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Weierstrass transform associated with the Hankel operator *

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Abstract

Using reproducing kernels for Hilbert spaces, we give best approximation for the Weierstrass transform associated with the Hankel transform. Also, estimates of extremal functions are checked.

1 Introduction

The Hankel transform $H_\mu,\ \mu \geqslant -1/2$, is defined for all integrable functions on $[0, +\infty[$ with respect to the measure $\dfrac{r^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dr$ as

$$\mathcal{H}_\mu(\lambda) = \frac{1}{2^\mu\Gamma(\mu+1)}\int_0^{+\infty} f(r)j_\mu(\lambda r)r^{2\mu+1}dr,$$

where j_μ is the modified Bessel function of the first kind and index μ.

Many harmonic analysis results related to the transform $\mathcal{H}_\mu$ are established in [6, 9, 13, 18, 20, 21].

Our purpose in this work is to define and study the Weierstrass transform $\mathcal{W}_{\mu,t}$ associated with the Hankel transform $\mathcal{H}_\mu$.

This transform is defined by

$$\mathcal{W}_{\mu,t}(f)(r) \quad = \quad \int_0^{+\infty} \mathcal{E}_t(r,s)f(s)\frac{s^{2\mu+1}}{2^\mu\Gamma(\mu+1)}ds,$$

where $\mathcal{E}_t(r,s),\ t > 0$ is the heat kernel associated with the Hankel transform which will be defined later. This integral transform which generalizes the usual Weierstrass transform [11, 15, 16], solves the heat problem

$$\begin{cases} \ell_\mu(u)(r,t) = \frac{\partial u}{\partial t}(r,t), \\ u(r,0) = f(r), \end{cases}$$

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where ℓ_μ is the singular differential operator defined on $]0,+\infty[$ by

$$\ell_\mu = \frac{\partial^2}{\partial r^2} + \frac{2\mu+1}{r}\frac{\partial}{\partial r}.$$

Building on the ideas of Saitoh, Matsuura, Fujiwara and Yamada [5, 12, 14, 15, 16], and using the theory of reproducing kernels [2], we give a best approximation of this transform and nice estimates of the associated extremal function.

Let $L^2([0,+\infty[,\frac{r^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dr)$ be the Hilbert space of square integrable functions on $[0,+\infty[$ with respect to the measure $\frac{r^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dr$, and $\langle .|.\rangle_\mu$ its inner product.

For $\nu\in\mathbb{R}$, we consider the Sobolev type space $\mathbf{H}^\mu_\nu([0,+\infty[)$, consisting of functions $f\in L^2([0,+\infty[,\frac{r^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dr)$ such that the function

$$\lambda\longmapsto (1+\lambda^2)^{\nu/2}\mathcal{H}_\mu(f)(\lambda),$$

belongs to the space $L^2([0,+\infty[,\frac{r^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dr)$. Then for $\nu>\mu+1$, $\mathbf{H}^\mu_\nu([0,+\infty[)$ is the Hilbert space when equipped with the inner product

$$\langle f|g\rangle_\nu = \int_0^{+\infty}(1+\lambda^2)^\nu\mathcal{H}_\mu(f)(\lambda)\overline{\mathcal{H}_\mu(g)(\lambda)}\frac{\lambda^{2\mu+1}}{2^\mu\Gamma(\mu+1)}d\lambda.$$

Moreover, the kernel

$$\mathcal{K}_\nu(r,s) = \int_0^{+\infty}\frac{j_\mu(\lambda r)j_\mu(\lambda s)}{(1+\lambda^2)^\nu}\frac{\lambda^{2\mu+1}}{2^\mu\Gamma(\mu+1)}d\lambda,$$

is a reproducing kernel of the space $\mathbf{H}^\mu_\nu([0,+\infty[)$, where

$$j_\mu(z) = \frac{2^\mu\Gamma(\mu+1)}{z^\mu}J_\mu(z) = \Gamma(\mu+1)\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(\mu+n+1)}(\frac{z}{2})^{2n},\quad z\in\mathbb{C},$$

and J_μ is the Bessel function of the first kind and index μ. Using the properties of the Hankel transform $\mathcal{H}_\mu$ and its connection with the convolution product, we show that the Weierstrass transform $\mathcal{W}_{\mu,t}$ is a bounded linear operator from $\mathbf{H}^\mu_\nu([0,+\infty[)$ into $L^2([0,+\infty[,\frac{r^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dr)$ and that for all $f\in\mathbf{H}^\mu_\nu([0,+\infty[)$,

$$\|\mathcal{W}_{\mu,t}(f)\|_{2,\mu}\leqslant ||f||_\nu.$$

Next, for $\xi>0$, we define on the space $\mathbf{H}^\mu_\nu([0,+\infty[)$, the new inner product by setting

$$\langle f|g\rangle_{\nu,\xi} = \xi\langle f|g\rangle_\nu + \langle\mathcal{W}_{\mu,t}(f)|\mathcal{W}_{\mu,t}(g)\rangle_\mu.$$

We show that $\mathbf{H}^{\mu}_{\nu}([0,+\infty[)$ equipped with the inner product $\langle .|.\rangle_{\nu,\xi}$, is a Hilbert space and we exhibit a reproducing kernel, that is

$$\mathcal{K}_{\nu,\xi}(r,s)=\int_0^{+\infty}\frac{j_\mu(\lambda r)j_\mu(\lambda s)}{\xi(1+\lambda^2)^\nu+e^{-2t\lambda^2}}\frac{\lambda^{2\mu+1}}{2^\mu\Gamma(\mu+1)}d\lambda.$$

The last section of this paper is devoted to study the extremal function. More precisely, for all $\nu>\mu+1$, $\xi>0$ and $g\in L^2([0,+\infty[,\dfrac{r^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dr)$, the infimum of

$$\Big\{\xi||f||^2_\nu+||g-\mathcal{W}_{\mu,t}(f)||^2_{2,\mu};\quad f\in\mathbf{H}^{\mu}_{\nu}([0,+\infty[)\Big\},$$

is attained at one function $f^*_{\xi,g}$, called the extremal function. We establish also, the following estimates

- For all $f\in\mathbf{H}^{\mu}_{\nu}([0,+\infty[)$ and $g=\mathcal{W}_{\mu,t}(f)$,

$$\lim_{\xi\to0^+}\left\|f^*_{\xi,g}-f\right\|_\nu=0.$$

- For all $f\in\mathbf{H}^{\mu}_{\nu}([0,+\infty[)$ and $g=\mathcal{W}_{\mu,t}(f)$,

$$\lim_{\xi\to0^+}f^*_{\xi,g}(r)=f(r),\quad\text{uniformly.}$$

2 The Hankel transform

We denote by

- $d\gamma_\mu$ the measure defined on $[0,+\infty[$ by

$$d\gamma_\mu(r)=\frac{r^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dr,$$

- $L^p\left(d\gamma_\mu\right)$, $p\in[1,+\infty]$, the space of measurable functions f on $[0,+\infty[$ satisfying

$$\begin{array}{lll}\|f\|_{p,\mu}=\left(\displaystyle\int_0^{+\infty}|f(r)|^p\,d\gamma_\mu(r)\right)^{\frac{1}{p}}<+\infty, & \text{if } p\in[1,+\infty[;\\ \|f\|_{\infty,\mu}=\underset{r\in[0,+\infty[}{ess\ sup}\ |f(r)|<+\infty, & \text{if } p=+\infty.\end{array}$$

- $\langle .|.\rangle_\mu$ the inner product on $L^2\left(d\gamma_\mu\right)$ defined by

$$\langle f|g\rangle_\mu=\int_0^{+\infty}f(r)\overline{g(r)}d\gamma_\mu(r).$$

- $\mathcal{C}_{*,0}(\mathbb{R})$ the space of even continuous functions f on $\mathbb{R}$ such that

$$\lim_{|r|\to+\infty}f(r)=0.$$

Let ℓ_μ be the Bessel operator defined on $]0, +\infty[$ by

$$\ell_\mu(u) = u'' + \frac{2\mu+1}{r}u',$$

then for all $\lambda \in \mathbb{C}$, the following problem,

$$\begin{cases} \ell_\mu(u) = -\lambda^2 u, \\ u(0) = 1, \\ u'(0) = 0, \end{cases}$$

admits a unique solution given by $j_\mu(\lambda.)$, where

$$j_\mu(z) = \frac{2^\mu\Gamma(\mu+1)}{z^\mu}J_\mu(z) = \Gamma(\mu+1)\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(\mu+n+1)}(\frac{z}{2})^{2n}, \quad z \in \mathbb{C}, \quad (2.1)$$

and J_μ is the Bessel function of the first kind and index μ [1, 3, 10, 22].

The eigenfunction j_μ satisfies the following properties

- The function j_μ has the Mehler integral representation, for all $x \in \mathbb{R}$

$$j_\mu(x) = \begin{cases} \dfrac{2\Gamma(\mu+1)}{\sqrt{\pi}\Gamma(\mu+(1/2))}\displaystyle\int_0^1 (1-t^2)^{\mu-1/2}\cos x\,t\,dt, & \text{if } \mu > -1/2; \\ \cos x, & \text{if } \mu = -1/2. \end{cases}$$

- For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$

$$\left|j_\mu^{(n)}(x)\right| \leqslant 1. \quad (2.2)$$

- The function j_μ satisfies the product formula [10, 22], for all $r, s \in [0, +\infty[$

$$j_\mu(r)j_\mu(s) = \begin{cases} \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+1/2)\Gamma(1/2)}\displaystyle\int_0^\pi j_\mu(\sqrt{r^2+s^2+2rs\cos\theta})(\sin\theta)^{2\mu}d\theta, & \text{if } \mu > -1/2; \\ \dfrac{j_{-1/2}(r+s)+j_{-1/2}(r-s)}{2}, & \text{if } \mu = -1/2. \end{cases}$$

Using the product formula, we will define and study the Hankel translation operator and the convolution product.

Definition 2.1 1) *For all $r \in [0, +\infty[$, the Hankel translation operator τ_r^μ is defined on $L^p(d\gamma_\mu)$ by*

$$\tau_r^\mu(f)(s) = \begin{cases} \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+1/2)\Gamma(1/2)}\displaystyle\int_0^\pi f(\sqrt{r^2+s^2+2rs\cos\theta})(\sin\theta)^{2\mu}d\theta, & \textit{if } \mu > -1/2; \\ \dfrac{f(r+s)+f(|r-s|)}{2}, & \textit{if } \mu = -1/2. \end{cases}$$

2) *The convolution product of* $f, g \in L^1(d\gamma_\mu)$ *is defined by*

$$f *_\mu g(r) = \int_0^{+\infty} \tau_r^\mu(f)(s)g(s)d\gamma_\mu(s).$$

The following assumptions hold

• The product formula can be written

$$\forall (r,s) \in [0,+\infty[\times[0,+\infty[, \quad \tau_r^\mu(j_\mu(\lambda.))(s) = j_\mu(\lambda r)j_\mu(\lambda s).$$

• For all $f \in L^p(d\gamma_\mu)$, $p \in [1,+\infty]$, and for all $r \in [0,+\infty[$ the function $\tau_r^\mu(f)$ belongs to the space $L^p(d\gamma_\mu)$ and

$$\|\tau_r^\mu(f)\|_{p,\mu} \leqslant \|f\|_{p,\mu}. \tag{2.3}$$

• Let $p, q, r \in [1,+\infty]$ be such that $1/p+1/q = 1+1/r$. For all $f \in L^p(d\gamma_\mu)$ and $g \in L^q(d\gamma_\mu)$, the function $f *_\mu g$ belongs to $L^r(d\gamma_\mu)$ and we have the following Young inequality

$$\|f *_\mu g\|_{r,\mu} \leqslant \|f\|_{p,\mu}\|g\|_{q,\mu}. \tag{2.4}$$

• For all $f \in L^1(d\gamma_\mu)$ and $\lambda \in [0,+\infty[$, the function $\tau_\lambda^\mu(f)$ belongs to $L^1(d\gamma_\mu)$ and we have

$$\int_0^{+\infty} \tau_\lambda^\mu(f)(r)d\gamma_\mu(r) = \int_0^{+\infty} f(r)d\gamma_\mu(r). \tag{2.5}$$

Definition 2.2 *The Hankel transform* $\mathcal{H}_\mu$ *is defined on* $L^1(d\gamma_\mu)$ *by [17]*

$$\mathcal{H}_\mu(f)(\lambda) = \int_0^{+\infty} f(r)j_\mu(r\lambda)d\gamma_\mu(r), \quad \lambda \in \mathbb{R},$$

where j_μ *is the modified Bessel function defined by the relation (2.1).*

The Hankel transform satisfies the following properties

• For all $f \in L^1(d\gamma_\mu)$ the function $H_\mu(f)$ belongs to the space $\mathcal{C}_{*,0}(\mathbb{R})$ and

$$\|H_\mu(f)\|_{\infty,\mu} \leqslant \|f\|_{1,\mu}.$$

• For all $f \in L^1(d\gamma_\mu)$ and $r \in [0,\infty[$

$$\mathcal{H}_\mu\big(\tau_r^\mu(f)\big)(\lambda) = j_\mu(r\lambda)\mathcal{H}_\mu(f)(\lambda). \tag{2.6}$$

• For $f, g \in L^1(d\gamma_\mu)$

$$\mathcal{H}_\mu(f *_\mu g) = \mathcal{H}_\mu(f)\mathcal{H}_\mu(g).$$

Theorem 2.3 *(Inversion formula for* $\mathcal{H}_\mu$*) Let* $f \in L^1(d\gamma_\mu)$ *such that* $\mathcal{H}_\mu(f) \in L^1(d\gamma_\mu)$*, then for almost every* $r \in [0,+\infty[$*, we have*

$$f(r) = \int_0^{+\infty} \mathcal{H}_\mu(f)(\lambda)j_\mu(\lambda r)d\gamma_\mu(\lambda).$$

Theorem 2.4 *(Plancherel theorem) The Hankel transform $\mathcal{H}_\mu$ can be extended to an isometric isomorphism from $L^2(d\gamma_\mu)$ onto itself. In particular for all $f, g \in L^2(d\gamma_\mu)$, we have (Parseval equality)*

$$\int_0^{+\infty} f(r)\overline{g(r)}d\gamma_\mu(r) = \int_0^{+\infty} \mathcal{H}_\mu(f)(\lambda)\overline{\mathcal{H}_\mu(g)(\lambda)}d\gamma_\mu(\lambda).$$

Remark 2.5 *i) Let $f \in L^1(d\gamma_\mu)$ and $g \in L^2(d\gamma_\mu)$, by the relation (2.4), the function $f *_\mu g$ belongs to $L^2(d\gamma_\mu)$, moreover*

$$\mathcal{H}_\mu(f *_\mu g) = \mathcal{H}_\mu(f)\mathcal{H}_\mu(g).$$

ii) For all $f, g \in L^2(d\gamma_\mu)$ the function $f *_\mu g$ belongs to the space $\mathcal{C}_{*,0}(\mathbb{R})$ and we have

$$f *_\mu g = \mathcal{H}_\mu\big(\mathcal{H}_\mu(f)\mathcal{H}_\mu(g)\big). \tag{2.7}$$

3 Weierstrass transform associated with the Hankel operator

In this section, we will define and study the Weierstrass transform associated with $\mathcal{H}_\mu$. For this we define some Hilbert spaces and we exhibit their reproducing kernels.

Let ν be a real number, $\nu > \mu + 1$. We denote by

• $\mathbf{H}^\mu_\nu([0, +\infty[)$ the subspace of $L^2(d\gamma_\mu)$ formed by the functions f, such that the maps

$$\lambda \longmapsto (1 + \lambda^2)^{\nu/2}\mathcal{H}_\mu(f)(\lambda),$$

belongs to $L^2(d\gamma_\mu)$.

• $\langle .|. \rangle_\nu$ the inner product on $\mathbf{H}^\mu_\nu([0, +\infty[)$ defined by

$$\langle f|g\rangle_\nu = \int_0^{+\infty} (1 + \lambda^2)^\nu \mathcal{H}_\mu(f)(\lambda)\overline{\mathcal{H}_\mu(g)(\lambda)}d\gamma_\mu(\lambda).$$

• $||.||_\nu$ the norm of $\mathbf{H}^\mu_\nu([0, +\infty[)$ defined by

$$||f||_\nu = \sqrt{\langle f|f\rangle_\nu}.$$

Remark 3.1 *For $\nu > \mu + 1$, the function*

$$\lambda \longmapsto \frac{1}{(1 + \lambda^2)^{\nu/2}},$$

belongs to $L^2(d\gamma_\mu)$. Hence for all $f \in \mathbf{H}^\mu_\nu([0, +\infty[)$, the function $\mathcal{H}_\mu(f)$ belongs to $L^1(d\gamma_\mu)$, then by inversion formula 2.3, we have for almost every $r \in [0, +\infty[$

$$f(r) = \int_0^{+\infty} \mathcal{H}_\mu(\lambda)j_\mu(\lambda r)d\gamma_\mu(\lambda).$$

Proposition 3.2 *For $\nu > \mu + 1$ the function $\mathcal{K}_\nu$ defined on $[0, +\infty[\times[0, +\infty[$ by*

$$\mathcal{K}_\nu(r, s) = \int_0^{+\infty} \frac{j_\mu(\lambda r) j_\mu(\lambda s)}{(1 + \lambda^2)^\nu} d\gamma_\mu(\lambda),$$

is a reproducing kernel of the space $\mathbf{H}^\mu_\nu([0, +\infty[)$, that is

i) For all $s \in [0, +\infty[$, the function

$$r \longmapsto \mathcal{K}_\nu(r, s),$$

belongs to $\mathbf{H}^\mu_\nu([0, +\infty[)$.

ii) (The reproducing property) For all $f \in \mathbf{H}^\mu_\nu([0, +\infty[)$ and $s \in [0, +\infty[$,

$$\langle f|\mathcal{K}_\nu(., s)\rangle_\nu = f(s).$$

Proof. i) From Remark 3.1 and the relation (2.2), we deduce that for all $s \in [0, +\infty[$, the function

$$\lambda \longmapsto \frac{j_\mu(\lambda s)}{(1 + \lambda^2)^\nu},$$

belongs to $L^1(d\gamma_\mu) \cap L^2(d\gamma_\mu)$. Then, the function $\mathcal{K}_\nu$ is well defined and by Theorem 2.3, we have

$$\mathcal{K}_\nu(r, s) = \mathcal{H}_\mu\Big(\frac{j_\mu(\lambda s)}{(1 + \lambda^2)^\nu}\Big)(r).$$

By Plancherel theorem, it follows that for all $s \in [0, +\infty[$, the function $\mathcal{K}_\nu(., s)$ belongs to $L^2(d\gamma_\mu)$, and we have

$$\mathcal{H}_\mu\Big(\mathcal{K}_\nu(., s)\Big)(\lambda) = \frac{j_\mu(\lambda s)}{(1 + \lambda^2)^\nu}. \tag{3.8}$$

Again, by the relation (2.2) and Remark 3.1, it follows that the function

$$\lambda \longmapsto (1 + \lambda^2)^{\nu/2}\mathcal{H}_\mu\big(\mathcal{K}_\nu(., s)\big)(\lambda),$$

belongs to $L^2(d\gamma_\mu)$.

ii) Let f be in $\mathbf{H}^\mu_\nu([0, +\infty[)$. For every $s \in [0, +\infty[$, we have

$$\langle f|\mathcal{K}_\nu(., s)\rangle_\nu = \int_0^{+\infty} (1+\lambda^2)^\nu \mathcal{H}_\mu(f)(\lambda)\overline{\mathcal{H}_\mu(\mathcal{K}_\nu(., s))(\lambda)} d\gamma_\mu(\lambda),$$

and by the relation (3.8), we get

$$\langle f|\mathcal{K}_\nu(., s)\rangle_\nu = \int_0^{+\infty} \mathcal{H}_\mu(f)(\lambda) j_\mu(\lambda s) d\gamma_\mu(\lambda).$$

The result follows from Remark 3.1.

The heat equation associated with the Hankel transform is given by

$$\ell_\mu u(r,t) = \frac{\partial}{\partial t} u(r,t), \tag{3.9}$$

where ℓ_μ is the Bessel operator defined above.

Let E be the kernel defined by

$$\begin{aligned} E(r,t) &= \int_0^{+\infty} e^{-t\lambda^2} j_\mu(r\lambda) d\gamma_\mu(\lambda) \\ &= \frac{e^{-r^2/4t}}{(2t)^{\mu+1}}. \end{aligned} \tag{3.10}$$

Then, the kernel E solves the equation (3.9).

Definition 3.3 *The heat kernel associated with the Hankel transform is defined by*

$$\begin{aligned} \mathcal{E}_t(r,s) &= \tau_r^\mu\Big(E(.,t)\Big)(s) \\ &= \frac{e^{-(r^2+s^2)/4t}}{(2t)^{\mu+1}} j_\mu(\frac{irs}{2t}). \end{aligned} \tag{3.11}$$

Then, we have the following properties

i) For all $t > 0$, $\mathcal{E}_t \geqslant 0$.

ii) From the relations (2.3),(2.6),(3.10) and (3.11), for all $t > 0$, $r \in [0,+\infty[$, the function $\mathcal{E}_t(r,.)$ belongs to $L^1(d\gamma_\mu)$ and for all $\lambda \in [0,+\infty[$, we have

$$\mathcal{H}_\mu\big(\mathcal{E}_t(r,.)\big)(\lambda) = e^{-t\lambda^2} j_\mu(\lambda r).$$

iii) From the relations (2.3), (2.5), (3.10) and (3.11), for all $t > 0$ and $s \in [0,+\infty[$, the function $\mathcal{E}_t(.,s)$ belongs to $L^1(d\gamma_\mu)$ and we have

$$\int_0^{+\infty} \mathcal{E}_t(r,s) d\gamma_\mu(r) = \int_0^{+\infty} E(r,t) d\gamma_\mu(r) = 1.$$

iv) For all $s \in [0,+\infty[$, the function

$$(r,t) \longmapsto \mathcal{E}_t(r,s),$$

solves the heat equation (3.9).

In the following, we shall define the Weierstrass transform associated with the Hankel transform and we establish some properties that we use later.

Definition 3.4 *The Weierstrass transform associated with the Hankel transform is defined on* $L^2(d\gamma_\mu)$, *by*

$$\begin{aligned}\mathcal{W}_{\mu,t}(f)(r) &= \big(E(.,t)*_\mu f\big)(r) \qquad (3.12)\\ &= \int_0^{+\infty}\mathcal{E}_t(r,s)f(s)d\gamma_\mu(s).\end{aligned}$$

For the classical Weierstrass transform, one can see [11, 15, 16].

Proposition 3.5 *i) For all* $f\in L^2(d\gamma_\mu)$, *the function* $\mathcal{W}_{\mu,t}(f)$ *solves the heat equation (3.9), with the initial condition*

$$\lim_{t\to 0^+}\mathcal{W}_{\mu,t}(f)=f,\quad \text{in } L^2(d\gamma_\mu).$$

ii) For all $t>0$ *and* $\nu>\mu+1$, *the transform* $\mathcal{W}_{\mu,t}$ *is a bounded linear operator from* $\mathbf{H}^\mu_\nu([0,+\infty[)$ *into* $L^2(d\gamma_\mu)$ *and for all* $f\in\mathbf{H}^\mu_\nu([0,+\infty[)$ *we have*

$$\|\mathcal{W}_{\mu,t}(f)\|_{2,\mu}\leqslant ||f||_\nu.$$

Proof. i) From the relations (3.11), (3.12), the derivative's theorem and the fact that for all $s\in[0,+\infty[$, the function $(r,t)\longmapsto\mathcal{E}_t(r,s)$ solves the heat equation (3.9), we deduce that the function $\mathcal{W}_{\mu,t}(f)$ is a solution of (3.9).

The family $\big(E(.,t)\big)_{t>0}$ is an approximate identity, in particular for all $f\in L^2(d\gamma_\mu)$

$$\lim_{t\to 0}E(.,t)*_\mu f=f \qquad \text{in} \quad L^2(d\gamma_\mu).$$

ii) From the relations (2.4) and (3.12), for all $f\in L^2(d\gamma_\mu)$, we have

$$\begin{aligned}\|\mathcal{W}_{\mu,t}(f)\|_{2,\mu}&=\|E(.,t)*_\mu f\|_{2,\mu}\\ &\leqslant\|E(.,t)\|_{1,\mu}\|f\|_{2,\mu}\\ &=\|f\|_{2,\mu}=\|\mathcal{H}_\mu(f)\|_{2,\mu}\leqslant\|f\|_\nu\,.\end{aligned}$$

Notations. For all positive real numbers ξ, t and for $\nu>\mu+1$, we denote by

• $\langle.|.\rangle_{\nu,\xi}$, the inner product defined on the space $\mathbf{H}^\mu_\nu([0,+\infty[)$ by

$$\langle f|g\rangle_{\nu,\xi}=\xi\langle f|g\rangle_\nu+\langle\mathcal{W}_{\mu,t}(f)|\mathcal{W}_{\mu,t}(g)\rangle_\mu.$$

• $\mathbf{H}^\mu_{\nu,\xi}([0,+\infty[)$, the space $\mathbf{H}^\mu_\nu([0,+\infty[)$ equipped with the inner product $\langle.|.\rangle_{\nu,\xi}$ and the norm

$$||f||^2_{\nu,\xi}=\xi||f||^2_\nu+||\mathcal{W}_{\mu,t}(f)||^2_{2,\mu}.$$

Then, we have the following main result [11, 15].

Theorem 3.6 *For all $\xi, t > 0$ and $\nu > \mu + 1$, the Hilbert space $\mathbf{H}^{\mu}_{\nu,\xi}([0,+\infty[)$ admits the following reproducing kernel,*

$$\mathcal{K}_{\nu,\xi}(r,s) = \int_0^{+\infty} \frac{j_\mu(r\lambda)j_\mu(s\lambda)}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}} d\gamma_\mu(\lambda),$$

that is

i) For all $s \in [0,+\infty[$, the function $r \longmapsto \mathcal{K}_{\nu,\xi}(r,s)$ belongs to $\mathbf{H}^{\mu}_{\nu,\xi}([0,+\infty[)$.

ii) (The reproducing property.) For all $f \in \mathbf{H}^{\mu}_{\nu,\xi}([0,+\infty[)$ and $s \in [0,+\infty[$,

$$\langle f|\mathcal{K}_{\nu,\xi}(.,s)\rangle_{\nu,\xi} = f(s).$$

Proof. i) Let $s \in [0,+\infty[$. From the inequality (2.2), we have

$$\frac{\big|j_\mu(\lambda s)\big|}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}} \leqslant \frac{1}{\xi(1+\lambda^2)^\nu}.$$

Then by the hypothesis $\nu > \mu + 1$, we deduce that for all $s \in [0,+\infty[$, the function

$$\lambda \longmapsto \frac{j_\mu(\lambda s)}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}},$$

belongs to $L^1(d\gamma_\mu) \cap L^2(d\gamma_\mu)$, and by the Plancherel theorem, the function

$$r \longmapsto \mathcal{K}_{\nu,\xi}(r,s) = \mathcal{H}_\mu\Big(\frac{j_\mu(\lambda s)}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}}\Big)(r) \tag{3.13}$$

belongs to $L^2(d\gamma_\mu)$, moreover the function

$$\lambda \longmapsto (1+\lambda^2)^{\nu/2}\mathcal{H}_\mu\Big(\mathcal{K}_{\nu,\xi}(.,s)\Big)(\lambda) = \frac{(1+\lambda^2)^{\nu/2} j_\mu(\lambda s)}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}},$$

belongs to $L^2(d\gamma_\mu)$.

This proves that for all $s \in [0,+\infty[$, the function $\mathcal{K}_{\nu,\xi}(.,s)$ belongs to the space $\mathbf{H}^{\mu}_{\nu,\xi}([0,+\infty[)$.

ii) Let f be in $\mathbf{H}^{\mu}_{\nu,\xi}([0,+\infty[)$. By the relation (3.13), we get

$$\langle f|\mathcal{K}_{\nu,\xi}(.,s)\rangle_\nu = \int_0^{+\infty} (1+\lambda^2)^\nu \mathcal{H}_\mu(f)(\lambda) \times \left(\overline{\frac{j_\mu(\lambda s)}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}}}\right) d\gamma_\mu(\lambda). \tag{3.14}$$

On the other hand, we have

$$\mathcal{W}_{\mu,t}\Big(\mathcal{K}_{\nu,\xi}(.,s)\Big)(r) = \Big(E(.,t) *_\mu \mathcal{K}_{\nu,\xi}(.,s)\Big)(r),$$

and by the relations (2.7), (3.10) and (3.13), we get

$$\mathcal{W}_{\mu,t}\Big(\mathcal{K}_{\nu,\xi}(.,s)\Big)(r) = \mathcal{H}_\mu\Big(\frac{j_\mu(\lambda s)e^{-t\lambda^2}}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}}\Big)(r). \tag{3.15}$$

By the same way

$$\mathcal{W}_{\mu,t}(f)(r) \quad = \quad \mathcal{H}_\mu\Big(e^{-t\lambda^2}\mathcal{H}_\mu(f)\Big)(r). \tag{3.16}$$

Thus,

$$\langle \mathcal{W}_{\mu,t}(f)|\mathcal{W}_{\mu,t}\Big(\mathcal{K}_{\nu,\xi}(.,s)\Big)\rangle_\mu = \langle \mathcal{H}_\mu\Big(e^{-t\lambda^2}\mathcal{H}_\mu(f)\Big)|\mathcal{H}_\mu\Big(\frac{j_\mu(\lambda s)e^{-t\lambda^2}}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}}\Big)\rangle_\mu.$$

Using the Parseval formula, we get

$$\langle \mathcal{W}_{\mu,t}(f)|\mathcal{W}_{\mu,t}\Big(\mathcal{K}_{\nu,\xi}(.,s)\Big)\rangle_\mu = \langle e^{-t\lambda^2}\mathcal{H}_\mu(f)|\frac{j_\mu(\lambda s)e^{-t\lambda^2}}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}}\rangle_\mu. \tag{3.17}$$

Combining the relations (3.14) and (3.17), we obtain

$$\langle f|\mathcal{K}_{\nu,\xi}(.,s)\rangle_{\nu,\xi} = \int_0^{+\infty} \mathcal{H}_\mu(f)(\lambda)j_\mu(\lambda s)d\gamma_\mu(\lambda).$$

The desired result arises from Remark 3.1.

4 The Extremal Function

This section contains the main result of this paper, that is the existence and unicity of the extremal function related to the generalized Weierstrass transform studied in the previous section.

Theorem 4.1 *Let $\nu > \mu + 1$, $\xi > 0$ and $g \in L^2(d\gamma_\mu)$. Then there is a unique function $f^*_{\xi,g} \in \mathbf{H}^\mu_\nu([0,+\infty[)$, where the infimum of*

$$\Big\{\xi||f||^2_\nu + ||g - \mathcal{W}_{\mu,t}(f)||^2_{2,\mu}, \quad f \in \mathbf{H}^\mu_\nu([0,+\infty[)\Big\},$$

is attained. Moreover, the extremal function $f^*_{\xi,g}$ is given by

$$f^*_{\xi,g}(r) = \int_0^{+\infty} g(s)\mathbf{Q}_\xi(r,s)d\gamma_\mu(s), \tag{4.18}$$

where,

$$\mathbf{Q}_\xi(r,s) = \int_0^{+\infty} \frac{e^{-t\lambda^2}j_\mu(\lambda r)j_\mu(\lambda s)}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}}d\gamma_\mu(\lambda). \tag{4.19}$$

Proof. The existence and unicity of the extremal function $f^*_{\xi,g}$ is given by [11, 15, 16]. On the other hand, we have

$$f^*_{\xi,g}(s) = \langle g|\mathcal{W}_{\mu,t}\Big(\mathcal{K}_{\nu,\xi}(.,s)\Big)\rangle_\mu,$$

and by (3.15), we obtain

$$\begin{aligned} f^*_{\xi,g}(s) &= \langle g|\mathcal{H}_\mu\Big(\frac{j_\mu(\lambda s)e^{-t\lambda^2}}{\xi(1+\lambda^2)^\nu+e^{-2t\lambda^2}}\Big)\rangle_\mu \\ &= \int_0^{+\infty} g(r)\Big(\int_0^{+\infty}\frac{e^{-t\lambda^2}j_\mu(\lambda r)j_\mu(\lambda s)}{\xi(1+\lambda^2)^\nu+e^{-2t\lambda^2}}d\gamma_\mu(\lambda)\Big)d\gamma_\mu(r) \\ &= \int_0^{+\infty} g(r)\mathbf{Q}_\xi(r,s)d\gamma_\mu(r). \end{aligned}$$

Corollary 4.2 *Let $\nu > \mu+1,\quad \xi > 0$ and $g \in L^2(d\gamma_\mu)$. The extremal function $f^*_{\xi,g}$, satisfies the following inequality*

$$\|f^*_{\xi,g}\|^2_{2,\mu} \leqslant \frac{\Gamma(\nu-\mu-1)}{\xi 2^{2\mu+4}\Gamma(\nu)}\int_0^{+\infty} e^{r^2}|g(r)|^2 d\gamma_\mu(r).$$

Proof. We have

$$f^*_{\xi,g}(s) = \int_0^{+\infty} e^{-\frac{r^2}{2}}e^{\frac{r^2}{2}}g(r)\mathbf{Q}_\xi(r,s)d\gamma_\mu(r).$$

By Hölder's inequality we get

$$|f^*_{\xi,g}(s)|^2 \leqslant \Big(\int_0^{+\infty} e^{-r^2}d\gamma_\mu(r)\Big)\Big(\int_0^{+\infty} e^{r^2}|g(r)|^2\Big|\mathbf{Q}_\xi(r,s)\Big|^2 d\gamma_\mu(r)\Big).$$

Integrating over $[0,+\infty[$ with respect to the measure $d\gamma_\mu(s)$, we obtain

$$\|f^*_{\xi,g}\|^2_{2,\mu} \leqslant \Big(\int_0^{+\infty} e^{-r^2}d\gamma_\mu(r)\Big)\Big(\int_0^{+\infty} e^{r^2}|g(r)|^2\,\|\mathbf{Q}_\xi(r,.)\|^2_{2,\mu}\,d\gamma_\mu(r)\Big).$$

However, by the relation (4.19)

$$\mathbf{Q}_\xi(r,s) = \mathcal{H}_\mu\Big(\frac{e^{-t\lambda^2}j_\mu(\lambda r)}{\xi(1+\lambda^2)^\nu+e^{-2t\lambda^2}}\Big)(s), \tag{4.20}$$

then by the Plancherel theorem

$$\|\mathbf{Q}_\xi(r,.)\|^2_{2,\mu} = \int_0^{+\infty}\frac{e^{-2t\lambda^2}\,|j_\mu(\lambda r)|^2}{\big|\xi(1+\lambda^2)^\nu+e^{-2t\lambda^2}\big|^2}d\gamma_\mu(\lambda).$$

Since, $a^2+b^2 \geqslant 2ab,\;\; a,b \geqslant 0$, and in virtue of the relation (2.2), it follows that

$$\|\mathbf{Q}_\xi(r,.)\|^2_{2,\mu} \leqslant \frac{1}{4\xi}\int_0^{+\infty}\frac{d\gamma_\mu(\lambda)}{(1+\lambda^2)^\nu}. \tag{4.21}$$

We complete the proof by using the relations (4.20) and (4.21), and the fact that

$$\int_0^{+\infty} e^{-r^2}d\gamma_\mu(r) = \frac{1}{2^{\mu+1}},$$

and

$$\int_0^{+\infty}\frac{d\gamma_\mu(\lambda)}{(1+\lambda^2)^\nu} = \frac{\Gamma(\nu-\mu-1)}{2^{\mu+1}\Gamma(\nu)}.$$

Corollary 4.3 *Let $\nu > \mu + 1$. For all $g_1, g_2 \in L^2(d\gamma_\mu)$, we have*

$$\left\| f^*_{\xi,g_1} - f^*_{\xi,g_2} \right\|_\nu \leqslant \frac{\|g_1 - g_2\|_{2,\mu}}{2\sqrt{\xi}}.$$

Proof. Let $\nu > \mu + 1$. For all $r \in [0, +\infty[$, the function

$$\lambda \longmapsto \frac{e^{-t\lambda^2}}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}} j_\mu(\lambda r),$$

belongs to $L^1(d\gamma_\mu) \cap L^2(d\gamma_\mu)$.
From the relation (4.18) and the fact that

$$\mathbf{Q}_\xi(r,s) = \mathcal{H}_\mu\Big(\frac{e^{-t\lambda^2} j_\mu(\lambda s)}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}}\Big)(r),$$

we deduce that for all $g \in L^2(d\gamma_\mu)$ and $s \in [0, +\infty[$, we have

$$f^*_{\xi,g}(s) = \int_0^{+\infty} g(r)\mathcal{H}_\mu\Big(\frac{e^{-t\lambda^2} j_\mu(\lambda s)}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}}\Big)(r) d\gamma_\mu(r).$$

Applying Parseval's equality, we get

$$\begin{aligned} f^*_{\xi,g}(s) &= \int_0^{+\infty} \mathcal{H}_\mu(g)(\lambda) \frac{e^{-t\lambda^2} j_\mu(\lambda s)}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}} d\gamma_\mu(\lambda) \\ &= \mathcal{H}_\mu\Big(\mathcal{H}_\mu(g) \frac{e^{-t\lambda^2}}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}}\Big)(s), \end{aligned}$$

which implies that

$$\mathcal{H}_\mu(f^*_{\xi,g})(\lambda) = \mathcal{H}_\mu(g)(\lambda) \frac{e^{-t\lambda^2}}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}}, \tag{4.22}$$

then for all $g_1, g_2 \in L^2(d\gamma_\mu)$,

$$\left\| f^*_{\xi,g_1} - f^*_{\xi,g_2} \right\|_\nu^2 = \int_0^{+\infty} \frac{(1+\lambda^2)^\nu e^{-2t\lambda^2} \Big|\mathcal{H}_\mu(g_1 - g_2)(\lambda)\Big|^2}{\Big(\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}\Big)^2} d\gamma_\mu(\lambda).$$

Applying again the fact that $a^2 + b^2 \geqslant 2ab, \ \ a, b \geqslant 0$, we obtain

$$\begin{aligned} \left\| f^*_{\xi,g_1} - f^*_{\xi,g_2} \right\|_\nu^2 &\leqslant \frac{1}{4\xi} \int_0^{+\infty} |\mathcal{H}_\mu(g_1 - g_2)(\lambda)|^2 \, d\gamma_\mu(\lambda) \\ &= \frac{1}{4\xi} \|g_1 - g_2\|_{2,\mu}^2 . \end{aligned}$$

Corollary 4.4 *Let $\nu > \mu + 1$. For every $f \in \mathbf{H}_\nu^\mu([0,+\infty[)$ and $g = \mathcal{W}_{\mu,t}(f)$, we have*

$$\lim_{\xi \to 0^+} \left\| f_{\xi,g}^* - f \right\|_\nu = 0.$$

Moreover, $(f_{\xi,g}^*)_{\xi>0}$ converges uniformly to f as $\xi \to 0^+$.

Proof. Let $f \in \mathbf{H}_\nu^\mu([0,+\infty[)$ and $g = \mathcal{W}_{\mu,t}(f)$. From Proposition 3.5, the function g belongs to $L^2(d\gamma_\mu)$. Applying the relations (3.16) and (4.22), we obtain

$$\mathcal{H}_\mu(f_{\xi,g}^* - f)(\lambda) = \frac{-\xi(1+\lambda^2)^\nu \mathcal{H}_\mu(f)(\lambda)}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}}. \tag{4.23}$$

Consequently,

$$\left\| f_{\xi,g}^* - f \right\|_\nu^2 = \int_0^{+\infty} \frac{\xi^2(1+\lambda^2)^{2\nu}}{\left(\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}\right)^2} (1+\lambda^2)^\nu \left|\mathcal{H}_\mu(f)(\lambda)\right|^2 d\gamma_\mu(\lambda).$$

Using the dominated convergence theorem and the fact that

$$\frac{\xi^2(1+\lambda^2)^{3\nu} \left|\mathcal{H}_\mu(f)(\lambda)\right|^2}{\left(\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}\right)^2} \leqslant (1+\lambda^2)^\nu \left|\mathcal{H}_\mu(f)(\lambda)\right|^2,$$

and $f \in \mathbf{H}_\nu^\mu([0,+\infty[)$, we deduce that

$$\lim_{\xi \to 0^+} \left\| f_{\xi,g}^* - f \right\|_\nu = 0.$$

From Remark 3.1, the function $\mathcal{H}_\mu(f)$ belongs to $L^1(d\gamma_\mu) \cap L^2(d\gamma_\mu)$, then by inversion formula and the relation (4.23), we get

$$(f_{\xi,g}^* - f)(r) = \int_0^{+\infty} \frac{-\xi(1+\lambda^2)^\nu \mathcal{H}_\mu(f)(\lambda)}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}} j_\mu(\lambda r) d\gamma_\mu(\lambda).$$

So, for all $r \in [0,+\infty[$,

$$\left|(f_{\xi,g}^* - f)(r)\right| \leqslant \int_0^{+\infty} \frac{\xi(1+\lambda^2)^\nu \left|\mathcal{H}_\mu(f)(\lambda)\right|}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}} d\gamma_\mu(\lambda).$$

Again, by dominated convergence theorem and the fact that

$$\frac{\xi(1+\lambda^2)^\nu \left|\mathcal{H}_\mu(f)(\lambda)\right|}{\xi(1+\lambda^2)^\nu + e^{-2t\lambda^2}} \leqslant \left|\mathcal{H}_\mu(f)(\lambda)\right|,$$

we deduce that

$$\sup_{r \in [0,+\infty[} \left|(f_{\xi,g}^* - f)(r)\right| \longrightarrow 0, \ as\ \xi \longrightarrow 0^+.$$

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