mathscr{M}_q(\mathbb{R}^2)$.

Proof. Let $T \in \mathscr{M}'_p(\mathbb{R}^2)$; $T = \sum_{k=0}^{m} \Delta_\alpha^k(T_{f_k})$ and $\phi \in \mathcal{M}_r(\mathbb{R}^2)$. From Lemma 5.3, the function $T * \phi$ belongs to the space $\mathscr{M}_q(\mathbb{R}^2)$ and for all $l \in \mathbb{N}$

$$\gamma_{l,q}(T * \phi) = \max_{0 \leqslant k \leqslant l} \|\Delta_\alpha^k(T * \phi)\|_{q,\nu_\alpha} = \max_{0 \leqslant k \leqslant l} \|T * \Delta_\alpha^k(\phi)\|_{q,\nu_\alpha}$$

According to the relation (5.3), it follows that

$$\begin{aligned} \gamma_{l,q}(T * \phi) &\leqslant \Big(\sum_{k=0}^{m} \|f_k\|_{p',\nu_\alpha} \Big) \max_{0 \leqslant k \leqslant l} \gamma_{m,r}(\Delta_\alpha^k(\phi)) \\ &\leqslant \Big(\sum_{k=0}^{m} \|f_k\|_{p',\nu_\alpha} \Big) \gamma_{m+l,r}(\phi). \end{aligned}$$

□

Definition 5.5. Let $1 \leqslant p, q, r < +\infty$, such that (5.4) holds. The convolution product of $T \in \mathscr{M}'_p(\mathbb{R}^2)$ and $S \in \mathscr{M}'_q(\mathbb{R}^2)$ is defined for all $\phi \in \mathcal{M}_r(\mathbb{R}^2)$, by

$$\langle S * T, \phi \rangle = \langle S, T * \phi \rangle.$$

From this Definition and Proposition 5.4, we deduce the following result

Proposition 5.6. *Let $1 \leqslant p, q, r < +\infty$ such that (5.4) holds. Then, for all $T \in \mathscr{M}'_p(\mathbb{R}^2)$ and $S \in \mathscr{M}'_q(\mathbb{R}^2)$, the functional $S * T$ is continuous on $\mathcal{M}_r(\mathbb{R}^2)$.*

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