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On Quasi-Hadamard Product of Certain Classes of Analytic Functions *

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Abstract

The main purpose of this paper is to derive some quasi-Hadamard product properties for certain classes of analytic functions which are defined by means of the Salagean operator. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

1 Introduction

Let $\mathcal{T}$ denote the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geqq 0)\,, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For $0 \leqq \alpha < 1$, we denote by $\mathcal{ST}^*(\alpha)$ and $\mathcal{CT}(\alpha)$ the usual subclasses of $\mathcal{T}$ consisting of functions with negative coefficients which are, respectively, *starlike of order* α and *convex of order* α in $\mathbb{U}$.

A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{ST}(\alpha, \beta)$ if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha \qquad (0 \leqq \alpha < 1;\ \beta \geqq 0;\ z \in \mathbb{U}). \tag{1.2}$$

Also, a function $f \in \mathcal{T}$ is said to be in the class $\mathcal{UCT}(\alpha, \beta)$ if it satisfies the inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right| + \alpha \qquad (0 \leqq \alpha < 1;\ \beta \geqq 0;\ z \in \mathbb{U}). \tag{1.3}$$

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Salagean [9] once introduced the following operator which called the Salagean operator

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z),$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \qquad (n \in \mathbb{N} := \{1, 2, \dots\}).$$

We note that

$$D^n f(z) = z - \sum_{k=2}^{\infty} k^n a_k z^k \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.4}$$

A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{ST}_n(\alpha, \beta)$, if it satisfies the following inequality:

$$\Re\left(\frac{D^{n+1} f(z)}{D^n f(z)}\right) > \beta \left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| + \alpha \qquad (0 \leqq \alpha < 1;\ \beta \geqq 0;\ n \in \mathbb{N}_0;\ z \in \mathbb{U}). \tag{1.5}$$

Making use of the similar arguments as Bharati *et al.* [2], we get the following necessary and sufficient condition of the class $\mathcal{ST}_n(\alpha, \beta)$.

A function $f \in \mathcal{ST}_n(\alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} k^n [k(1+\beta) - (\alpha+\beta)] a_k \leqq 1 - \alpha. \tag{1.6}$$

The result is sharp for the function f given by

$$f(z) = z - \frac{1-\alpha}{k^n [k(1+\beta) - (\alpha+\beta)]} z^k \qquad (z \in \mathbb{U}). \tag{1.7}$$

In view of (1.6), we have the following inclusion relationships for any positive integer n:

$$\mathcal{ST}_n(\alpha, \beta) \subset \mathcal{ST}_{n-1}(\alpha, \beta) \subset \cdots \subset \mathcal{ST}_2(\alpha, \beta) \subset \mathcal{ST}_1(\alpha, \beta) \subset \mathcal{ST}_0(\alpha, \beta).$$

Indeed, by specializing the parameters n, α and β, we obtain the following subclasses studied by various authors.

(1) $\mathcal{ST}_0(\alpha, \beta) \equiv \mathcal{ST}(\alpha, \beta)$ and $\mathcal{ST}_1(\alpha, \beta) \equiv \mathcal{UCT}(\alpha, \beta)$ (see [2]).

(2) $\mathcal{ST}_0(0, \beta) \equiv \beta - \mathcal{ST}$ and $\mathcal{ST}_1(0, \beta) \equiv \beta - \mathcal{UCT}$ (see [6]).

(3) $\mathcal{ST}_0(\alpha, 0) \equiv \mathcal{ST}^*(\alpha)$ and $\mathcal{ST}_1(\alpha, 0) \equiv \mathcal{CT}(\alpha)$ (see [10]).

Let $f_j \in \mathcal{T}$ $(j = 1, 2, \dots, s)$ be given by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \qquad (a_{k,j} \geqq 0;\ j = 1, 2, \dots, s).$$

Then the quasi-Hadamard product (or convolution) of these functions is defined by

$$(f_1 * \cdots * f_s)(z) = z - \sum_{k=2}^{\infty} \left(\prod_{j=1}^{s} a_{k,j}\right) z^k. \tag{1.8}$$

We note that, several interesting properties and characteristics of the quasi-Hadamard product of two or more functions can be found in the recent investigations by (for example) Raina and Bansal [7], Raina and Prajapat [8], Aouf [1], Darwish and Aouf [3], El-Ashwah and Aouf [4], and the references cited therein. In this paper, we aim at proving some new quasi-Hadamard product properties for the function classes $\mathcal{ST}_n(\alpha,\beta)$. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

2 Main Results

Throughout this paper, we assume that

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \in \mathcal{ST}_n(\alpha,\beta) \qquad (j = 1, 2, \ldots, s),$$

and

$$g_j(z) = z - \sum_{k=2}^{\infty} b_{k,j} z^k \in \mathcal{ST}_m(\alpha,\beta) \qquad (j = 1, 2, \ldots, t).$$

We begin by presenting the following quasi-Hadamard product results for the class $\mathcal{ST}_n(\alpha,\beta)$.

Theorem 2.1 *Let $f_j \in \mathcal{ST}_n(\alpha_j,\beta)$ $(j = 1, 2, \ldots, s)$. Then*

$$(f_1 * \cdots * f_s)(z) \in \mathcal{ST}_n(\delta,\beta),$$

where

$$\delta = 1 - \frac{(1+\beta)\prod_{j=1}^{s}(1-\alpha_j)}{2^{(s-1)n}\prod_{j=1}^{s}(2+\beta-\alpha_j) - \prod_{j=1}^{s}(1-\alpha_j)}. \tag{2.1}$$

The result is sharp for the functions f_j $(j = 1, 2, \ldots, s)$ given by

$$f_j = z - \frac{1-\alpha_j}{2^n(2+\beta-\alpha_j)} z^2 \qquad (j = 1, 2, \ldots, s). \tag{2.2}$$

Proof. We use the principle of mathematical induction in the proof of Theorem 2.1.

Let $f_j \in \mathcal{ST}_n(\alpha_j,\beta)$ $(j = 1, 2)$, we need to find the largest δ such that

$$\sum_{k=2}^{\infty} \frac{k^n[k(1+\beta) - (\delta+\beta)]}{1-\delta} a_{k,1} a_{k,2} \leqq 1.$$

Since $f_j \in \mathcal{ST}_n(\alpha_j,\beta)$ $(j = 1, 2)$, in view of (1.6), we have

$$\sum_{k=2}^{\infty} \frac{k^n[k(1+\beta) - (\alpha_j+\beta)]}{1-\alpha_j} a_{k,j} \leqq 1 \qquad (j = 1, 2).$$

Furthermore, by the Cauchy-Schwartz inequality, we get

$$\sum_{k=2}^{\infty} \frac{k^n\sqrt{[k(1+\beta)-(\alpha_1+\beta)][k(1+\beta)-(\alpha_2+\beta)]}}{\sqrt{(1-\alpha_1)(1-\alpha_2)}}\sqrt{a_{k,1}a_{k,2}} \leqq 1.$$

Thus, it suffices to show that

$$\begin{aligned}&\frac{k^n[k(1+\beta)-(\delta+\beta)]}{1-\delta}a_{k,1}a_{k,2}\\ &\quad\leqq \frac{k^n\sqrt{[k(1+\beta)-(\alpha_1+\beta)][k(1+\beta)-(\alpha_2+\beta)]}}{\sqrt{(1-\alpha_1)(1-\alpha_2)}}\sqrt{a_{k,1}a_{k,2}} \quad (k \in \mathbb{N}^* := \mathbb{N}\setminus\{1\}),\end{aligned}$$

or equivalently,

$$\sqrt{a_{k,1}a_{k,2}} \leqq \frac{(1-\delta)\sqrt{[k(1+\beta)-(\alpha_1+\beta)][k(1+\beta)-(\alpha_2+\beta)]}}{[k(1+\beta)-(\delta+\beta)]\sqrt{(1-\alpha_1)(1-\alpha_2)}} \qquad (k \in \mathbb{N}^*).$$

By noting that

$$\sqrt{a_{k,1}a_{k,2}} \leqq \frac{\sqrt{(1-\alpha_1)(1-\alpha_2)}}{k^n\sqrt{[k(1+\beta)-(\alpha_1+\beta)][k(1+\beta)-(\alpha_2+\beta)]}} \qquad (k \in \mathbb{N}^*),$$

consequently, we need only to prove that

$$\frac{(1-\alpha_1)(1-\alpha_2)}{k^n[k(1+\beta)-(\alpha_1+\beta)][k(1+\beta)-(\alpha_2+\beta)]} \leqq \frac{(1-\delta)}{[k(1+\beta)-(\delta+\beta)]} \qquad (k \in \mathbb{N}^*),$$

which is equivalent to

$$\delta \leqq 1-\frac{(k-1)(1+\beta)(1-\alpha_1)(1-\alpha_2)}{k^n[k(1+\beta)-(\alpha_1+\beta)][k(1+\beta)-(\alpha_2+\beta)]-(1-\alpha_1)(1-\alpha_2)} := A(k) \quad (k \in \mathbb{N}^*).$$

Since $A(k)$ is an increasing function of k ($k \in \mathbb{N}^*$), letting $k = 2$, we obtain

$$\delta = 1-\frac{(1+\beta)(1-\alpha_1)(1-\alpha_2)}{2^n(2+\beta-\alpha_1)(2+\beta-\alpha_2)-(1-\alpha_1)(1-\alpha_2)},$$

which implies that (2.1) holds for $s = 2$.

We now suppose that (2.1) holds for $s = 2, 3, \ldots, t$. Then

$$(f_1 * \cdots * f_t)(z) \in \mathcal{ST}_n(\gamma, \beta),$$

where

$$\gamma = 1-\frac{(1+\beta)\prod_{j=1}^{t}(1-\alpha_j)}{2^{(t-1)n}\prod_{j=1}^{t}(2+\beta-\alpha_j)-\prod_{j=1}^{t}(1-\alpha_j)}.$$

Then, by means of the above technique, we can show that

$$(f_1 * \cdots * f_{t+1})(z) \in \mathcal{ST}_n(\delta, \beta),$$

where

$$\delta = 1 - \frac{(1+\beta)(1-\alpha_{t+1})(1-\gamma)}{2^n(2+\beta-\alpha_{t+1})(2+\beta-\gamma)-(1-\alpha_{t+1})(1-\gamma)}.$$

Since

$$1-\gamma = \frac{(1+\beta)\prod_{j=1}^{t}(1-\alpha_j)}{2^{(t-1)n}\prod_{j=1}^{t}(2+\beta-\alpha_j)-\prod_{j=1}^{t}(1-\alpha_j)}$$

and

$$2+\beta-\gamma = \frac{(1+\beta)2^{(t-1)n}\prod_{j=1}^{t}(2+\beta-\alpha_j)}{2^{(t-1)n}\prod_{j=1}^{t}(2+\beta-\alpha_j)-\prod_{j=1}^{t}(1-\alpha_j)},$$

thus, we have

$$\delta = 1 - \frac{(1+\beta)\prod_{j=1}^{t+1}(1-\alpha_j)}{2^{tn}\prod_{j=1}^{t+1}(2+\beta-\alpha_j)-\prod_{j=1}^{t+1}(1-\alpha_j)},$$

which shows that (2.1) holds for $s = t+1$.

Finally, for the functions f_j $(j = 1, 2, \ldots, s)$ given by (2.2), we have

$$(f_1 * \cdots * f_s)(z) = z - \left(\prod_{j=1}^{s}\frac{1-\alpha_j}{2^n(2+\beta-\alpha_j)}\right)z^2 := z - A_2 z^2.$$

It follows that

$$\sum_{k=2}^{\infty}\frac{k^n[k(1+\beta)-(\delta+\beta)]}{1-\delta}A_k = 1.$$

This evidently completes the proof of Theorem 2.1.

By setting $\alpha_j = \alpha$ $(j = 1, 2, \ldots, s)$ in Theorem 2.1, we get

Corollary 2.1 *Let $f_j \in \mathcal{ST}_n(\alpha,\beta)$ $(j = 1, 2, \ldots, s)$. Then*

$$(f_1 * \cdots * f_s)(z) \in \mathcal{ST}_n(\delta,\beta),$$

where

$$\delta = 1 - \frac{(1+\beta)(1-\alpha)^s}{2^{(s-1)n}(2+\beta-\alpha)^s-(1-\alpha)^s}. \tag{2.3}$$

The result is sharp for the functions f_j $(j = 1, 2, \ldots, s)$ given by

$$f_j = z - \frac{1-\alpha}{2^n(2+\beta-\alpha)}z^2 \qquad (j = 1, 2, \ldots, s). \tag{2.4}$$

Next, by similarly applying the method of proof of Theorem 2.1, we easily get the following result.

Theorem 2.2 *Let $f_j \in \mathcal{ST}_n(\alpha,\beta_j)$ $(j = 1, 2, \ldots, s)$. Then*

$$(f_1 * \cdots * f_s)(z) \in \mathcal{ST}_n(\alpha,\eta),$$

where

$$\eta = \frac{2^{(s-1)n}\prod_{j=1}^{s}(2+\beta_j-\alpha)}{(1-\alpha)^{s-1}} + \alpha - 2. \tag{2.5}$$

The result is sharp for the functions f_j $(j = 1, 2, \ldots, s)$ given by

$$f_j = z - \frac{1-\alpha}{2^n(2+\beta_j-\alpha)} z^2 \qquad (j = 1, 2, \ldots, s). \tag{2.6}$$

Putting $\beta_j = \beta$ $(j = 1, 2, \ldots, s)$ in Theorem 2.2, we get

Corollary 2.2 *Let $f_j \in \mathcal{ST}_n(\alpha, \beta)$ $(j = 1, 2, \ldots, s)$. Then*

$$(f_1 * \cdots * f_s)(z) \in \mathcal{ST}_n(\alpha, \eta),$$

where

$$\eta = \frac{2^{(s-1)n}(2+\beta-\alpha)^s}{(1-\alpha)^{s-1}} + \alpha - 2.$$

The result is sharp for the functions f_j $(j = 1, 2, \ldots, s)$ given by (2.4).

Theorem 2.3 *Let $f_j \in \mathcal{ST}_n(\alpha_j, \beta)$ $(j = 1, 2, \ldots, s)$ and suppose that*

$$F(z) = z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{s} a_{k,j}^t\right) z^k \qquad (t > 1;\ z \in \mathbb{U}). \tag{2.7}$$

Then $F \in \mathcal{ST}_n(\delta_s, \beta)$, where

$$\delta_s = 1 - \frac{s(1+\beta)(1-\alpha)^t}{2^{n(t-1)}(2+\beta-\alpha)^t - s(1-\alpha)^t} \qquad \left(\alpha = \min_{1\leqq j\leqq s}\{\alpha_j\}\right), \tag{2.8}$$

and

$$2^{n(t-1)}(2+\beta-\alpha)^t \geqq s(2+\beta)(1-\alpha)^t.$$

The result is sharp for the functions f_j $(j = 1, 2, \ldots, s)$ given by (2.2).

Proof. Since $f_j \in \mathcal{ST}_n(\alpha_j, \beta)$, in view of (1.6), we obtain

$$\sum_{k=2}^{\infty}\left(\frac{k^n[k(1+\beta)-(\alpha_j+\beta)]}{1-\alpha_j}\right) a_{k,j} \leqq 1 \qquad (j = 1, 2, \ldots, s).$$

By virtue of the Cauchy-Schwarz inequality, we get

$$\sum_{k=2}^{\infty}\left(\frac{k^n[k(1+\beta)-(\alpha_j+\beta)]}{1-\alpha_j}\right)^t a_{k,j}^t \leqq \left(\sum_{k=2}^{\infty}\frac{k^n[k(1+\beta)-(\alpha_j+\beta)]}{1-\alpha_j} a_{k,j}\right)^t \leqq 1. \tag{2.9}$$

It follows from (2.9) that

$$\sum_{k=2}^{\infty}\left(\frac{1}{s}\sum_{j=1}^{s}\left(\frac{k^n[k(1+\beta)-(\alpha_j+\beta)]}{1-\alpha_j}\right)^t a_{k,j}^t\right) \leqq 1.$$

By setting

$$\alpha = \min_{1 \leqq j \leqq s} \{\alpha_j\},$$

suppose also that

$$\delta_s \leqq 1 - \frac{s(k-1)(1+\beta)(1-\alpha)^t}{k^{n(t-1)}[k(1+\beta)-(\alpha+\beta)]^t - s(1-\alpha)^t} \qquad (k \in \mathbb{N}^*).$$

By virtue of (1.6), we get

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left(\left(\frac{k^n[k(1+\beta)-(\delta_s+\beta)]}{1-\delta_s} \right) \left(\sum_{j=1}^{s} a_{k,j}^t \right) \right) \\
&\qquad \leqq \sum_{k=2}^{\infty} \left(\frac{1}{s} \left(\frac{k^n[k(1+\beta)-(\alpha+\beta)]}{1-\alpha} \right)^t \left(\sum_{j=1}^{s} a_{k,j}^t \right) \right) \\
&\qquad \leqq \sum_{k=2}^{\infty} \left(\frac{1}{s} \sum_{j=1}^{s} \left(\frac{k^n[k(1+\beta)-(\alpha_j+\beta)]}{1-\alpha_j} \right)^t a_{k,j}^t \right) \leqq 1.
\end{aligned}$$

Now let

$$D(k) = 1 - \frac{s(k-1)(1+\beta)(1-\alpha)^t}{k^{n(t-1)}[k(1+\beta)-(\alpha+\beta)]^t - s(1-\alpha)^t} \qquad (k \in \mathbb{N}^*).$$

Since $D(k)$ is an increasing function of k $(k \in \mathbb{N}^*)$, we readily have

$$\delta_s = D(2) = 1 - \frac{s(1+\beta)(1-\alpha)^t}{2^{n(t-1)}(2+\beta-\alpha)^t - s(1-\alpha)^t}.$$

By noting that

$$2^{n(t-1)}(2+\beta-\alpha)^t \geqq s(2+\beta)(1-\alpha)^t,$$

we can see that

$$0 \leqq \delta_s < 1.$$

The result is sharp for the functions f_j $(j = 1, 2, \ldots, s)$ given by (2.2). The proof of Theorem 2.3 is thus completed.

Taking $t = 2$ and $\alpha_j = \alpha$ $(j = 1, 2, \ldots, s)$ in Theorem 2.3, we get the following result.

Corollary 2.3 *Let $f_j \in \mathcal{ST}_n(\alpha, \beta)$ $(j = 1, 2, \ldots, s)$ and suppose that*

$$F(z) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{s} a_{k,j}^2 \right) z^k \qquad (z \in \mathbb{U}). \tag{2.10}$$

Then $F \in \mathcal{ST}_n(\delta_s, \beta)$, where

$$\delta_s = 1 - \frac{s(1+\beta)(1-\alpha)^2}{2^n(2+\beta-\alpha)^2 - s(1-\alpha)^2},$$

and

$$2^n(2+\beta-\alpha)^2 \geqq s(2+\beta)(1-\alpha)^2.$$

The result is sharp for the functions f_j $(j = 1, 2, \ldots, s)$ given by (2.4).

By similarly applying the method of proof of Theorem 2.3, we easily get the following result.

Theorem 2.4 *Let $f_j \in \mathcal{ST}_n(\alpha, \beta_j)$ $(j = 1, 2, \ldots, s)$ and function F be defined by* (2.7). *Then $F \in \mathcal{ST}_n(\alpha, \eta_s)$, where*

$$\eta_s = \frac{2^{n(t-1)}(2+\beta-\alpha)^t}{s(1-\alpha)^{t-1}} + \alpha - 2 \qquad \left(\beta = \min_{1 \leqq j \leqq s} \{\beta_j\}\right),$$

and

$$2^{n(t-1)}(2+\beta-\alpha)^t \geqq s(2-\alpha)(1-\alpha)^{t-1}.$$

The result is sharp for the functions f_j $(j = 1, 2, \ldots, s)$ given by (2.6).

Taking $t = 2$ and $\beta_j = \beta$ $(j = 1, 2, \ldots, s)$ in Theorem 2.4, we get the following result.

Corollary 2.4 *Let $f_j \in \mathcal{ST}_n(\alpha, \beta)$ $(j = 1, 2, \ldots, s)$ and function F be defined by* (2.10). *Then $F \in \mathcal{ST}_n(\alpha, \eta_s)$, where*

$$\eta_s = \frac{2^n(2+\beta-\alpha)^2}{s(1-\alpha)} + \alpha - 2,$$

and

$$2^n(2+\beta-\alpha)^2 \geqq s(2-\alpha)(1-\alpha).$$

The result is sharp for the functions f_j $(j = 1, 2, \ldots, s)$ given by (2.4).

Finally, we derive some quasi-Hadamard product results for $\mathcal{ST}_n(\alpha, \beta)$ and $\mathcal{ST}_m(\alpha, \beta)$.

Theorem 2.5 *Let $f_j \in \mathcal{ST}_n(\alpha, \beta)$ $(j = 1, 2, \ldots, s)$ and $g_j \in \mathcal{ST}_m(\alpha, \beta)$ $(j = 1, 2, \ldots, t)$. Then*

$$(f_1 * \cdots * f_s * g_1 * \cdots * g_t)(z) \in \mathcal{ST}_{sn+tm+s+t-1}(\alpha, \beta).$$

Proof. Suppose that

$$h(z) := (f_1 * \cdots * f_s * g_1 * \cdots * g_t)(z).$$

Then we have

$$h(z) = z - \sum_{k=2}^{\infty} \left(\prod_{j=1}^{s} a_{k,j} \prod_{j=1}^{t} b_{k,j} \right) z^k. \tag{2.11}$$

We need to show that

$$\sum_{k=2}^{\infty}\left(k^{sn+tm+s+t-1}[k(1+\beta)-(\alpha+\beta)]\left(\prod_{j=1}^{s}a_{k,j}\prod_{j=1}^{t}b_{k,j}\right)\right)\leqq 1-\alpha. \quad (2.12)$$

Since $f_j \in \mathcal{ST}_n(\alpha,\beta)$, in view of (1.6), we have

$$\sum_{k=2}^{\infty}k^n[k(1+\beta)-(\alpha+\beta)]a_{k,j}\leqq 1-\alpha \qquad (j=1,2,\ldots,s). \quad (2.13)$$

Therefore, we get

$$a_{k,j}\leqq \frac{1-\alpha}{k^n[k(1+\beta)-(\alpha+\beta)]} \qquad (j=1,2,\ldots,s;\ k\in\mathbb{N}^*).$$

Since

$$\frac{1-\alpha}{k(1+\beta)-(\alpha+\beta)}\leqq\frac{1}{k} \qquad (0\leqq\alpha<1;\ \beta\geqq 0;\ k\in\mathbb{N}^*),$$

which implies that

$$a_{k,j}\leqq k^{-(1+n)} \qquad (j=1,2,\ldots,s;\ k\in\mathbb{N}^*). \quad (2.14)$$

Similarly, for $g_j \in \mathcal{ST}_m(\alpha,\beta)$, we have

$$\sum_{k=2}^{\infty}k^m[k(1+\beta)-(\alpha+\beta)]b_{k,j}\leqq 1-\alpha \qquad (j=1,2,\ldots,t). \quad (2.15)$$

Therefore, we get

$$b_{k,j}\leqq \frac{1-\alpha}{k^m[k(1+\beta)-(\alpha+\beta)]} \qquad (j=1,2,\ldots,t;\ k\in\mathbb{N}^*),$$

which implies that

$$b_{k,j}\leqq k^{-(1+m)} \qquad (j=1,2,\ldots,t;\ k\in\mathbb{N}^*). \quad (2.16)$$

By using (2.14) for $j=1,2,\ldots,s$, (2.16) for $j=1,2,\ldots,t-1$, and (2.15) for $j=t$, we obtain

$$\begin{aligned}&\sum_{k=2}^{\infty}\left\{k^{sn+tm+s+t-1}[k(1+\beta)-(\alpha+\beta)]\left(\prod_{j=1}^{s}a_{k,j}\prod_{j=1}^{t}b_{k,j}\right)\right\}\\ &\leqq\sum_{k=2}^{\infty}\left\{k^{sn+tm+s+t-1}\,[k(1+\beta)-(\alpha+\beta)]\,k^{-s(1+n)}k^{-(t-1)(1+m)}\right\}b_{k,t}\\ &=\sum_{k=2}^{\infty}k^m[k(1+\beta)-(\alpha+\beta)]b_{k,t}\leqq 1-\alpha.\end{aligned}$$

Therefore, we know that

$$h \in \mathcal{ST}_{sn+tm+s+t-1}(\alpha, \beta).$$

The proof of Theorem 2.5 is evidently completed.

Setting $\beta = 0$ in Theorem 2.5, we get the following result.

Corollary 2.5 *Let* $f_j \in \mathcal{ST}_n(\alpha, 0)$ $(j = 1, 2, \dots, s)$ *and* $g_j \in \mathcal{ST}_m(\alpha, 0)$ $(j = 1, 2, \dots, t)$. *Then*

$$(f_1 * \cdots * f_s * g_1 * \cdots * g_t)(z) \in \mathcal{ST}^*(\alpha).$$

Taking into account the quasi-Hadamard product of functions

$$f_j \ (j = 1, 2, \dots, s)$$

only, in the proof of the above theorem, and using (2.14) for $j = 1, 2, \dots, s-1$, and (2.13) for $j = s$, we obtain

Corollary 2.6 *Let* $f_j \in \mathcal{ST}_n(\alpha, \beta)$ $(j = 1, 2, \dots, s)$. *Then*

$$(f_1 * \cdots * f_s)(z) \in \mathcal{ST}_{s(n+1)-1}(\alpha, \beta).$$

Remark 2.1 By specializing the parameters $m = 0$ and $n = 1$ in Theorem 2.5, we get the corresponding results obtained by Frasin [5].

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