r-Almost Newton-Yamabe solitons on Legendrian submanifolds of Sasakian space forms

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Abstract. In this research paper, we develop the geometrical bearing on Legendrian submanifolds of Sasakian space forms in terms of r-almost Newton-Yamabe Soliton with the potential function $\psi : M^n \longrightarrow \mathbb{R}$. Also, we examine the certain conditions for L-minimal and totally geodesic Legendrian submanifolds of Sasakian space form admitting the r-almost Newton-Yamabe Soliton. Finally, we illustrate some examples based on this study.

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1 Introduction

In the last twenty years, geometric flows are most significant tools to explain the geometric structures in Riemannian geometry. A specific class of solutions on which the metric evolves by dilation and diffeomorphisms plays a vital part in the study of singularities of the flows as they appear as possible singularity models. They are often called soliton solutions.

The theory of Yamabe flow, which was firstly introduced by Hamilton in his famous research work [13], as a tool for constructing metrics of constant scalar curvature on a Riemannian manifold (M^n, g), $n \geq 3$. The Yamabe flow is an evolution equation for metrics on Riemannian manifolds is given by

$$\frac{\partial}{\partial t} g(t) = -\rho g(t), \quad g(0) = g_0, \tag{1.1}$$

where ρ is the scalar curvature corresponding to Riemannian metric g and t is time, which is used to deform a metric by smoothing out its singularities.

A Yamabe soliton is a spacial solution of the Yamabe flow that moves by one parameter family of diffeomorphism generated by fixed vector field X on M^n with a constant λ satisfying the following equation

$$\frac{1}{2}\mathcal{L}_U g = (\rho - \lambda)g, \tag{1.2}$$

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where $\mathcal{L}_U g$ is the Lie derivative of the metric g, in direction of vector field U. According to Pigola et al. [16] if we replace the constant λ in (1.2) with a smooth function $\lambda \in C^{\infty}(M)$, called soliton function. In this more general framework we refer to equation (1.2) as being the fundamental equation of an *almost Yamabe soliton*. If $\lambda > 0$, $\lambda < 0$ or $\lambda = 0$, then the (M^n, g) is called *Yamabe expander* , *Yamabe shrinker* or *Yamabe steady soliton*, respectively.

In the particular situation when the vector field X is the gradient of a smooth function $\psi : M^n \longrightarrow \mathbb{R}$, the manifold will be called a *gradient almost Yamabe soliton*. The function ψ is called the *potential function* of the gradient almost Yamabe soliton. In this case equation (1.2) becomes

$$Hess\psi = (\rho - \lambda)g, \tag{1.3}$$

where $Hess\psi$ stands for the Hessian of the potential function ψ. The almost gradient Yamabe soliton equation (1.3) links geometric information about the curvature of the manifold through the scalar curvature tensor and the geometry of the level sets of the potential function by means of their second fundamental form. Hence, study almost gradient Yamabe solitons under some curvature conditions is an interesting topic.

An Einstein manifold [4] with constant potential function is called a trivial gradient Ricci soliton. Gradient Yamabe solitons [9] play an important role in Yamabe flow as they correspond to self-similar solutions, and often arise as singularity models [13].

It is worth to remark that they arise from the *Ricci-Bourguignon flow* recently discovered by Cantino and Mazzieri ([5], [6]). In this more general setting, we call (1.2) as being fundamental equation of an *almost Ricci soliton* [16].

Many geometers extensively studied the above mentioned solitons which is closely related to this topic, for further details see ([5],[7], [11], [18], [19], [20]).

On the one hand isometric immersions of an almost Ricci soliton in to Riemannian manifold discussed by Barros et al. [2]. In this case, if a Riemannian manifold has non-positive sectional curvature, they established that an almost Ricci soliton is a Ricci soliton with a vector field of integrable norm, then the manifold can not be minimal. Furthermore, in [24] Wylie demonstrated that if a shrinking Ricci soliton, conceding bounded norm of a vector U on a manifold, then that manifold must be compact. In particular if Riemannian manifold is a space form of non-positive sectional curvature, then such immersions can not be minimal. Cunah et al, [8] have studied the immersed almost Ricci soliton under Newton transformation P_r with second-order differential operators L_r and introduced the new notion r-almost Newton-Ricci soliton, for some $0 \leq r \leq n$. Recently, in 2020, Siddiqi [21] discussed about Newton-Ricci-Bourguignon almost solitons on Lagrangin submanifolds of complex space form.

On other hand *Symplectic geometry* covers different classes of symplectic manifolds, contact manifolds and relation between them. The local structures such as Hamiltonian dynamics and some special types of submanifolds mainly Lagrangian submanifolds (symplectic case) and Legendrian submanifolds (contact case). Symplectic geometry and contact geometry is a relatively new field in mathematics and has connections to algebraic geometry, dynamical systems, geometric topology, and theoretical physics.

The differential geometry of Legendrian submanifolds has been an important geometric object of the contact geometry. Contact manifold is an odd-dimensional manifold equipped with a completely non-integrable field of tangent hyperplanes and a Legendrian submanifold is a submanifold everywhere tangent to this hyperplane field, which moreover is of maximal dimension. Contact geometry has its roots in classical mechanics. For instance, in optics, wave fronts of light waves propagating in a space admit natural lifts to Legendrian submanifolds of the associated space of contact elements (tangent hyperplanes). Furthermore, contact geometry is the odd-dimensional counterpart of symplectic geometry which studies symplectic manifolds, that is, manifolds that locally looks like the phase space of a mechanical system.

In the 1990's Y.G. Oh [15] introduced the study of Hamiltonian minimal (H-minimal) Lagrangian submanifolds in Kähler manifold. This is a nice extensions of the notion of minimal submanifold, and has been studied by many geometers.([12], [14], [23]). On the other hand, there is notion of Sasakian manifold which is an odd-dimensional counterpart Kähler manifolds. In Sasakian manifold, we consider Legendrian minimal (L-minimal) Legendrian submanifold which corresponds to (H-minimal) Lagrangian manifold in Kählar manifold [15] .

There were two notions of Hamiltonian deformations [15] and Lagrangian deformations in Lagrangian Geometry. In contrast, there is only a notion of Legendrian deformations in Legendrian Geometry. Analogous to the Lagrangian submanifolds in a complex space form, we consider a Legendrian submanifold in Sasakian space form. Such a submanifold has been deeply studied over the past of several decades.

Therefore the present research article inspired by the above literature, in this framework we have to explore the study of r-almost Newton-Yamabe soliton on Legendrian submanifolds of Sasakian space form.

2 Sasakian space form

A $(2m+1)$-dimensional differentiable manifold M^{2m+1} is called a contact manifold [1] if there exits a globally defined 1-form η such that $\eta \wedge (d\eta)^m \neq 0$. On a contact manifold there exists a unique vector filed ζ satisfying

$$d\eta(\zeta, X) = 0, \quad \eta(\zeta) = 0$$

for all $U \in T(M^{2m+1})$.

Let M^{2m+1} be a $(2m+1)$-dimensional Riemannian manifold. M is called an almost contact manifold if it is equipped with an almost contact structure (φ, ξ, η), where φ is a $(1,1)$-tensor field, ξ a unit vector field, η a one-form dual to ξ satisfying [1]

$$\varphi^2 = -I + \eta \otimes \zeta, \quad \eta \circ \varphi = 0, \tag{2.1}$$

$$\varphi(\zeta) = 0, \quad \eta(\zeta) = 1, \quad g(U, \zeta) = \eta(U). \tag{2.2}$$

It is well-known that there exists a Riemannian metric g such that

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V) \tag{2.3}$$

$$g(\varphi U, V) = -g(U, \varphi V), \tag{2.4}$$

where $U, V \in \chi(M)$. Moreover, if the almost contact structure (φ, ζ, η) is normal, i.e.

$$(\nabla_U \varphi)V = g(U, V)\zeta - \eta(V)U \tag{2.5}$$

$$\nabla_U \zeta = -\varphi U \tag{2.6}$$

for any vectors U, V on M, where ∇ denotes the Levi-Civita connection with respect to g, then M is said to be a Sasakian manifold it is satisfies $[\varphi, \varphi] + 2d\eta \otimes \zeta = 0$ on M^{2m+1}, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. For more details and background, see [1] and [25].

Let $(M^{2m+1}, \varphi, \zeta, \eta, g)$ be a Sasakian manifold is a contact manifold with the contact structure $(\eta, g, \zeta, \varphi)$ and L^n an m-dimensional manifold.

An immersion $\varphi : L^m \longrightarrow M^{2m+1}$ is called Legendrian immersion of an m-dimensional compact smooth manifold L into a $(2m+1)$-dimensional contact manifold (M, η) such that [23]

1. Legendrian $\Longleftrightarrow \varphi * \eta = 0$,
2. $dim(L) = m$.
3. An Legendrain immersion is called L minimal $\Longleftrightarrow div \varphi H = 0$, where H is the mean curvature vector of φ.

We may choose an almost contact metric structure (ζ, g) on M compatible with the contact structure η. A Legendrian deformation of φ is defined as a one-parameter smooth family $\{\varphi_t\}$ of Legendrian immersions $\varphi : L \longrightarrow M$ with $\varphi_0 = \varphi$.

3 Legendrian submanifolds of Sasakian space form

A plane of T_pM at p is called φ-section if it is spanned by U and φU, where U is orthonormal to ζ. The curvature of φ-section is called φ-sectional curvature.

A $(2m + 1)$-Sasakian space form is defined as a $(2m + 1)$-Sasakian manifold with constant φ-sectional curvature c and is denoted by $M^{2n+1}(c)$. As examples of Sasakian space form, $\mathbb{R}^{2n+1}$ and $\mathbb{S}^{2n+1}$ are equipped with Sasakian space form structures(more details in [1] and [25]). The curvature of a Sasakian space form $M^{2n+1}(c)$ is given by [25]

$$\begin{aligned} \bar{R}(U, V)W = {} & \frac{c+3}{4}[g(V, W)U - g(U, W)V] \\ & + \frac{c-1}{4}[\eta(U)\eta(W)V - \eta(V)\eta(W) + g(U, W)\eta(V)\zeta \\ & - g(V, W)\eta(U)\zeta + g(\varphi V, W)\varphi U - g(\varphi U, W)\varphi V - 2g(U, \varphi V)\varphi W] \end{aligned} \tag{3.1}$$

for any $U, V, W \in T(M)$.

Let L^m be an m-dimensional submanifold of a Sasakian space form $M^{2m+1}(c)$. If the one-form η constrained in M is zero, then we say M is a Legendrian submanifold. It is well-known that for such a submanifold φ maps any tangent vector to M at any $p \in M$ into the normal vector space $T_p^{\perp}(M)$ i.e. $\varphi(T_pM) \subset T_p^{\perp}M$. Actually, a Legendrian submanifold is a special *C-totally real submanifold* (i.e. the unit vector field ζ is orthonormal to M) [12] . Therefore we obtain from (2.5) and (2.6) that for any $U, V \in T(M)$,

$$g(\varphi U, \varphi V) = g(U, V), \quad \eta(U) = g(U, \zeta) = 0. \tag{3.2}$$

4 r-almost Newton-Yamabe soliton

Let $\varphi : L^m \longrightarrow M^{m+p}$ be an Legendrian immersion into an $(2m+1)$-dimensional Sasakian manifold M. We call L^m is an r-almost Newton-Yamabe soliton, for some $0 \leq r \leq m$, if there exist a smooth function $\psi : L^m \longrightarrow \mathbb{R}$ such that [6]

$$P_r \circ Hess\psi = (\rho - \lambda)g, \tag{4.1}$$

where λ is a smooth function on L^m and $P_r \circ Hess\psi$ stands for tensor given by

$$P_r \circ Hess\psi(U, V) = g(P_r \nabla_U \nabla_\psi, V), \tag{4.2}$$

for tangent vectors fields $U, V \in \chi(M)$. For $r = 0$, equation (4.1) reduces to the definition of a gradient almost Yamabe soliton.

The Gauss equation implies that

$$R(U, V, W, W') = (\bar{R}(U, V)W)^T) + g(BU, W)BV - g(BV, W)BU \tag{4.3}$$

for every tangent vector fields $U, V, W \in \chi(L)$, where $(.)^T$ denotes the tangential components of a vector field in $\chi(L)$ along L^m. Here B stands for second fundamental form (or shape operator) of L^m in M^{m+1} with respect to a fixed orientation related to the second fundamental form h by

$$g(h(U, V), \alpha) = g(B_\alpha U, Y), \tag{4.4}$$

where α is a normal vector field on L^m.

$\bar{R}$ and R denotes the curvature tensors of $M^{m+1}(c)$ and L^m, respectively. In particular, the scalar curvature ρ of the submanifold L^m satisfies

$$\rho = \sum_{i,j}^{m} g(\bar{R}(E_i, E_j)E_j, E_i) + m^2 \|H\|^2 - \|B\|^2, \tag{4.5}$$

where $\{E_1, ...E_m\}$ is an orthonormal frame on TM and $|.|$ denotes the Hilbert-Schmidt norm. When $M^{2m+1}(c)$ is a Sasakian space form of constant sectional curvature c, we have the identity

$$\tau = \frac{m(m-1)(c+3)}{4} + m^2 \|H\|^2 - \|B\|^2. \tag{4.6}$$

Associated to second fundamental form B of the submanifold L^m there are m algebraic invariants, which are the elementary symmetric functions ρ_r of its principal curvatures $k_1, ...k_m$, given by

$$\rho_0 = 1, \quad \rho_r = \sum_{i_1<...<i_r} k_1, ...k_m. \tag{4.7}$$

The r-th mean curvature H_r of the immersion is define by $\binom{m}{r}H_r = \rho_r$.
If $r = 0$, we have $H_1 = \frac{1}{m}Tr(A) = H$ the mean curvature of L^m.

For each $0 \leq r \leq m$, we defines the Newton transformation $P_r : \chi(L) \longrightarrow \chi(L)$ of the submanifold M^m be setting $P_0 = I$ (the identity operator) and for $0 \leq r \leq m$, by the recurrence relation

$$P_r = \sum_{j=0}^{r} (-1)^{r-j} \binom{m}{j} H_j A^{r-j}, \tag{4.8}$$

where B^j denotes the composition of B with itself, j times ($B^0 = I$). Let us recall that associated to each Newton transformation P_r one has the second order linear differential operator $\mathcal{L}_r : C^\infty(L) \longrightarrow C^\infty(L)$ defined by

$$\mathcal{L}_r u = Tr(P_r \circ Hessu). \tag{4.9}$$

When $r = 0$, we note that $\mathcal{L}_0$ is just the Laplacian operator. Moreover, it is not difficult to see that

$$\begin{aligned} div_M(P_r \nabla u) &= \sum_{i=1}^{m} g(\nabla_{E_i} P_r)\nabla_u, E_i) + \sum_{i=1}^{m} g(P_r(\nabla_{E_i} \nabla_u), E_i) \\ &= g(div_M P_r, \nabla_u) + \mathcal{L}_r u, \end{aligned} \tag{4.10}$$

where the divergence of P_r on L^m is given by

$$div_M P_r = Tr(\nabla P_r) = \sum_{i=1}^{m} (\nabla_{E_i} P_r) E_i. \tag{4.11}$$

In particular, when the ambient space has constant sectional curvature equation (4.10) reduces to

$$\mathcal{L}_r u = div_M(P_r \nabla u), \tag{4.12}$$

because $div_M P_r = 0$ (see [15] for more details).

Our aim, it also will be appropriate to deal with the so called traceless second fundamental form of the submanifold, which is is given by

$$\Phi = BHI, \qquad Tr(\Phi) = 0. \tag{4.13}$$

and

$$\|\Phi\|^2 = Tr(\Phi^2) = \|B\|^2 - m\,\|H\|^2 \geq 0. \tag{4.14}$$

with equality if and only if M^n is totally umbilical.

In order to establish our results let us mention the following maximum principle due to Caminha et al. for more details see [10]. We follows that, for each $p \geq 1$ use the notation

$$\mathcal{L}^p(L) = \left\{ u : L^m \longrightarrow \mathbb{R}; \int_L |u|^p \, dL < +\infty \right\}. \tag{4.15}$$

Also, we have the following lemma:

Lemma 4.1. *Let U be a smooth vector field on the n-dimensional, complete, non compact, oriented Riemannian manifold M^n, such that $div_M U$ does not change sign on M^n. If $|X| \in \mathcal{L}^1(M)$, then $div_M U = 0$.*

The following results further generalized Theorem 1.2 in [2].

Theorem 4.2. *If the data (g, ψ, λ, r) be complete r-almost Newton-Yamabe soliton on Legendrian submanifold L^m in Sasakian space from $M^{2m+1}(c)$ of constant sectional curvature c, with bounded second fundamental form and potential function $\psi : L^m \longrightarrow \mathbb{R}$ such that $|\nabla \psi| \in \mathcal{L}^1(L)$. Then we have*

1. If $(c+3) \leq 0$, $\lambda > 0$ and then L^m can not be L-minimal,
2. If $(c+3) < 0$, $\lambda \geq 0$ and then L^m can not be L-minimal.
3. If $c = -3$, $\lambda \geq 0$ and L^m is L-minimal, then L^m is isometric to the $\mathbb{R}^m$.

Proof. We know that the ambient space has constant sectional curvature, by equation (4.12) the operator $\mathcal{L}_r$ is a divergent type operator. On the other side, since L^m has bounded second fundamental form it follows from (4.8) that the Newton transformation P_r has bounded norm. In particular,

$$|P_r \nabla \psi| \leq |P_r| \, |\nabla \psi| \in \mathcal{L}^1(L), \tag{4.16}$$

Using (1) and (2), let us consider by contradiction that L^m is minimal. Then, equation (4.6) jointly with the considering $(c+3) \leq 0$ $(c+3 < 0)$ imply that the scalar curvature of L^m satisfies $\rho \leq 0 (\rho < 0)$. Hence, contracting (4.1) we have $\mathcal{L}_r \psi = m(\lambda - \rho) > 0$ in both case, which contradicts Lemma (4.1), since the fact after mentioned. This completes the proof of the first two assertions.

For the (3) assertion, since the ambient space has constant sectional curvature $c = -3$ and L^m is minimal, then the equation (4.6) becomes as

$$\rho = - \|B\|^2 \leq 0. \tag{4.17}$$

So, since $\lambda \geq 0$ we have that $\mathcal{L}_r(\psi) = m(\lambda - \rho) \geq 0$. Now, using the fact that $\mathcal{L}_r u = div_M(P_r \nabla u)$ and $|P_r \nabla \psi| \in \mathcal{L}^1(L)$, we have again from Lemma (4.1) that $\mathcal{L}_r \psi = 0$ on L^m. Hence, we conclude that $0 \geq m\rho = m\lambda \geq 0$, that is, $\rho = \lambda = 0$. This implies that $\|B\|^2 = 0$. Therefore, the r-almost Newton-Yamabe soliton L^m must be geodesic and flat.

$\square$

In order to prove our next theorems we will need the following lemmas, which corresponds to Theorem 3 [2].

Lemma 4.3. *Let u be a non-negative smooth subharmonic function on a complete Riemannian manifold M^n. If $u \in \mathcal{L}^p(M)$, for some $p > 1$, the u is constant.*

Further, we are in condition to establish the following result, which holds when the ambient space is an arbitrary Riemannian manifold.

Theorem 4.4. *Let the data (g, ψ, λ, r) be complete r-almost Newton-Yamabe soliton on Legendrian submanifold L^m in a Sasakian space form $M^{2m+1}(c)$ of sectional curvature K, such that P_r is bounded from above (in the sense of quadratic forms) and its potential function $\psi : L^m \longrightarrow \mathbb{R}$ is non-negative and $\psi \in \mathcal{L}^p(L)$ for some $p > 1$. Then we have*

1. *If $K \leq -3$, $\lambda > 0$ then L^m can not be L-minimal,*
2. *If $K < -3$, $\lambda \geq 0$ then L^m can not be L-minimal,*
3. *If $K \leq -3$, $\lambda \geq 0$ and L^m is L-minimal, then L^m is flat and totally geodesic.*

Proof. For proving (1), we begin with a contradiction that L^m is minimal our assumption on the sectional curvature of the ambient space and equation (4.5) imply that $\tau \leq 0$. Hence, contracting equation (4.1) we have

$$\mathcal{L}_r\psi = m(\rho - \lambda) > 0. \tag{4.18}$$

Thus, since we are considering that P_r is bounded from above, there exists a positive constant ω such that

$$\omega\Delta\psi \geq \mathcal{L}_r\psi > 0. \tag{4.19}$$

In particular, from Lemma (4.3) we get that ψ must be constant, which gives a contradiction. Finally, reasoning as in the proof of Theorem (4.2) we can easily obtain (2) and (3).

In our next results we generalized Theorem 1.5 of [2] for the case when $U = \nabla\psi$, giving conditions for an r- almost Newton-Yamabe soliton on Legendrian submanifold in Sasakian space form to be totally umbilical since it has bounded second fundamental form. Therefore, we prove the following theorem:

Theorem 4.5. *If the data (g, ψ, λ, r) be complete r-almost Newton-Yamabe soliton on Legendrian submanifold L^m in Sasakian space $M^{2m+1}(c)$ of constant sectional curvature c, with bounded second fundamental form and potential function $\psi : L^m \longrightarrow \mathbb{R}$ such that $|\nabla\psi| \in \mathcal{L}^1(L)$. Then we have*

1. *$\lambda \geq \frac{(m-1)(c+3)}{4} + mH^2$, then L^m is totally geodesic, with $\lambda = \frac{(m-1)(c+3)}{4}$, and scalar curvature $\rho = m\frac{(m-1)(c+3)}{4}$,*

2. *If L^m is compact and $\lambda \geq \frac{(m-1)(c+3)}{4} + mH^2$, then L^m is isometric to a Euclidean sphere,*

3. *If $\lambda \geq \frac{(m-1)(c+3+H^2)}{4}$, then L^m is totally umbilical. In particular, the scalar curvature $\tau = m\frac{(m-1)(c+3)}{4}K_L$ is constant, where $K_L = \frac{4\lambda}{(n-1)}$ is the sectional curvature of L^m.*

Proof. To prove (1), using the equations (4.1) and (4.6), we obtain

$$\mathcal{L}_r\psi = m[\lambda + \frac{(m-1)(c+3)}{4} - m\,\|H\|^2] + \|B\|^2\,. \tag{4.20}$$

□

Then, for our consideration on λ, we get that $\mathcal{L}_r\psi$ is non-negative function on L^m. By Lemma (4.1) we find that $\mathcal{L}_r\psi$ vanishes identically. Hence, from equation (4.20) we arrive at that L^m is totally geodesic and $\lambda = \frac{(m-1)(c+3)}{4}$. Moreover, it is clear form (4.6) that $\rho = \frac{m(m-1)(m+3)}{4}$, which complete the proof of (1).

If L^m is compact, as it is totally geodesic, then the ambient space must be necessarily a sphere $\mathbb{S}^{2m+1}$ and M^m is isometric to the Euclidean sphere $\mathbb{S}^m$, proving (2).

For the assertion (3), we start with equation (4.20) that can be written in terms of the traceless second fundamental form Φ as

$$\mathcal{L}_r\psi = m[\lambda - \frac{(m-1)(c+3+H^2)}{4}] + \|\Phi\|^2\,. \tag{4.21}$$

Therefore, our assumption on λ gives $\mathcal{L}_r\psi \geq 0$. Then by applying Lemma (4.1) once again we have $\mathcal{L}_r\psi = 0$. This implies that $|\Phi|^2$, that is, L^m is a totally umbilical. In particular κ of L^m is constant and L^m has constant sectional curvature given by $K_M = \frac{c+3+\kappa^2}{4}$. This combined with (4.21) , we obtain that

$$\lambda = \frac{(m-1)(c+3+H^2)}{4} = \frac{(m-1)(c+3+\kappa^2)}{4} \tag{4.22}$$

$$= (m-1)K_L,$$

which implies that $\rho = m(m-1)K_L$, as desired.

□

Now, we have the following consequence of the Theorem (4.5):

Theorem 4.6. *Let the data (g, ψ, λ, r) be complete r-almost Newton-Yamabe soliton on Legendrian submanifold L^m in Sasakian space form $M^{2m+1}(c)$ with constant sectional curvature c. If $\lambda = (m-1)H^2$, then L^m is isometric to $\mathbb{S}^m$.*

From Theorem 1.6 of [2] which states that a nontrivial almost Yamabe soliton L^m, minimally immersed in $\mathbb{S}^{n+1}$ with $\rho \geq m(m-2)$ and such that the nor of the second fundamental form obtain its maximum, must be isometric to $\mathbb{S}^n$. Now, applying Theorem (4.5) we obtain an generalization of this results.

Theorem 4.7. *Let the data (g, ψ, λ, r) be complete r-almost Newton-Yamabe soliton on Legendrian submanifold L^m in Sasakian space form $M^{2m+1}(c)$ with constant sectional curvature c. Consider that $\rho \geq n(n-2)$, the norm of the second fundamental form attains its maximum and $\lambda \geq \lambda = (m-1)$. Then, L^m is isometric to $\mathbb{S}^n$.*

Proof. Since the immersions is minimal with $\rho \geq m(m-2)$, from (4.6) we arrive at

$$\|B\|^2 = m(m-1) - \rho \leq n.$$

From Simons's formula [22], we obtain

$$\Delta \|B\|^2 = \|\nabla B\|^2 + (n - \|B\|^2) \|B\|^2 \geq 0. \tag{4.23}$$

Thus, we can apply Hopf's strong maximum principle to get that $\nabla B = 0$ on M^m. Therefore, Proposition 1 of citeref17 assures that L^m must be compact and, hence, the results from Theorem (4.5). □

Another application of Theorem (4.4), we can also obtain the following theorem:

Theorem 4.8. *Let the data (g, ψ, λ, r) be complete r-almost Newton-Yamabe soliton on Legendrian submanifold L^m in Sasakian space form $M^{2m+1}(c)$ with constant sectional curvature c, such that P_r is bounded from above and its potential function $\psi : L^m \longrightarrow \mathbb{R}$ is non-negative and $\psi \in \mathcal{L}^p(L)$ for some $p > 1$. Then we have*

1. *$\lambda \geq \frac{(m-1)(c+3)}{4} + mH^2$, then L^m is totally geodesic, with $\lambda = \frac{(m-1)(c+3)}{4}$, and scalar curvature $\rho = \frac{m(m-1)(c+3)}{4}$.*

2. *If $\lambda \geq \frac{(m-1)(c+3+H^2)}{4}$, then L^m is totally umbilical. In particular, the scalar curvature $\rho = \frac{m(m-1)(K_M+3)}{4}$ is constant, where $K_M = \frac{4\lambda}{(m-1)} - 3$ is the sectional curvature of L^m.*

Proof. Let us begin observing that by equation (4.20) and assumption on λ we get

$$\mathcal{L}_r\psi = m[\lambda - \frac{(m-1)(c+3)}{4} - m\|H\|^2] + \|B\|^2 \geq 0. \tag{4.24}$$

Since we are assuming that P_r is bounded from above, there is a positive constant ω such that

$$\omega\Delta\psi \geq \mathcal{L}_r\psi \geq 0. \tag{4.25}$$

Using Lemma (4.3), we have that ψ must constant. Therefore $\mathcal{L}_r\psi = 0$, and equation (4.24) we conclude that L^m is totally geodesic, $\lambda = \frac{(m-1)(c+3)}{4}$ and $\rho = \frac{m(m-1)(c+3)}{4}$, proving assertion (1), reasoning as in Theorem (4.5), it is easy to prove assertion (2). □

5 Some examples

Example 5.1. For the case of minimal Legendrian submanifolds in $\mathbb{S}^{2m+1}(1)$. Let us consider the standard immersion of $\mathbb{L}^m$ in $\mathbb{S}^{2m+1}(1)$, which we know that its is totally geodesic. In particular, $P_r = 0$ for all $1 \leq r \leq m$, and choosing $\lambda = \frac{(m-1)}{m}$, we obtain that the immersion satisfies equation (4.1).

Example 5.2. Let $\mathbb{S}^{2m+1}(1)$ be the unit sphere in the Euclidean space $\mathbb{R}^{m+1}$ and $\psi : \mathbb{S}^{2m+1}(1) \hookrightarrow \mathbb{R}^{m+1}$ the natural embedding with induced metric g on $\mathbb{S}^{2m+1}(1)$, then $(\mathbb{S}^{2m+1}(1), \varphi, \xi, \eta, g)$ is a contact metric manifold.It is well known that this contact metric structure gives a Sasakian structure on $\mathbb{S}^{2m+1}(1)$ and its a Sasakian space form with constant φ-sectional curvature $c = 1$.

Let $i : L^m \longrightarrow \mathbb{S}^{2m+1}(1) \subset \mathbb{R}^{m+1}$ be an immersion of a smooth m-dimensional manifold L^m in to unit sphere.
pasn For a constant $t \in \mathbb{R}^{2m+2}$, according to [3], by choosing the functions $\bar{f}_t$ on $\mathbb{R}^{2m+2}$ such that

$$\bar{f}_l(t) = -g(t, l) + 2m - 1$$

and

$$\psi_l(t) = -\bar{f}_l + c, \quad \bar{f}_l := i * \tilde{f}_l \in C^\infty(S^{2m+1}),$$

where $l \in \mathbb{S}^{2m+1}(1)$, $t \neq 0$, $c \in \mathbb{R}^{2m+2}$ and $t = (t_1, ...t_{2m+1}) \in \mathbb{S}^{2m+1}$ is the position vector, we have that $(\mathbb{S}^{2m+1}, g, \nabla\psi_l, \lambda_l)$ satisfies

$$Hess\psi_l = (\rho - \lambda_l)g. \tag{5.1}$$

On the other hand, it is well know that $\mathbb{S}^{2m+1}$ is totally umbilical with r-th mean curvature $H_r = 1$ and second fundamental form $B = I$. In particular, for every $0 \leq r \leq m$ the Newton tensor are given by

$$P_r = \alpha I, \tag{5.2}$$

where $\alpha = \sum_{j=0}^{r}(-1)^{r-j}\binom{m}{j}$. Hence, taking smooth function $\psi = \alpha^{-1}\psi_l$ we get that subamnifold satisfied equation (4.3).

Example 5.3. We recall the *Gaussian soliton* is the Euclidean space $\mathbb{R}^m$ endowed with its standard metric $|.|$ admits the standard Sasakian structure and the potential function $\psi(x) = \frac{\lambda}{4}|x|^2$. It is well know that the horospheres of the hyperbolic space $\mathbb{H}^{m+1}$ are totally umbilical hypersurface isometric to $\mathbb{R}^m$, having r-th mean curvature $H_r = 1$ and second fundamental form $B = I$. Hence, we can reason as in example (5.2) to verify that the horospheres $\mathbb{R}^m \hookrightarrow \mathbb{H}^{m+1}$ satisfies equation (4.3).

Example 5.4. The odd dimensional Euclidean space admits the standard Sasakian structure , we denote by $\mathbb{R}^{2m+1}(-3)$.In general, an immersion into $\mathbb{R}^{2n+1}(-3)$ which lies in some cylinders and minimal in the cylinder.

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