Some almost paracomplex structures on the tangent bundle with vertical rescaled Berger deformation metric

A. Zagane and N. Boussekkine

Abstract. In the present paper, we study some almost paracomplex structures on the tangent bundle with vertical rescaled Berger deformation metric and search conditions for these structures to be anti-paraKähler, quasi-anti-paraKähler.

M.S.C. 2010: 53C20, 53C55, 53C15, 53B35.
Key words: Horizontal lift and vertical lift; tangent bundles; vertical rescaled Berger deformation metric; almost paracomplex structure; anti-paraHermitian metric.

1 Introduction

The notion of almost paracomplex structure has been studied, since the first papers by P.K. Rashevskij [13], P. Libermann [9] and E.M. Patterson [12] until now, from several different points of view. Moreover, the papers related to it have appeared many times in a rather disperse way, and a survey of further results on paracomplex geometry (including para-Hermitian and para-Kähler geometry) can be found for instance in [3, 4]. Also, other further significant developments are due in some recent problems [1, 17], where certain aspects concerning the geometry of tangent and cotangent bundles are presented in [8, 11, 14]...

In this paper, we construct almost anti-paraHermitian structures on tangent bundle equipped with the vertical rescaled Berger deformation metric and investigate necessary and sufficient conditions for these structures to become anti-paraKähler, quasi-anti-paraKähler. Also we characterize some properties of almost anti-paraHermitian structures in context of almost product Riemannian manifolds are presented.

2 Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,m}}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and $\Im^1_0(M)$ be the module over $C^\infty(M)$ of C^∞ vector fields on M.

Balkan Journal of Geometry and Its Applications, Vol.26, No.1, 2021, pp. 124-140.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by :

$$\begin{aligned}\mathcal{V}_{(x,u)} &= Ker(d\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}|_{(x,u)},\ a^i \in \mathbb{R}\},\\ \mathcal{H}_{(x,u)} &= \{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)},\ a^i \in \mathbb{R}\},\end{aligned}$$

where $(x,u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Note that the map $X \to X^H$ is an isomorphism between the vector spaces T_xM and $\mathcal{H}_{(x,u)}$. Similarly, the map $X \to X^V$ is an isomorphism between the vector spaces T_xM and $\mathcal{V}_{(x,u)}$. Obviously, each tangent vector $Z \in T_{(x,u)}TM$ can be written in the form $Z = X^H + Y^V$, where $X, Y \in T_xM$ are uniquely determined vectors.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\}. \tag{2.2}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=\overline{1,m}}$ is a local adapted frame on TTM.

If U be a local vector field constant on each fiber T_xM, i.e., $(U = u = u^i \frac{\partial}{\partial x^i})$, the vertical lift U^V is called the canonical vertical vector field or Liouville vector field on TM.

If $w = w^i \frac{\partial}{\partial x^i} + \overline{w}^j \frac{\partial}{\partial x^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by

$$w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}, \tag{2.3}$$

$$w^v = (\overline{w}^k + w^i u^j \Gamma^k_{ij}) \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}. \tag{2.4}$$

Lemma 2.1. *[6, 20] Let (M,g) be a Riemannian manifold. The bracket operation of vertical and horizontal vector fields is given by the formulas*

1. $[X^H, Y^H]_{(x,u)} = [X,Y]^H_{(x,u)} - (R_x(X,Y)u)^V$,
2. $[X^H, Y^V]_{(x,u)} = (\nabla_X Y)^V_{(x,u)}$,
3. $[X^V, Y^V]_{(x,u)} = 0$,

for all vector fields $X, Y \in \Im^1_0(M)$ and $(x,u) \in TM$, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M,g).

3 Vertical rescaled Berger deformation metric

An almost product structure φ on a manifold M is a $(1,1)$ tensor field on M such that $\varphi^2 = id_M$, $\varphi \neq \pm id_M$ (id_M is the identity tensor field of type $(1,1)$ on M). The pair (M,φ) is called an almost product manifold.

A linear connection ∇ on (M, φ) such that $\nabla\varphi = 0$ is said an almost product connection. There exists an almost product connection on every almost product manifold. [5].

An almost paracomplex manifold is an almost product manifold (M, φ), such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even [4].

The integrability of an almost paracomplex structure is equivalent to the vanishing of the Nijenhuis tensor:

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X, Y].$$

A paracomplex structure is an integrable almost paracomplex structure. On the other hand, in order that an almost paracomplex structure be integrable, it is necessary and sufficient that we can introduce a torsion free linear connection such that $\nabla\varphi = 0$. [17, 15]

Let (M^{2m}, φ) be an almost paracomplex manifold. A Riemannian metric g is said anti-paraHermitian metric with respect to the paracomplex structure φ if

$$g(\varphi X, \varphi Y) = g(X, Y), \tag{3.1}$$

or equivalently (purity condition), (B-metric)[17]

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{3.2}$$

for all $X, Y \in \Im^1_0(M)$.

If (M^{2m}, φ) is an almost paracomplex manifold with an anti-paraHermitian metric g, then the triple (M^{2m}, φ, g) is said almost anti-paraHermitian manifold (an almost B-manifold)[17]. Moreover, (M^{2m}, φ, g) becomes anti-paraKähler manifold (B-manifold)[17] if φ is parallel with respect to the Levi-Civita connection ∇ of g, i.e., $(\nabla\varphi = 0)$.

A Tachibana operator ϕ_φ applied to the anti-paraHermitian metric (pure metric) g is given by

$$\begin{aligned}(\phi_\varphi g)(X, Y, Z) \;=\; & \varphi X(g(Y, Z)) - X(g(\varphi Y, Z)) + g((L_Y\varphi)X, Z) \\ & + g((L_Z\varphi)X, Y),\end{aligned} \tag{3.3}$$

for all $X, Y, Z \in \Im^1_0(M)$ [19].

In an almost anti-paraHermitian manifold, an anti-paraHermitian metric g is called paraholomorphic if

$$(\phi_\varphi g)(X, Y, Z) = 0, \tag{3.4}$$

for all $X, Y, Z \in \Im^1_0(M)$[17].

Since the anti-paraKähler condition $(\nabla\varphi = 0)$ is equivalent to paraholomorphicity condition of the anti-paraHermitian metric g, we have $(\phi_\varphi g) = 0$ [17, 15].

The purity conditions for a tensor field $\omega \in \Im^q_0(M)$ with respect to the paracomplex structure φ are given by

$$\omega(\varphi X_1, X_2, \cdots, X_q) = \omega(X_1, \varphi X_2, \cdots, X_q) = \cdots = \omega(X_1, X_2, \cdots, \varphi X_q),$$

for all $X_1, X_2, \cdots, X_q \in \Im_0^1(M)$ [17].

It is well known that if (M^{2m}, φ, g) is a anti-paraKähler manifold, the Riemannian curvature tensor is pure [17], and

$$\left\{ \begin{array}{ll} R(\varphi Y, Z) & = R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) & = R(Y, Z), \end{array} \right. \tag{3.5}$$

for all $Y, Z \in \Im_0^1(M)$.

Let (M^{2m}, φ, g) be a non-integrable almost anti-paraHermitian manifold. If

$$\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0.$$

for all $X, Y, Z \in \Im_0^1(M)$, where σ is the cyclic sum by three arguments, then the triple (M^{2m}, φ, g) is a quasi-anti-para-Kähler manifold [7, 10]. We know that

$$\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0 \Leftrightarrow \underset{X,Y,Z}{\sigma} (\phi_\varphi g)(X, Y, Z) = 0, \tag{3.6}$$

which was proven in [16].

Definition 3.1. Let (M^{2m}, φ, g) be an almost anti-paraHermitian manifold and $f : M \to]0, +\infty[$ be a strictly positive smooth function on M. We define the fiber-wise vertical *rescaled Berger deformation metric* de noted by $\tilde{g}$ on TM, by

$$\begin{array}{rcl} \tilde{g}(X^H, Y^H)_{(x,u)} & = & g_x(X, Y), \\ \tilde{g}(X^H, Y^V)_{(x,u)} & = & 0, \\ \tilde{g}(X^V, Y^V)_{(x,u)} & = & f(x)\big(g_x(X, Y) + \delta^2 g_x(X, \varphi u) g_x(Y, \varphi u)\big), \end{array}$$

for all $X, Y \in \Im_0^1(M)$ and $(x, u) \in TM$, where δ is some constant [2, 18]. Then f is called *twisting function*.

In the following, we consider $\lambda = 1 + \delta^2 r^2$ and $r^2 = g(u, u) = \|u\|^2$, where $\| \, . \, \|$ denotes the norm with respect to g.

Let U^V be the canonical vertical vector field. Then $\tilde{g}(X^V, \varphi U^V) = \lambda f g(X, \varphi u)$.

Lemma 3.1. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, we have:*

$$\begin{array}{rcl} (1)\ X^H \tilde{g}(Y^H, Z^H) & = & X g(Y, Z), \\ (2)\ X^V \tilde{g}(Y^H, Z^H) & = & 0, \\ (3)\ X^H \tilde{g}(Y^V, Z^V) & = & \dfrac{1}{f} X(f) \tilde{g}(Y^V, Z^V) + \tilde{g}((\nabla_X Y)^V, Z^V) + \tilde{g}(Y^V, (\nabla_X Z)^V), \\ (4)\ X^V \tilde{g}(Y^V, Z^V) & = & \delta^2 f \big[g(X, \varphi Y) g(Z, \varphi u) + g(Y, \varphi u) g(X, \varphi Z) \big], \end{array}$$

for all $X, Y, Z \in \Im_0^1(M)$.

Theorem 3.2. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric. Then the* *corresponding Levi-Civita connection $\widetilde{\nabla}$ satisfies the following:*

$$
\begin{aligned}
1.\ \widetilde{\nabla}_{X^H} Y^H &= (\nabla_X Y)^H - \frac{1}{2}(R(X,Y)u)^V,\\
2.\ \widetilde{\nabla}_{X^H} Y^V &= (\nabla_X Y)^V + \frac{1}{2f}X(f)Y^V + \frac{f}{2}(R(u,Y)X)^H,\\
3.\ \widetilde{\nabla}_{X^V} Y^H &= \frac{1}{2f}Y(f)X^V + \frac{f}{2}(R(u,X)Y)^H,\\
4.\ \widetilde{\nabla}_{X^V} Y^V &= \frac{-1}{2f}\tilde{g}(X^V,Y^V)(grad\, f)^H + \frac{\delta^2}{\lambda}g(X,\varphi Y)(\varphi U)^V,
\end{aligned}
$$

for all vector fields $X, Y \in \Im^1_0(M)$, where ∇ and R respectively denote the Levi-Civita connection and the curvature tensor of (M^{2m}, φ, g).

Proof. The proof of Theorem 3.2 follows directly from the Koszul formula and Lemma 3.1. □

4 Some almost paracomplex anti-paraHermitian structures

I. Let (M^{2m}, φ, g) be an anti-paraKähler manifold. We consider the almost paracomplex structure P on TM defined by

$$
\left\{\begin{array}{ll} PX^H &= X^H \\ PX^V &= -X^V \end{array}\right. \tag{4.1}
$$

for all $X \in \Im^1_0(M)$ [4].

Lemma 4.1. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost paracomplex structure P defined by (4.1). Then the triple $(TM, P, \tilde{g})$ is an almost anti-paraHermitian manifold.*

Proof. From Definition 3.1 and (4.1), it is easy to see that the vertical rescaled Berger deformation metric $\tilde{g}$ is anti-paraHermitian metric (pure metric) with respect to the almost paracomplex structure P. □

Proposition 4.2. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost paracomplex structure P defined by (4.1). Then we infer:*

1. $(\phi_P \tilde{g})(X^H, Y^H, Z^H) = 0,$
2. $(\phi_P \tilde{g})(X^V, Y^H, Z^H) = 0,$
3. $(\phi_P \tilde{g})(X^H, Y^V, Z^H) = 2fg(R(X,Z)u, Y),$
4. $(\phi_P \tilde{g})(X^H, Y^H, Z^V) = 2fg(R(X,Y)u, Z),$
5. $(\phi_P \tilde{g})(X^V, Y^V, Z^H) = 0,$

6. $(\phi_P\tilde{g})(X^V, Y^H, Z^V) = 0,$
7. $(\phi_P\tilde{g})(X^H, Y^V, Z^V) = 2X(f)\tilde{g}(Y^V, Z^V),$
8. $(\phi_P\tilde{g})(X^V, Y^V, Z^V) = 0,$

for all $X, Y, Z \in \Im^1_0(M)$.

Proof. We calculate the Tachibana operator ϕ_P applied to the anti-paraHermitian metric $\tilde{g}$. This operator is characterized by (3.3), and from Lemma 3.1 we have

$$\begin{aligned}
1.\,(\phi_P\tilde{g})(X^H, Y^H, Z^H) &= (PX^H)\tilde{g}(Y^H, Z^H) - X^H\tilde{g}(PY^H, Z^H) \\
&\quad +\tilde{g}\big((L_{Y^H}P)X^H, Z^H\big) + \tilde{g}\big(Y^H, (L_{Z^H}P)X^H\big) \\
&= X^H\tilde{g}(Y^H, Z^H) - X^H\tilde{g}(Y^H, Z^H) \\
&\quad +\tilde{g}\big(L_{Y^H}PX^H - P(L_{Y^H}X^H), Z^H\big) \\
&\quad +\tilde{g}\big(Y^H, L_{Z^H}PX^H - P(L_{Z^H}X^H)\big) \\
&= \tilde{g}\big([Y^H, X^H] - P[Y^H, X^H], Z^H\big) \\
&\quad +\tilde{g}\big(Y^H, [Z^H, X^H] - P[Z^H, X^H]\big) \\
&= \tilde{g}\big([Y^H, X^H], Z^H\big) - \tilde{g}\big(P[Y^H, X^H], Z^H\big) \\
&\quad +\tilde{g}\big(Y^H, [Z^H, X^H]\big) - \tilde{g}\big(Y^H, P[Z^H, X^H]\big) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
2.\,(\phi_P\tilde{g})(X^V, Y^H, Z^H) &= (PX^V)\tilde{g}(Y^H, Z^H) - X^V\tilde{g}(PY^H, Z^H)) \\
&\quad +\tilde{g}\big((L_{Y^H}P)X^V, Z^H\big) + \tilde{g}\big(Y^H, (L_{Z^H}P)X^V\big) \\
&= -X^V\tilde{g}(Y^H, Z^H) - X^V\tilde{g}(Y^H, Z^H) \\
&\quad +\tilde{g}\big(-[Y^H, X^V] - P[Y^H, X^V], Z^H\big) \\
&\quad +\tilde{g}\big(Y^H, -[Z^H, X^V] - P[Z^H, X^V]\big) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
3.\,(\phi_P\tilde{g})(X^H, Y^V, Z^H) &= (PX^H)\tilde{g}(Y^V, Z^H) - X^H\tilde{g}(PY^V, Z^H) \\
&\quad +\tilde{g}\big((L_{Y^V}P)X^H, Z^H\big) + \tilde{g}\big(Y^V, (L_{Z^H}P)X^H\big) \\
&= \tilde{g}\big([Y^V, X^H] - P[Y^V, X^H], Z^H\big) \\
&\quad +\tilde{g}\big(Y^V, [Z^H, X^H] - P[Z^H, X^H]\big) \\
&= \tilde{g}\big(Y^V, -2(R(Z, X)u)^V\big) \\
&= 2\tilde{g}\big((R(X, Z)u)^V, Y^V\big) \\
&= 2f\big(g(R(X, Z)u, Y) + \delta^2 g(R(X, Z)u, \varphi u)g(Y, \varphi u)\big) \\
&= 2fg(R(X, Z)u, Y).
\end{aligned}$$

Because the Riemann curvature R of an anti-paraKähler manifold is pure, this means

$$g(R(X, Z)u, \varphi u) \quad = \quad g(R(\varphi X, Z)u, u) = 0.$$

$$
\begin{aligned}
4.\,(\phi_P\tilde g)(X^H,Y^H,Z^V) &= (PX^H)\tilde g(Y^H,Z^V)-X^H\tilde g(PY^H,Z^V)\\
&\quad+\tilde g\big((L_{Y^H}P)X^H,Z^V\big)+\tilde g\big(Y^H,(L_{Z^V}P)X^H\big)\\
&= \tilde g\big([Y^H,X^H]-P[Y^H,X^H],Z^V\big)\\
&\quad+\tilde g\big(Y^H,[Z^V,X^H]-P[Z^V,X^H]\big)\\
&= -2\tilde g\big((R(Y,X)u)^V,Z^V\big)\\
&= 2\tilde g\big((R(X,Y)u)^V,Z^V\big)\\
&= 2fg(R(X,Y)u,Z).
\end{aligned}
$$

The other formulas are obtained by a similar calculation. □

Theorem 4.3. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, $(TM,\tilde g)$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost paracomplex structure P defined by* (4.1)*,then the triple $(TM,P,\tilde g)$ is an anti-paraKähler manifold if and only if M is flat and f is constant.*

Proof. For all $X,Y,Z\in\Im_0^1(M)$ and $h,k,l\in\{H,V\}$

$$
(\phi_P\tilde g))(X^h,Y^k,Z^l)=0 \;\Leftrightarrow\; \begin{cases} g(R(X,Z)u,Y) &=0\\ g(R(X,Y)u,Z) &=0\\ X(f) \end{cases}
\;\Leftrightarrow\; \begin{cases} R &=0\\ f &= constant \end{cases}
$$

□

Theorem 4.4. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, $(TM,\tilde g)$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost paracomplex structure P defined by* (4.1)*, then the triple $(TM,P,\tilde g)$ is a quasi-anti-paraKähler manifold if and only if f is constant.*

Proof. From (3.6) and Proposition 4.2 we have, for all $X,Y,Z\in\Im_0^1(M)$,

1. $\underset{X^H,Y^H,Z^H}{\sigma}(\phi_P\tilde g)(X^H,Y^H,Z^H)=0,$
2. $\underset{X^V,Y^H,Z^H}{\sigma}(\phi_P\tilde g)(X^V,Y^H,Z^H)=2g(R(Z,Y)u,X)+2g(R(Y,Z)u,X)=0,$
3. $\underset{X^V,Y^V,Z^H}{\sigma}(\phi_P\tilde g)(X^V,Y^V,Z^H)=2Z(f)\tilde g(X^V,Y^V),$
4. $\underset{X^V,Y^V,Z^V}{\sigma}(\phi_P\tilde g)(X^V,Y^V,Z^V)=0,$

then, $(TM,P,\tilde g)$ is a quasi-anti-paraKähler manifold if and only if f is constant. □

II. Now consider the almost product structure P defined by (4.1). We define a tensor field S of type $(1,2)$ and linear connection $\widehat\nabla$ on TM by,

$$
S(\widetilde X,\widetilde Y)=\frac12\big[(\widetilde\nabla_{P\widetilde Y}P)\widetilde X+P\big((\widetilde\nabla_{\widetilde Y}P)\widetilde X\big)-P\big((\widetilde\nabla_{\widetilde X}P)\widetilde Y\big)\big]. \tag{4.2}
$$

$$\widehat{\nabla}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - S(\widetilde{X},\widetilde{Y}). \tag{4.3}$$

for all $\widetilde{X},\widetilde{Y} \in \Im^1_0(TM)$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $(TM,\tilde{g})$ given by Theorem 3.2. Then $\widehat{\nabla}$ is an almost product connection on TM (see [5, p.151] for more details).

Lemma 4.5. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, $(TM,\tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost product structure P defined by* (4.1). *Then the tensor field S satisfies*

$$\begin{array}{rcl}
(1)\ S(X^H,Y^H) &=& -\dfrac{1}{2}(R(X,Y)u)^V,\\
(2)\ S(X^H,Y^V) &=& -\dfrac{1}{f}X(f)Y^V + \dfrac{f}{2}(R(u,Y)X)^H,\\
(3)\ S(X^V,Y^H) &=& \dfrac{1}{2f}Y(f)X^V - f(R(u,X)Y)^H,\\
(4)\ S(X^V,Y^V) &=& -\dfrac{1}{2f}\tilde{g}(X^V,Y^V)(grad\,f)^H,
\end{array}$$

for all $X,Y \in \Im^1_0(M)$.

Proof. (1) Using (4.1) and (4.2), we have

$$\begin{aligned}
S(X^H,Y^H) &= \frac{1}{2}\big[(\widetilde{\nabla}_{PY^H}P)X^H + P\big((\widetilde{\nabla}_{Y^H}P)X^H\big) - P\big((\widetilde{\nabla}_{X^H}P)Y^H\big)\big]\\
&= \frac{1}{2}\big[\widetilde{\nabla}_{Y^H}X^H - P(\widetilde{\nabla}_{Y^H}X^H) + P\big(\widetilde{\nabla}_{Y^H}X^H\big)\\
&\quad -\widetilde{\nabla}_{Y^H}X^H - P\big(\widetilde{\nabla}_{X^H}Y^H\big) + \widetilde{\nabla}_{X^H}Y^H\big]\\
&= \frac{1}{2}\big[-P\big(\widetilde{\nabla}_{X^H}Y^H\big) + \widetilde{\nabla}_{X^H}Y^H\big]\\
&= \frac{1}{2}\big[-(\nabla_X Y)^H - \frac{1}{2}(R(X,Y)u)^V\\
&\quad +(\nabla_X Y)^H - \frac{1}{2}(R(X,Y)u)^V\big]\\
&= -\frac{1}{2}(R(X,Y)u)^V.
\end{aligned}$$

(2) By a similar calculation to (1), we get

$$\begin{aligned}
S(X^H,Y^V) &= \frac{1}{2}\big[(\widetilde{\nabla}_{PY^V}P)X^H + P\big((\widetilde{\nabla}_{Y^V}P)X^H\big) - P\big((\widetilde{\nabla}_{X^H}P)Y^V\big)\big]\\
&= \frac{1}{2}\big[-\widetilde{\nabla}_{Y^V}X^H + P(\widetilde{\nabla}_{Y^V}X^H) + P\big(\widetilde{\nabla}_{Y^V}X^H\big)\\
&\quad -\widetilde{\nabla}_{Y^V}X^H + P\big(\widetilde{\nabla}_{X^H}Y^V\big) + \widetilde{\nabla}_{X^H}Y^V\big]\\
&= \frac{1}{2}\big[2P(\widetilde{\nabla}_{Y^V}X^H) - 2\widetilde{\nabla}_{Y^V}X^H + P\big(\widetilde{\nabla}_{X^H}Y^V\big) + \widetilde{\nabla}_{X^H}Y^V\big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\big[-\frac{1}{f}X(f)Y^V + f(R(u,Y)X)^H - \frac{1}{f}X(f)Y^V \\
&\quad -f(R(u,Y)X)^H - (\nabla_X Y)^V - \frac{1}{2f}X(f)Y^V + \frac{f}{2}(R(u,Y)X)^H \\
&\quad +(\nabla_X Y)^V + \frac{1}{2f}X(f)Y^V + \frac{f}{2}(R(u,Y)X)^H\big] \\
&= -\frac{1}{f}X(f)Y^V + \frac{f}{2}(R(u,Y)X)^H.
\end{aligned}
$$

The other formulas are obtained by similar calculations. □

Theorem 4.6. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, $(TM,\tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost product structure P defined by* (4.1)*. Then the almost product connection $\widehat{\nabla}$ defined by* (4.3) *is as follows,*

$$
\begin{aligned}
(1)\ \widehat{\nabla}_{X^H}Y^H &= (\nabla_X Y)^H, \\
(2)\ \widehat{\nabla}_{X^H}Y^V &= (\nabla_X Y)^V + \frac{3}{2f}X(f)Y^V, \\
(3)\ \widehat{\nabla}_{X^V}Y^H &= \frac{3f}{2}(R(u,X)Y)^H, \\
(4)\ \widehat{\nabla}_{X^V}Y^V &= \frac{\delta^2}{\lambda}g(X,\varphi Y)(\varphi U)^V,
\end{aligned}
$$

for all $X,Y \in \Im_0^1(M)$.

Proof. The proof of Theorem 4.6 follows directly from Theorem 3.2, Lemma 4.5 and formula (4.3). □

Lemma 4.7. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, $(TM,\tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost product structure P defined by* (4.1) *and $\widehat{T}$ denote the torsion tensor of $\widehat{\nabla}$. Then we have:*

$$
\begin{aligned}
(1)\ \widehat{T}(X^H,Y^H) &= (R(X,Y)u)^V, \\
(2)\ \widehat{T}(X^H,Y^V) &= \frac{3}{2f}X(f)Y^V - \frac{3f}{2}(R(\varphi u,Y)X)^H, \\
(3)\ \widehat{T}(X^V,Y^H) &= -\frac{3}{2f}Y(f)X^V + \frac{3f}{2}(R(\varphi u,X)Y)^H, \\
(4)\ \widehat{T}(X^V,Y^V) &= 0,
\end{aligned}
$$

for all $X,Y \in \Im_0^1(M)$.

Proof. The proof of Lemma 4.7 follows directly from Lemma 4.5 and formula

$$
\begin{aligned}
\widehat{T}(\widetilde{X},\widetilde{Y}) &= \widehat{\nabla}_{\widetilde{X}}\widetilde{Y} - \widehat{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}] \\
&= S(\widetilde{Y},\widetilde{X}) - S(\widetilde{X},\widetilde{Y})
\end{aligned}
$$

for all $\widetilde{X},\widetilde{Y} \in \Im_0^1(TM)$. □

From Lemma 4.7 we obtain

Theorem 4.8. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(TM, \tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost product structure P defined by (4.1), then $\widehat{\nabla}$ is symmetric if and only if M is flat and f is constant. In this case, the Levi-Civita connection $\widetilde{\nabla}$ and the almost product connection $\widehat{\nabla}$ coincide with each other.*

III. Let (M^{2m}, φ, g) be an anti-paraKähler manifold. We Consider the almost paracomplex structure Q on TM defined by

$$\begin{cases} QX^H &= X^V \\ QX^V &= X^H \end{cases} \tag{4.4}$$

for all $X \in \Im^1_0(M)$[4].

Theorem 4.9. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost paracomplex structure Q defined by (4.4), then*

(i) If $f = 1$, the vertical rescaled Berger deformation metric is anti-paraHermitian with respect to Q if and only if $\delta = 0$, i.e., the triple $(TM, Q, \tilde{g})$ is an almost anti-paraHermitian manifold, then $\tilde{g}$ reduces to the Sasaki metric.

(ii) In the case of $f \neq 1$ The vertical rescaled Berger deformation metric is never anti-paraHermitian with respect to Q.

Proof. For the purity condition, we put for all $X, Y \in \Im^1_0(M)$ and $k, h \in \{H, V\}$:

$$A(X^k, Y^h) = \tilde{g}(QX^k, Y^h) - \tilde{g}(X^k, QY^h).$$

$$\begin{aligned} (i)\ A(X^H, Y^H) &= \tilde{g}(QX^H, Y^H) - \tilde{g}(X^H, QY^H) = 0, \\ (ii)\ A(X^H, Y^V) &= \tilde{g}(QX^H, Y^V) - \tilde{g}(X^H, QY^V) \\ &= f\big[g(X,Y) + \delta^2 g(X, \varphi u) g(Y, \varphi u)\big] - g(X,Y) = 0 \\ &= (f-1) g(X,Y) + f\delta^2 g(X, \varphi u) g(Y, \varphi u) = 0, \\ (iii)\ A(X^V, Y^V) &= \tilde{g}(QX^V, Y^V) - \tilde{g}(X^V, QY^V) = 0, \end{aligned}$$

From this, if $f = 1$, then $A(X^k, Y^h) = 0$ if and only if $\delta = 0$. □

IV. Let (M^{2m}, φ, g) be an almost anti-paraKähler manifold. We define a tensor field $P_\varphi \in \Im^1_1(TM)$ by,

$$\begin{cases} P_\varphi X^H &= X^H + \eta g(X, \varphi u)(\varphi U)^H \\ P_\varphi X^V &= -X^V + \mu g(X, \varphi u)(\varphi U)^V \end{cases} \tag{4.5}$$

for all $X \in \Im^1_0(M)$, where $\eta, \mu : \mathbb{R} \to \mathbb{R}$ are smooth functions.

If $\eta = \mu = 0$, then P_φ is the almost paracomplex structure defined by (4.1).

In the following, we consider $\eta \neq 0$ and $\mu \neq 0$. Note that

$$\begin{cases} P_\varphi (\varphi U)^H &= (1 + \eta r^2)(\varphi U)^H \\ P_\varphi (\varphi U)^V &= (-1 + \mu r^2)(\varphi U)^V \end{cases} \tag{4.6}$$

such that $r^2 = g(u, u)$.

Lemma 4.10. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric. Then the endomorphism P_φ defined by (4.5) is an almost paracomplex structure if and only if $\eta = -\dfrac{2}{r^2}$ and $\mu = \dfrac{2}{r^2}$, i.e.,*

$$\begin{cases} P_\varphi X^H &= X^H - \dfrac{2}{r^2} g(X, \varphi u)(\varphi U)^H \\ P_\varphi X^V &= -X^V + \dfrac{2}{r^2} g(X, \varphi u)(\varphi U)^V \end{cases} \tag{4.7}$$

for all $X \in \Im_0^1(M)$ and $r^2 = g(u, u)$.

Proof. 1) Let $X \in \Im_0^1(M)$,

$$\begin{aligned} P_\varphi^2(X^H) &= P_\varphi(P_\varphi(X^H)) \\ &= P_\varphi(X^H + \eta g(X, \varphi u)(\varphi U)^H) \\ &= X^H + \eta g(X, \varphi u)(\varphi U)^H + \eta g(X, \varphi u)(1 + \eta r^2)(\varphi U)^H \\ &= X^H + \eta(2 + \eta r^2) g(X, \varphi u)(\varphi U)^H. \end{aligned} \tag{4.8}$$

$$\begin{aligned} P_\varphi^2(X^V) &= P_\varphi(P_\varphi(X^V)) \\ &= P_\varphi(-X^V + \mu g(X, \varphi u)(\varphi U)^V) \\ &= X^V - \mu g(X, \varphi u)(\varphi U)^V + \mu g(X, \varphi u)(-1 + \mu r^2)(\varphi U)^V \\ &= X^V + \mu(-2 + \mu r^2) g(X, \varphi u)(\varphi U)^V. \end{aligned} \tag{4.9}$$

From (4.8) and (4.9), then $P_\varphi^2 = Id_{TM}$ equivalent to $\eta = -\dfrac{2}{r^2}$ and $\mu = \dfrac{2}{r^2}$. □

Theorem 4.11. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(TM, \tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost paracomplex structure P_φ defined by (4.7). Then the triple $(TM, P_\varphi, \tilde{g})$ is an almost anti-paraHermitian manifold.*

Proof. For purity condition, we put for all $X, Y \in \Im_0^1(M)$ and $k, h \in \{H, V\}$:

$$A(X^k, Y^h) = \tilde{g}(P_\varphi X^k, Y^h) - \tilde{g}(X^k, P_\varphi Y^h).$$

$$\begin{aligned} (i)\ A(X^H, Y^H) &= \tilde{g}(P_\varphi X^H, Y^H) - \tilde{g}(X^H, P_\varphi Y^H) \\ &= \tilde{g}(X^H - \frac{2}{r^2} g(X, \varphi u)(\varphi U)^H, Y^H) \\ &\quad - \tilde{g}(X^H, Y^H - \frac{2}{r^2} g(Y, \varphi u)(\varphi U)^H) \\ &= \tilde{g}(X^H, Y^H) - \frac{2}{r^2} g(X, \varphi u) g(Y, \varphi u) \\ &\quad - \tilde{g}(X^H, Y^H) + \frac{2}{r^2} g(Y, \varphi u) g(X, \varphi u) \\ &= 0. \end{aligned}$$

$$
\begin{aligned}
(ii)\ A(X^V,Y^V) &= \tilde{g}(P_\varphi X^V,Y^V)-\tilde{g}(X^V,P_\varphi Y^V)\\
&= \tilde{g}(-X^V+\frac{2}{r^2}g(X,\varphi u)(\varphi U)^V,Y^V)\\
&\quad -\tilde{g}(X^V,-Y^V+\frac{2}{r^2}g(Y,\varphi u)(\varphi U)^V)\\
&= -\tilde{g}(X^V,Y^V)+\frac{2}{r^2}g(X,u)f\lambda g(Y,\varphi u)\\
&\quad +\tilde{g}(X^V,Y^V)-\frac{2}{r^2}g(Y,\varphi u)f\lambda g(X,\varphi u)\\
&= 0.\\
(iii)\ A(X^H,Y^V) &= \tilde{g}(P_\varphi X^H,Y^V)-\tilde{g}(X^H,P_\varphi Y^V)\\
&= \tilde{g}(X^H-\frac{2}{r^2}g(X,\varphi u)(\varphi U)^H,Y^V)\\
&\quad -\tilde{g}(X^H,-Y^V+\frac{2}{r^2}g(Y,\varphi u)(\varphi U)^V)\\
&= 0.
\end{aligned}
$$

□

Lemma 4.12. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, $(TM,\tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric and $\widetilde{\nabla}$ denote the corresponding Levi-Civita connection of $\tilde{g}$. Then we have:*

$$
\begin{aligned}
1.\ \widetilde{\nabla}_{X^H}(\varphi U)^H &= -\frac{1}{2}(R(X,\varphi u)u)^V,\\
2.\ \widetilde{\nabla}_{X^H}(\varphi U)^V &= \frac{1}{2f}X(f)(\varphi U)^V,\\
3.\ \widetilde{\nabla}_{X^V}(\varphi U)^H &= (\varphi X)^H+\frac{1}{2f}g(\varphi u,grad\, f)X^V+\frac{f}{2}(R(u,X)\varphi u)^H,\\
4.\ \widetilde{\nabla}_{X^V}(\varphi U)^V &= (\varphi X)^V-\frac{\lambda}{2}g(X,\varphi u)(grad\, f)^H+\frac{\delta^2}{\lambda}g(X,u)(\varphi U)^V,
\end{aligned}
$$

for all vector fields $X\in\Im_0^1(M)$.

Proof. The proof of lemma 4.12 follows directly from Theorem 3.2. □

Proposition 4.13. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, $(TM,\tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric, the almost paracomplex structure P_φ defined by (4.7) and $\widetilde{\nabla}$ denote the corresponding Levi-Civita connection of $\tilde{g}$. Then we have:*

$$
\begin{aligned}
1.\ (\widetilde{\nabla}_{X^H}P_\varphi)Y^H &= -(R(X,Y)u)^H-\frac{2}{r^2}g(Y,\varphi\nabla_X U)(\varphi U)^H\\
&\quad +\frac{1}{r^2}g(Y,\varphi u)(R(Y,\varphi u)u)^V,\\
2.\ (\widetilde{\nabla}_{X^H}P_\varphi)Y^V &= \frac{2}{r^2}g(Y,\varphi\nabla_X U)(\varphi U)^V-\frac{f}{r^2}g(R(u,Y)X,\varphi u)(\varphi U)^H,
\end{aligned}
$$

$$
\begin{aligned}
3.\ (\widetilde{\nabla}_{X^V} P_\varphi)Y^H \quad = \quad & f(R(u,X)Y)^H - \frac{f}{r^2} g(Y,\varphi u)(R(u,X)\varphi u)^H \\
& -\frac{2}{r^2} g(Y,\varphi u)(\varphi X)^H + \frac{f}{2} g(R(u,X)Y,\varphi u)(\varphi U)^H \\
& +\big[\frac{4}{r^4} g(X,u)g(Y,\varphi u) - \frac{2}{r^2} g(Y,\varphi X)\big](\varphi U)^H \\
& +\big[\frac{1}{f} Y(f) - \frac{1}{fr^2} g(Y,\varphi u) g(\varphi u, grad\, f)\big] X^V \\
& -\frac{1}{fr^2} Y(f) g(X,\varphi u)(\varphi X)^V, \\
4.\ (\widetilde{\nabla}_{X^V} P_\varphi)Y^V \quad = \quad & \big[g(X,Y) - \frac{1}{r^2} g(X,\varphi u) g(Y,\varphi u)\big](grad\, f)^H \\
& -\frac{1}{r^2}\big[g(X,Y) + \delta^2 g(X,\varphi u) g(Y,\varphi u)\big] g(\varphi u, grad\, f)(\varphi U)^H \\
& +\big[\frac{2r^2\delta^2 - 4\lambda}{\lambda r^4} g(X,u) g(Y,\varphi u) + \frac{2}{r^2\lambda} g(X,\varphi Y)\big](\varphi U)^V \\
& +\frac{2}{r^2} g(Y,\varphi u)(\varphi X)^V,
\end{aligned}
$$

for all vector fields $X \in \Im^1_0(M)$.

Proof. The proof of Proposition 4.13 follows directly from Theorem 3.2 and from the formula $\widetilde{\nabla}_{\widetilde{X}} P_\varphi \widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}(P_\varphi \widetilde{Y}) - P_\varphi \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}$. □

Hence, we deduce:

Theorem 4.14. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, $(TM,\tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost paracomplex structure P_φ defined by (4.7). Then the triple $(TM,P_\varphi,\tilde{g})$ is never an almost anti-paraHermitian manifold.*

V. Let (M^{2m},φ,g) be an almost anti-paraHermitian manifold. We define a tensor field $Q_\varphi \in \Im^1_1(TM)$ by,

$$
\left\{
\begin{aligned}
Q_\varphi X^H &= \frac{1}{\sqrt{f}}(X^V + \eta g(X,\varphi u)(\varphi U)^V) \\
Q_\varphi X^V &= \sqrt{f}(X^H + \mu g(X,\varphi u)(\varphi U)^H)
\end{aligned}
\right.
\tag{4.10}
$$

for all $X \in \Im^1_0(M)$, where $\eta,\mu : \mathbb{R} \to \mathbb{R}$ are smooth functions.

If $\eta = \mu = 0$, then Q_φ is the almost paracomplex structure defined by (4.4). In the following, we consider $\eta \neq 0$ and $\mu \neq 0$.
Note that

$$
\left\{
\begin{aligned}
Q_\varphi(\varphi U)^H &= \frac{1}{\sqrt{f}}(1+\eta r^2)(\varphi U)^V \\
Q_\varphi(\varphi U)^V &= \sqrt{f}(1+\mu r^2)(\varphi U)^H
\end{aligned}
\right.
\tag{4.11}
$$

Lemma 4.15. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric. Then* *the endomorphism* Q_φ *defined by* (4.5) *is an almost paracomplex structure if and only if*

$$\eta + \mu + \eta\mu r^2 = 0. \tag{4.12}$$

Proof. 1) Let $X \in \Im_0^1(M)$,

$$\begin{aligned} Q_\varphi^2(X^H) &= Q_\varphi(Q_\varphi(X^H)) \\ &= \frac{1}{\sqrt{f}} Q_\varphi(X^V + \eta g(X,\varphi u)(\varphi U)^V) \\ &= X^H + \mu g(X,\varphi u)(\varphi U)^H + \eta g(X,\varphi u)(1+\mu r^2)(\varphi U)^H \\ &= X^H + (\eta + \mu + \eta\mu r^2) g(X,\varphi u)(\varphi U)^H. \end{aligned} \tag{4.13}$$

$$\begin{aligned} Q_\varphi^2(X^V) &= Q_\varphi(Q_\varphi(X^V)) \\ &= \sqrt{f} Q_\varphi(X^H + \mu g(X,\varphi u)(\varphi U)^H) \\ &= X^V + \eta g(X,\varphi u)(\varphi U)^V + \mu g(X,\varphi u)(1+\eta r^2)(\varphi U)^V \\ &= X^V + (\eta + \mu + \eta\mu r^2) g(X,\varphi u)(\varphi U)^V. \end{aligned} \tag{4.14}$$

From (4.13) and (4.14), then $Q_\varphi^2 = Id_{TM}$ equivalent to $\eta + \mu + \eta\mu r^2 = 0$. □

Theorem 4.16. *Let* (M^{2m}, φ, g) *be an anti-paraKähler manifold,* $(TM, \tilde{g})$ *be its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost paracomplex structure* Q_φ *defined by* (4.10) *and* (4.12). *The triple* $(TM, Q_\varphi, \tilde{g})$ *is an almost anti-paraHermitian manifold if and only if*

$$\mu = \lambda\eta + \delta^2, \tag{4.15}$$

where $\lambda = 1 + \delta^2 r^2$.

Proof. For the purity condition, we put for all $X, Y \in \Im_0^1(M)$ and $k, h \in \{H, V\}$:

$$A(X^k, Y^h) = \tilde{g}(Q_\varphi X^k, Y^h) - \tilde{g}(X^k, Q_\varphi Y^h).$$

$$\begin{aligned} (i)\ A(X^H, Y^H) &= \tilde{g}(Q_\varphi X^H, Y^H) - \tilde{g}(X^H, Q_\varphi Y^H) \\ &= \tilde{g}(\frac{1}{\sqrt{f}}(X^V + \eta g(X,\varphi u)(\varphi U)^V), Y^H) \\ &\quad -\tilde{g}(X^H, \frac{1}{\sqrt{f}}(Y^V + \eta g(Y,\varphi u)(\varphi U)^V)) \\ &= 0. \\ (ii)\ A(X^V, Y^V) &= \tilde{g}(Q_\varphi X^V, Y^V) - \tilde{g}(X^V, Q_\varphi Y^V) \\ &= \tilde{g}(\sqrt{f}(X^H + \mu g(X,\varphi u)(\varphi U)^H), Y^V) \\ &\quad -\tilde{g}(X^V, \sqrt{f}(Y^H + \mu g(Y,\varphi u)(\varphi U)^H)) \\ &= 0. \end{aligned}$$

$$\begin{aligned}
(iii)\ A(X^H,Y^V) &= \tilde{g}(Q_\varphi X^H,Y^V)-\tilde{g}(X^H,Q_\varphi Y^V)\\
&= \tilde{g}(\frac{1}{\sqrt{f}}(X^V+\eta g(X,\varphi u)(\varphi U)^V),Y^V)\\
&\quad -\tilde{g}(X^H,\sqrt{f}(Y^H+\mu g(Y,\varphi u)(\varphi U)^H))\\
&= \frac{1}{\sqrt{f}}\tilde{g}(X^V,Y^V)+\frac{1}{\sqrt{f}}\eta g(X,\varphi u)\tilde{g}((\varphi U)^V),Y^V)\\
&\quad -\sqrt{f}\tilde{g}(X^H,Y^H)-\sqrt{f}\mu g(Y,\varphi u)\tilde{g}(X^H,(\varphi U)^H))\\
&= \sqrt{f}(g(X,Y)+\delta^2 g(X,\varphi u)g(Y,\varphi u))\\
&\quad +\sqrt{f}\eta\lambda g(X,\varphi u)g(Y,\varphi u)\\
&\quad -\sqrt{f}g(X,Y)-\sqrt{f}\mu g(X,\varphi u)g(Y,\varphi u)\\
&= \sqrt{f}(\delta^2+\lambda\eta-\mu)g(X,\varphi u)g(Y,\varphi u).
\end{aligned}$$

Then $A(X^H,Y^V)=0$ equivalent to $\mu=\lambda\eta+\delta^2$. □

By equations (4.12) and (4.15), we have

$$\begin{cases} \eta+\mu+\eta\mu r^2=0\\ \mu=\lambda\eta+\delta^2\end{cases} \Leftrightarrow \begin{cases} \eta=\dfrac{\varepsilon-\sqrt{\lambda}}{r^2\sqrt{\lambda}}\\ \mu=\dfrac{\varepsilon\sqrt{\lambda}-1}{r^2}\end{cases}$$

where $\varepsilon=\pm 1$.

We shall study the integrability of Q_φ. As we know, the integrability of Q_φ is equivalent to the vanishing of the Nijenhuis tensor. The Nijenhuis tensor of Q_φ is given by

$$N_{Q_\varphi}(\widetilde{X},\widetilde{Y})=[Q_\varphi\widetilde{X},Q_\varphi\widetilde{Y}]-Q_\varphi[Q_\varphi\widetilde{X},\widetilde{Y}]-Q_\varphi[\widetilde{X},Q_\varphi\widetilde{Y}]+[\widetilde{X},\widetilde{Y}].$$

where $\widetilde{X},\widetilde{Y}\in\Im^1_0(TM)$.

Lemma 4.17. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric. Then the almost paracomplex structure Q_φ defined by* (4.10) *and* (4.12) *is integrable if and only if $N_{Q_\varphi}(X^H,Y^H)=0$, for all $X,Y\in\Im^1_0(M)$.*

Proof. We put $Q_\varphi X^V=Z^H$ and $Q_\varphi Y^V=W^H$, then we have

$$\begin{aligned}
N_{Q_\varphi}(X^V,Y^V) &= [Q_\varphi X^V,Q_\varphi Y^V]-Q_\varphi[Q_\varphi X^V,Y^V]-Q_\varphi[X^V,Q_\varphi Y^V]+[X^V,Y^V]\\
&= [Z^H,W^H]-Q_\varphi[Z^H,Q_\varphi W^H]-Q_\varphi[Q_\varphi Z^H,W^H]+[Q_\varphi Z^H,Q_\varphi W^H]\\
&= N_{Q_\varphi}(Z^H,W^H).
\end{aligned}$$

$$\begin{aligned}
N_{Q_\varphi}(X^V,W^H) &= [Q_\varphi X^V,Q_\varphi W^H]-Q_\varphi[Q_\varphi X^V,W^H]-Q_\varphi[X^V,Q_\varphi W^H]+[X^V,W^H]\\
&= [Z^H,Q_\varphi W^H]-Q_\varphi[Z^H,W^H]-Q_\varphi[Q_\varphi Z^H,Q_\varphi W^H]+[Q_\varphi Z^H,W^H]\\
&= -Q_\varphi[Q_\varphi Z^H,Q_\varphi W^H]+[Q_\varphi Z^H,W^H]+[Z^H,Q_\varphi W^H]-Q_\varphi[Z^H,W^H]\\
&= -Q_\varphi(N_{Q_\varphi}(Z^H,W^H)).
\end{aligned}$$

□

Lemma 4.18. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(TM, \tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric and the almost paracomplex structure Q_φ defined by* (4.10) *and* (4.12)*. Then*

$$\begin{aligned} N_{Q_\varphi}(X^H, Y^H) &= -(R(X,Y)u)^V + \frac{\eta}{f}[g(Y,\varphi u)(\varphi X)^V - g(X,\varphi u)(\varphi Y)^V] \\ &+ \frac{2\eta' - \eta^2}{f}[g(X,u)g(Y,\varphi u) - g(X,\varphi u)g(Y,u)](\varphi U)^V \\ &+ \frac{1}{2f}[X(f)Y^H - Y(f)X^H]. \end{aligned} \tag{4.16}$$

for all $X, Y \in \Im_0^1(M)$.

Proof. By straightforward calculations, and using the formulas

$$\begin{aligned} &(\varphi U)^V(\eta) = 2\eta' g(\varphi u, u), \quad (\varphi U)^V(g(Y,\varphi u)) = g(Y,u), \\ &[Y^V, (\varphi U)^V] = (\varphi Y)^V, \quad [Y^H, (\varphi U)^V] = 0, \end{aligned}$$

we obtain the result. □

Lemma 4.19. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric. Then the almost paracomplex structure Q_φ defined by* (4.10) *and* (4.12) *is integrable if and only if f is constant and*

$$\begin{aligned} (R(X,Y)u)^V &= \frac{\eta}{f}[g(Y,\varphi u)(\varphi X)^V - g(X,\varphi u)(\varphi Y)^V] \\ &+ \frac{2\eta' - \eta^2}{f}[g(X,u)g(Y,\varphi u) - g(X,\varphi u)g(Y,u)](\varphi U)^V. \end{aligned} \tag{4.17}$$

for all $X, Y \in \Im_0^1(M)$.

It is known that since (M^{2m}, φ, g) is anti-paraKähler, then the Riemannian curvature tensor of (M^{2m}, φ, g) satisfies the equality $R(\varphi X, Y)u = R(X, \varphi Y)u$. Then, according to (4.17), this identity is never satisfied. This shows that the almost paracomplex structure Q_φ do not integrable and the triple $(TM, Q_\varphi, \tilde{g})$ is never anti-paraKähler.

References

[1] D.V. Alekseevsky, C. Medori, A. Tomassini, *Para-Kähler Einstein metrics on homogeneous manifolds*, C. R. Acad. Sci. Paris, Ser. I 347 (2009), 69-72. https://doi.org/10.1016/j.crma.2008.11.016

[2] M. Altunbas, R. Simsek, A. Gezer, *A Study Concerning Berger type deformed Sasaki Metric on the Tangent Bundle*, Zh. Mat. Fiz. Anal.Geom. 15, 4 (2019), 435-447. https://doi.org/10.15407/mag15.04.435

[3] V. Cruceanu, P.M. Gadea, J. Munoz Masque, *Para-Hermitian and para-Kähler manifolds*, Quaderni Inst. Mat. Univ. Messina 1 (1995), 1-72.

[4] V. Cruceanu, P. Fortuny, P.M. Gadea , *A survey on paracomplex geometry*, Rocky Mountain J. Math. 26 (1996), 83-115. doi:10.1216/rmjm/1181072105

[5] M. De León, P.R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics*, North-Holland Mathematics Studies, 1989.

[6] P. Dombrowski, *On the Geometry of the tangent bundle*, J. Reine Angew. Math. 210, 73-88 (1962). https://doi.org/10.1515/crll.1962.210.73

[7] G.T. Ganchev, A.V. Borisov, *Note on the almost complex manifolds with a Norden metric*, C. R. Acad. Bulgarie Sci. 39, 5 (1986), 31-34.

[8] A. Gezer, M. Ozkan, *Notes on the tangent bundle with deformed complete lift metric*, Turkish J. Math. 38 (2014), 1038-1049. doi:10.3906/mat-1402-30

[9] P. Libermann, *Sur les structures presque paracomplexes*, C. R. Acad. Sci. Paris 234 (1952), 2517-2519.

[10] M. Manev, D. Mekerov, *On Lie groups as quasi-Kähler manifolds with Killing Norden metric*, Adv. Geom., **8**, 3 (2008), 343-352. https://doi.org/10.1515/ADVGEOM.2008.022

[11] F. Ocak, A.A. Salimov, *Geometry of the cotangent bundle with Sasakian metrics and its applications*, Proc. Indian Acad. Sci. (Math. Sci.) 124, 3 (2014), 427-436. https://doi.org/10.1007/s12044-014-0191-6

[12] E.M. Patterson, *Riemann extensions which have Kähler metrics*, Proc. Roy. Soc. Edinburgh Sect. A 64 (1954), 113-126.

[13] P.K. Rashevskij, *The scalar field in a stratified space*, Trudy Sem. Vektor. Tenzor. Anal. 6 (1948), 225-248.

[14] A.A. Salimov and F. Agca, *Some properties of Sasakian metrics in cotangent bundles*, Mediterr. J. Math. 8(2) (2011), 243-255. https://doi.org/10.1007/s00009-010-0080-x

[15] A.A. Salimov, A. Gezer, M. Iscan, *On para-Kähler-Norden structures on the tangent bundles*, Ann. Polon. Math. 103, 3 (2012), 247-261. DOI: 10.4064/ap103-3-3

[16] A.A. Salimov, M. Iscan, K. Akbulut, *Notes on para-Norden-Walker 4-manifolds*, Int. J. Geom. Methods Mod. Phys. 7, 8 (2010), 1331-1347. https://doi.org/10.1142/S021988781000483X

[17] A.A. Salimov, M. Iscan, F. Etayo,*Para-holomorphic B-manifold and its properties*, Topology Appl. 154, 4 (2007), 925-933. https://doi.org/10.1016/j.topol.2006.10.003

[18] A. Yampolsky, *On geodesics of tangent bundle with fiberwise deformed Sasaki metric over Kahlerian manifolds*, Zh. Mat. Fiz. Anal. Geom. 8, 2 (2012), 177-189.

[19] K.Yano, M. Ako, *On certain operators associated with tensor field*, Kodai Math. Sem. Rep. 20 (1968), 414-436. doi:10.2996/kmj/1138845745

[20] K. Yano, S. Ishihara, *Tangent and Cotangent Bundles*, Marcel Dekker. Inc., New York 1973.

Authors' address:

Abderrahim Zagane, Naima Boussekkine
Department of Mathematics,
University of Relizane, 48000, Relizane, Algeria.
E-mail:Zaganeabr2018@gmail.com , nboussekkine@yahoo.fr