Isomorphic-dilations of the skew-fields constructed over parallel lines in the Desargues affine plane

O. Zaka and J. F. Peters

Abstract. This paper considers dilations and translations of lines in the Desargues affine plane. A *dilation* of a line transforms each line into a parallel line whose length is a multiple of the length of the original line. In addition to the usual Playfair axiom for parallel lines in an affine plane, further conditions are given for distinct lines to be parallel in the Desargues affine plane. This paper introduces the dilation of parallel lines in a finite Desargues affine plane that is a bijection of the lines. Two main results are given in this paper, namely, each dilation in a finite Desarguesian plane is an isomorphism between skew fields constructed over isomorphic lines and each dilation in a finite Desarguesian plane occurs in a Pappian space.

1 Introduction

The foundations for the study of the connections between axiomatic geometry and algebraic structures were set forth by Hilbert [3], recently elaborated and extended in terms of the algebra of affine planes in, for example, [7], [5, §IX.3, p. 574](affine transformation of the plane transforms a lattice into a lattice) and in the Desargues affine plane (see, *e.g.*, [4, 21, 11, 9, 20, 6, 19]). Properties of geometric structures such as adjacency and proximity [2, §III.XVIII, 193] [17, 18, 15, 16] are preserved by affine transformations [1, §III.XVII, 130ff]. This paper focuses on the algebra arising from dilatations and translations entirely in the Desargues affine plane.

1.1 Contributions of this paper

In this article we show a geometric representation in the Desargues affine plane, of the isomorphism between the two whatever skew-fields (the case of the automorphisms in a skew-field we have shown in [4]). Earlier, we have shown that on each Desargues affine plane line, we can construct a skew-field simply and constructively, using simple elements of elementary geometry, and only the basic axioms of Desargues affine plane.

In this paper, we consider dilatations and translations entirely in the Desargues affine plane. We highlight the case when for a fixed point of dilatations it is not in any of the lines on which we have constructed the skew-field (the case when the fixed point is the point of the line where we constructed the skew-field is exhausted in the paper [4]).

We will prove that, "Each dilatation in a Desargues affine plane, is an isomorphism between skew-fields constructed over isomorphic lines of that plane".

In previous work [21], we have shown how we can transform a line in the Desargues affine plane into an additive Group of its points. We have also shown [11] how to construct a skew-field with a set of points on a line in the Desargues affine plane. In addition, for a line of in any Desargues affine plane, we construct a skew-field with the points of this line, by appropriately defined addition and multiplication of points in a line.

1.2 How this paper differs from what has been done previously on affine Desarguesian planes

We do not know that such a discussion was made earlier, in the Desargues affine plane. There are many works in the Desargues affine plane, which study dilations and translations. Also there are many articles that build a skew-field or a field (commutative skew-field) in the Desargues affine plane or in Pappian affine plane, in one way or another, use coordinate to achieve that. The novelty in this article is that, to achieve our results, we do not use coordinates, and follow simple techniques, of elementary geometry, for proofs.

The main result in this paper is that each dilation is an isomorphism between skew fields constructed over isomorphic lines (see Theorem 3.4)..
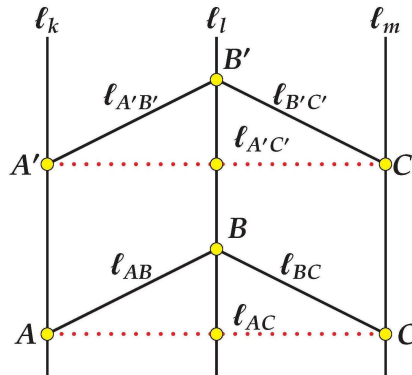

Figure 1: Parallel lines in Desargues Axiom: $\boldsymbol{\ell_{AC}} \parallel \boldsymbol{\ell_{A'C'}}$

2 Preliminaties

Let $\mathcal{P}$ be a nonempty space and let 2^{π} be a collection of subsets of $\mathcal{P}$ (denoted by $\mathcal{L}$). The elements p of $\mathcal{P}$ are points and an element ℓ of $\mathcal{L}$ is a line. We write $P \in \mathcal{P}$ and

$P \in \ell$, interchangeably. The geometric structure $(\mathcal{P}, \mathcal{L})$ is an **affine plane**, provided

1⁰ For each $\{P, Q\} \in \mathcal{P}$, there is exactly one $\ell \in \mathcal{L}$ such that $\{P, Q\} \in \ell$.

2⁰ For each $P \in \mathcal{P}, \ell \in \mathcal{L}, P \notin \ell$, there is exactly one $\ell' \in \mathcal{L}$ such that $P \in \ell'$ and $\ell \cap \ell' = \varnothing$ (Playfair Parallel Axiom [8]). Put another way, if $P \notin \ell$, then there is a unique line ℓ' on P missing ℓ [10].

3⁰ There is a 3-subset $\{P, Q, R\} \in \mathcal{P}$, which is not a subset of any ℓ in the plane. Put another way, there exist three non-collinear points $\mathcal{P}$ [10].

Desargues' Axiom [7, §3.9, pp. 60-61] [22]. Let $A, B, C, A', B', C' \in \mathcal{P}$ and let pairwise distinct lines $\ell_k, \ell_l, \ell_m, \ell_{AC}, \ell_{A'C'} \in \mathcal{L}$ such that

$$\ell_k \parallel \ell_l \parallel \ell_m \text{ and } \ell_A \parallel \ell_{A'} \text{ and } \ell_C \parallel \ell_{C'}.$$
$$A, B \in \ell_{AB}, A'B' \in \ell_{A'B'}, \text{ and } B, C \in \ell_{BC}, B'C' \in \ell_{B'C'}.$$
$$A \neq C, A' \neq C', \text{ and } \ell_{AB} \neq \ell_l, \ell_{BC} \neq \ell_l.$$

Then $\boldsymbol{\ell_{AC} \parallel \ell_{A'C'}}$. ∎

Example 2.1. The parallel lines $\ell_{AC}, \ell_{A'C'} \in \mathcal{L}$ in Desargues' Axiom are represented in Fig. 1. In other words, the base of $\triangle ABC$ is parallel with the base of $\triangle A'B'C'$, provided the restrictions on the points and lines in Desargues' Axiom are satisfied. ∎

A **Desargues affine plane** is an affine plane that satisfies Desargues' Axiom.

An **affine space** $\mathcal{P}$ is a nonempty set endowed with structure by the prescription of a real vector space $\vec{\mathcal{P}}$ together with an injective and transitive action of the additive group of $\vec{\mathcal{P}}$ on $\mathcal{P}$. The vector space $\vec{\mathcal{P}}$ is the translation space of $\mathcal{P}$ [23].

Let $\mathcal{P}, \mathcal{P}'$ be a pair of affine spaces, with associated vector spaces $\vec{\mathcal{P}}, \vec{\mathcal{P}}'$. A mapping $f : \mathcal{P} \longrightarrow \mathcal{P}'$ is affine [12], provided there is a linear mapping $\phi : \vec{\mathcal{P}} \longrightarrow \vec{\mathcal{P}}'$ such that

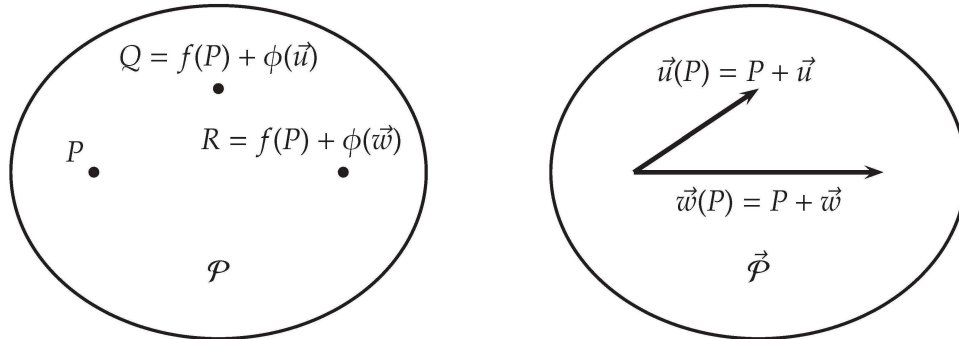$$f(P + \vec{l}) = f(P) + \phi(\vec{v}) \text{ for all } P \in \mathcal{P}, \vec{v} \in \vec{\mathcal{P}}.$$

Figure 2: Affine plane $\mathcal{P} \longmapsto$ (associated vector space) $\vec{\mathcal{P}}$

Example 2.2. Let $P \in \mathcal{P}, \vec{u} \in \vec{\mathcal{P}}$, then $\vec{u}(P) = P + \vec{v}$. A sample affine mapping of $\mathcal{P}$ to $\vec{\mathcal{P}}$ with respect to the point P is shown in Fig. 2. ∎

A space $(\mathcal{P}, \mathcal{L})$ is Pappian, provided

Theorem 2.1. [**Pappus**].
Let $P_1, P_2, P_3, P_4, P_5, P_6 \in \mathcal{P}$ and $|\ell_{P_1 P_3} \cap \ell_{P_2 P_3}| = 1$, then $\ell_{P_3 Q_4} \parallel \ell_{P_1 P_6}$

Every Desarguesian affine plane is isomorphic to a coordinate plane over a field [13] and every finite field is commutative [14, §3, p. 351] . From this, we obtain

Theorem 2.2. [**Tecklenburg**] [13].
Every finite Desarguesian affine plane is Pappian.

3 Main result. Dilations of lines in a finite Desarguesian plane as isomorphisms of skew-fields constructed over the lines

In this section, we will give the main result in this article.

Proposition 3.1. *[9] For every two parallel lines $\ell_1, \ell_2 \in \mathcal{L}$ in Desargues affine plane, exist a dilatation in this plane, $\delta : \ell_1 \longrightarrow \ell_2$. This dilatation δ is a bijection of lines ℓ_1 and ℓ_2.*

Lemma 3.2. *Each dilation, with a fixed point, in a finite Desargues affine plane $\mathcal{A}_\mathcal{D} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an isomorphism's between skew-fields constructed over isomorphic lines $\ell_1, \ell_2 \in \mathcal{L}$ of that plane.*

Proof. Consider a dilatation δ with a fixed point V in a finite Desargues affine plane. The case when this dilatation leads us to a line in itself is covered in [4], where the fixed point for dilatation is on the lines.

Consider a dilation $\delta : \ell_1 \longrightarrow \ell_2$ and a fixed point V related to dilatation δ of the Desargues affine plane such that $V \notin \ell_1$ and $V \notin \ell_2$. From properties of dilatations, we know that $\delta(\ell_1) = \ell_2 \parallel \ell_1$. We mark $\mathbf{K}_1 = (\ell_1, +, *)$ the skew-field constructed over the line ℓ_1 and $\mathbf{K}_2 = (\ell_2, +, *)$ the skew-field constructed over the line ℓ_2.

From Prop. 3.1, we have that $\delta : \ell_1 \longrightarrow \ell_2$ is a bijection. Now we see dilation δ as a function $\delta : \mathbf{K}_1 = (\ell_1, +, *) \longrightarrow \mathbf{K}_2 = (\ell_2, +, *)$ namely, as a function $\delta : \mathbf{K}_1 \longrightarrow \mathbf{K}_2$. Let's show that δ is an isomorphism of these two skew-fields. The dilation δ is a bijection of lines ℓ_1 and ℓ_2. Let us prove that δ, is a homomorphism between the skew-fields $\mathbf{K}_1$ and $\mathbf{K}_2$, so we have to show that:

$$\forall A, C \in \ell_1, \quad \begin{array}{l} \mathbf{1}. \delta(A + C) = \delta(A) + \delta(C). \\ \mathbf{2}. \delta(A * C) = \delta(A) * \delta(C). \end{array}$$

Case.1.. For points $A, C \in \ell_1$, have $\delta(A), \delta(C) \in \ell_2$. We show firstly that:

$$\delta(A + C) = \delta(A) + \delta(C).$$

During the construction of the skew-field over a line of Desargues affine plane, we choose two points (each affine plan has at least two points), which we write with $\mathbf{O}$ and $\mathbf{I}$ and call them **zero** and **one** respectively, these points play the role of **unitary**

elements regarding the two actions addition and multiplication respectively. In our case, the role of zero is the fixed point O. From the
addition algorithm (Algorithm 1 in [21] and [11]) we have:

$$\forall A, C \in \ell_1, \left[\begin{array}{l} \textbf{Step.1.} \exists B \notin OI \\\\ \textbf{Step.2.} \ell_{OI}^B \cap \ell_{OB}^A = D \\\\ \textbf{Step.3.} \ell_{CB}^D \cap OI = E \end{array} \right] \iff A + C = E.$$

From the construction of the $A + C$ point, we have that:

$$\ell^{OB} \parallel \ell^{AD}; \ell^{BC} \parallel \ell^{D(A+C)},$$

and

$$O, A, C, A + C \in \ell_1; B, D \in \ell_{AC}^B.$$

Hence,

(3.1) $$D(A + C) \parallel BC.$$

We take now the dilatation δ. Mark $\delta(O) = O", \delta(A) = A", \delta(C) = C", \delta(B) = B"$ we also have that $\delta(V) = V$ (see Fig. 3).
By dilation properties (see [9]), we have that:

$$O" \in \ell^{OV}, B" \in \ell^{VB}, C" \in \ell^{VC}, A" \in \ell^{VA},$$

also from the definition of dilation (see [9]), we have that:

$$OB \parallel \delta(O)\delta(B) = O"B"; BC \parallel \delta(B)\delta(C) = B"C".$$

Calculate now $\delta(A) + \delta(C)$ according to the addition algorithm (see [21] and [11]). During addition of points A and C we chose the point $B \notin \ell_1$, here we choose the point $\delta(B) = B" \notin \ell_2$ (we do this for ease of proof, as in [21], we have shown that the choice of auxiliary point B, or in our case $\delta(B) = B"$ is arbitrary), by applying the addition algorithm we have:

$$\forall \delta(A), \delta(C) \in \ell_2, \left[\begin{array}{l} \textbf{Step.1.} \exists B" \notin \ell_2 \\\\ \textbf{Step.2.} \ell_{\ell_2}^{B"} \cap \ell_{O"B"}^{A"} = D" \\\\ \textbf{Step.3.} \ell_{C"B"}^{D"} \cap \ell_2 = E" \end{array} \right] \iff A" + C" = E".$$

From this we have:

$$\left. \begin{array}{r} A" + C" \in \ell_{B"C"}^{D"} = \ell_{D(A+C)}^{D"} \\\\ A" + C" \in \ell_2 \end{array} \right\} \implies A" + C" = \ell_{D(A+C)}^{D"} \cap \ell_2$$

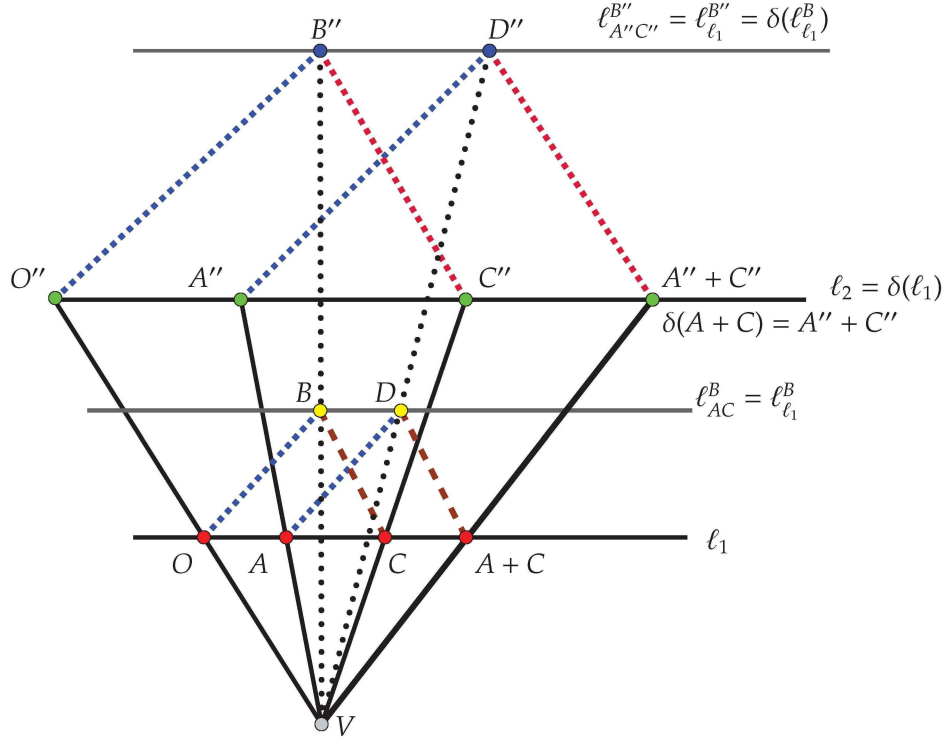$$\implies \delta(A) + \delta(C) = \ell_{D(A+C)}^{D"} \cap \ell_2.$$

Figure 3: The addition of two points in a line of Desargues affine plane under Dilation: $\delta(A + C) = \delta(A) + \delta(C)$.

On the other hand, we have

$$\delta : \ell_1 \to \ell_2 \text{ and}$$

$$\left. \begin{array}{c} D \mapsto D" = \delta(D) \\ A + C \mapsto \delta(A + C) \end{array} \right\} \implies D(A + C) \parallel D"\delta(A + C) = \delta(D)\,\delta(A + C).$$

From this result, have that

$$\left. \begin{array}{c} \delta(A + C) \in \ell_{D(A+C)}^{D"} \\ \delta(D), \delta(A + C) \in \ell_2 \end{array} \right\} \Rightarrow \delta(A + C) = \ell_{D(A+C)}^{D"} \cap \ell_2 = A" + C" = \delta(A) + \delta(C).$$

Hence, we achieve the required result, namely,

$$\delta(A + C) = \delta(A) + \delta(C).$$

This proof can also be made in the three-vertexes language (see [20])

From the above constructions and results, we have a similarity between the two three-vertexes

$$(O, C, B) \sim (O", C", B"),$$

we also have the similarity of

$$(A, A + C, D) \sim (A", A" + C", D").$$

But from the similarity of the three-vertexes in the Desargues affine plane, we also have:

$$(A, A + C, D) \sim (A", \delta(A + C), D")$$

or in other words

$$(A, A + C, D) \sim (\delta(A), \delta(A + C), \delta(D)),$$

from these we have

$$(A", A" + C", D") \sim (A", \delta(A + C), D").$$

Since these two similar three-vertexes have the same two vertices then the third vertice it will be the same, thus

$$A" + C" = \delta(A + C) \Longrightarrow \delta(A + C) = \delta(A) + \delta(C)$$

In parallelograms language, we would say from the above parallels that:

$$(O, A, D, B) \sim (O", A", D", B")$$

and

$$(C, A + C, D, B) \sim (\delta(C), \delta(A + C), \delta(D), \delta(B)) = (C", \delta(A + C), D", B")$$

But on the other hand, from the above parallelisms and from the addition of points $A"$ and $C"$ we would have the similarity of the parallelograms:

$$(C, A + C, D, B) \sim (C", A" + C", D", B"),$$

since the similarity of the n-vertexes in the Desargues affine plane is the relation of equivalence (see [20]) (such would be and for parallelograms), we have:

$$(C", \delta(A + C), D", B") \sim (C", A" + C", D", B").$$

or, equally

$$(\delta(C), \delta(A + C), \delta(D), \delta(B)) \sim (\delta(C), \delta(A) + \delta(C), \delta(D), \delta(B)).$$

Since these two parallel parallelograms have three equal vertices they would be equal, then the fourth vertices would coincide, thus

$$(C", \delta(A + C), D", B") = (C", A" + C", D", B") \Longrightarrow \delta(A + C) = \delta(A) + \delta(C).$$

Case.2. We see now that stands for multiplication properties:

$$\delta(A * C) = \delta(A) * \delta(C)$$

During the multiplication of points A and C, we choose a point I (necessarily different from the point O that we have assigned in the role of the element 'zero'. We do this freely, as each line in affine plan having at least 2-points) in the role of "**one**" (see Fig. 2), the dilatation δ gives us the point **I** in point $\delta(I) = I$", then from point A we construct the line ℓ_{BC}^{E} and thus determine multiplication as well:

(3.2) $$A * C = \ell_{BC}^{E} \cap \ell_1.$$
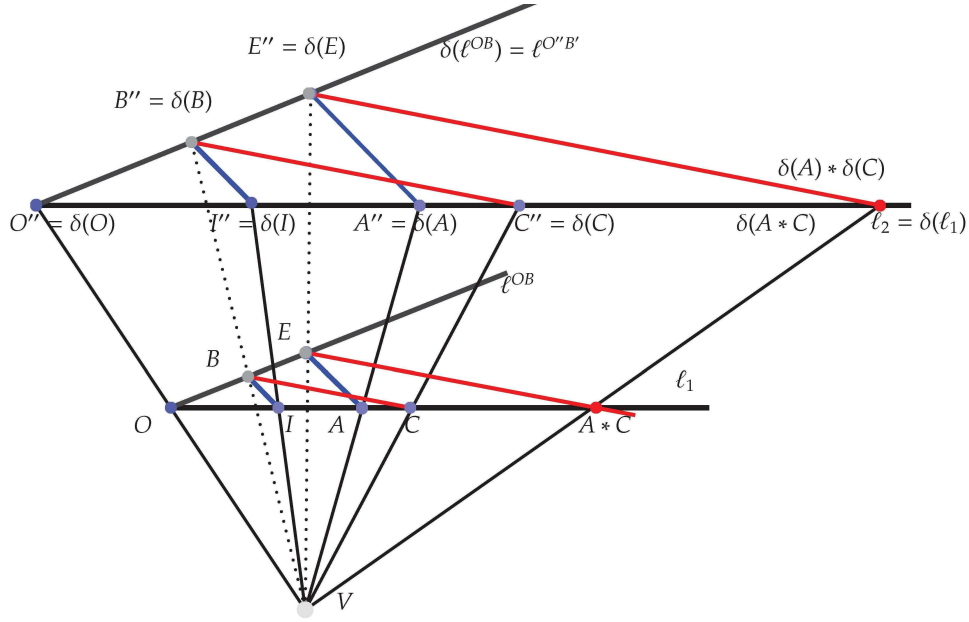

Figure 4: The multiplication of two points in a line of the Desargues affine plane under Dilatation: $\delta(A * C) = \delta(A) * \delta(C)$

By dilation properties have

$$O" \in \ell^{VO}, I" \in \ell^{VI}, C" \in \ell^{VC}, A" \in \ell^{VA}, B" \in \ell^{VB},$$

also for these points we have:

$$O", A", C" \in \ell_2 = \delta(\ell_1) \text{ and } B" \in \delta(OB).$$

Also by definition and dilation properties (see [11]), we have:

$$OB \parallel \delta(O)\,\delta(B) = O"B", BC \parallel \delta(B)\,\delta(C) = B"C", IB \parallel \delta(I)\,\delta(B) = I"B"$$

Calculate now $A" * C"$ according to multiplication Algorithm (see [21],[11]). During the multiplication of points A and C we chose, as auxiliary points, the point $B \notin \ell_1$, here we choose the point $B" = \delta(B) \notin \ell_2$ (we do this for ease of proof, as in [21], we

have shown that the choice of auxiliary point B, or in our case $\delta(B)$ is arbitrary), by applying the multiplication algorithm we have:

$$\forall A" = \delta(A), C" = \delta(C) \in \ell_2, \left\{ \begin{array}{l} \textbf{Step}.1. \exists B" \notin \ell_2 \\[2mm] \textbf{Step}.2. \ell^{A"}_{I"B"} \cap O"B" = E" \\[2mm] \textbf{Step}.3. \ell^{E"}_{B"C"} \cap \ell_2 = F" \end{array} \right\} \iff A" * C" = F".$$

Now multiply by Algorithm 2 of the multiplication definition (see [11]) to obtain

$$A" = \delta(A), C" = \delta(C) \in \ell_2.$$

From point $A"$ construct the line $\ell^{A"}_{I"B"}$ but since $IB \parallel I"B"$, we have that $\ell^{A"}_{I"B"} = \ell^{A"}_{IB}$ and $\ell^{A"}_{I"B"} \cap \ell^{O"B"} = E"$, we have

$$A"E" \parallel I"B" \parallel IB \parallel AE \implies A"E" \parallel AE.$$

Now from point $E"$ construct the line $\ell^{E"}_{B"C"}$, by the definition of multiplication, we have that:

$$E"(A" * C") \parallel B"C",$$

but

$$\left[\begin{array}{l} B"C" \parallel BC \\ BC \parallel E(A * C) \end{array} \right] \implies E"(A" * C") \parallel E(A * C).$$

See now $\delta(E)$, by dilatation properties have:

$$\delta(\ell_1) = \ell_2, \delta(OB) = O"B",$$

we consider two three-vertexes (O, A, E) and $(O", A", E")$, have a similarity of these two three-vertexes, because:

$$\left. \begin{array}{c} AE \parallel A"E", \\ OA \parallel O"A" (\ell_1 \parallel \ell_2), \\ OE \parallel O"E" \\ (E" \in O"B", E \in OB, OB \parallel O"B") \end{array} \right\} \implies (O, A, E) \sim (O", A", E")$$

Mark with $E''' = \delta(E)$, by dilatation properties and from the above parallels, and we have that,

$$AE \parallel A"E''' = A"\delta(E) \Rightarrow \delta(E) \in \ell^{A"}_{AE} = \ell^{A"}_{A"E"} \Rightarrow \delta(E) \in \ell^{A"}_{A"E"}$$

but

$$\delta(E) \in O"B" \Rightarrow \delta(E) = \ell^{A"}_{A"E"} \cap O"B" = E" \Rightarrow E" = E'''$$

Or with three-vertexes similarities

$$OA \parallel O"A", OE \parallel O"E''' \implies (O, A, E) \sim (O", A", E'''),$$

since similarity, in the Desargues affine plane, is a equivalence relation, have that:

$$(O", A", E") \sim (O", A", E''') \implies E" = E''' \implies E" = \delta(E).$$

so, by

$$E"(A" * C") \parallel E(A * C) \Rightarrow E(A * C) \parallel \delta(E)\left[\delta(A) * \delta(C)\right]$$

$$\Rightarrow \delta(A) * \delta(C) = \ell_{\delta(B)\delta(C)}^{\delta(E)} \cap \ell_2 = \ell_{E(A*C)}^{\delta(E)} \cap \ell_2.$$

See now, two three-vertexes $(A, E, A * C)$ and $(A", E", \delta(A * C))$, from above we have similarity

$$(A, E, A * C) \sim (A", E", A" * C").$$

From dilation properties, have that:

$$A" \in \ell^{VA}, E" \in \ell^{VE}, \delta(A * C) \in \ell^{V(A*C)}$$

and

$$AE \parallel A"E", A(A * C) \parallel A"\delta(A * C) \Longrightarrow \delta(A * C) \in \ell_2,$$

$$E(A * C) \parallel E"\delta(A * C) \Longrightarrow \delta(A * C) \in \ell_{E(A*C)}^{E"}$$

Thus,

$$\delta(A * C) = \ell_{E(A*C)}^{E"} \cap \ell_2 = \delta(A) * \delta(C),$$

so we proved that:

$$\delta(A * C) = \delta(A) * \delta(C).$$

$\square$

Lemma 3.3. *Each translation (dilatation which is different from $id_{\mathcal{P}}$ and has no fixed point) in a finite Desargues affine plane $\mathcal{A}_{\mathcal{D}} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an isomorphism's between skew-fields constructed over isomorphic lines $\ell_1, \ell_2 \in \mathcal{L}$, of that plane.*

Proof. Consider a translation φ (different from $id_{\mathcal{P}}$ and there is no fixed points) in the Desargues affine plane.

The case when this translation leads us to a line in itself (the translation has a direction according to line ℓ), we have dealt with [4].

Consider a translation $\varphi : \ell_1 \longrightarrow \ell_2$ in the Desargues affine plane. From properties of translations, know that $\varphi(\ell_1) = \ell_2 \parallel \ell_1$. This translation has a different direction from that of the lines ℓ_1, ℓ_2 which belong to the same equivalence class, according to the parallelism.

We mark $\mathbf{K}_1 = (\ell_1, +, *)$ the skew-field constructed over the line ℓ_1 and $\mathbf{K}_2 = (\ell_2, +, *)$ the skew-field constructed over the line ℓ_2 same way as in Lemma 3.2.

From Prop. 3.1, we have that $\varphi : \ell_1 \longrightarrow \ell_2$ is a bijection. Now we see this translations φ as a function $\varphi : \mathbf{K}_1 = (\ell_1, +, *) \longrightarrow \mathbf{K}_2 = (\ell_2, +, *)$ namely, as a function $\varphi : \mathbf{K}_1 \longrightarrow \mathbf{K}_2$. Next, we show that φ is an isomorphism of these two skew-fields. The translation φ is a bijection of lines ℓ_1 and ℓ_2. Let us prove that φ, is a homomorphism between the skew-fields $\mathbf{K}_1$ and $\mathbf{K}_2$, so we have to show that

$$\mathbf{1}. \varphi(A + C) = \varphi(A) + \varphi(C)$$
$$\forall A, C \in \ell_1,$$
$$\mathbf{2}. \varphi(A * C) = \delta(A) * \varphi(C)$$

Case 1 For points $A, C \in \ell_1$, we have $\varphi(A), \varphi(C) \in \ell_2$. We show first that:

$$\varphi(A + C) = \varphi(A) + \varphi(C).$$

During the construction of the skew-field over a line on the Desargues affine plane, we choose two points (each affine plan has at least two points) which we write with **O** and **I** and call them **zero** and **one** respectively, these points play the role of **unitary elements** regarding the two actions addition and multiplication respectively. In our case, the role of zero is the fixed point O. From the addition algorithm (Algorithm 1 in [21] and [11]) we have:

$$\forall A, C \in \ell_1, \left[\begin{array}{l} \textbf{Step.1.}\exists B \notin OI \\[6pt] \textbf{Step.2.}\ell_{OI}^B \cap \ell_{OB}^A = D \\[6pt] \textbf{Step.3.}\ell_{CB}^D \cap OI = E \end{array}\right] \Longleftrightarrow A + C = E$$

From the construction of the $A + C$ point, we have that:

$$\ell^{OB} \parallel \ell^{AD}; \ell^{BC} \parallel \ell^{D(A+C)};$$

and

$$O, A, C, A + C \in \ell_1; B, D \in \ell_{AC}^B = \ell_{\ell_1}^B,$$

Hence,

$$D(A + C) \parallel BC.$$

We take now the translation φ.

$$\varphi(O) = O', \varphi(A) = A', \varphi(C) = C', \varphi(B) = B'.$$

and we have that

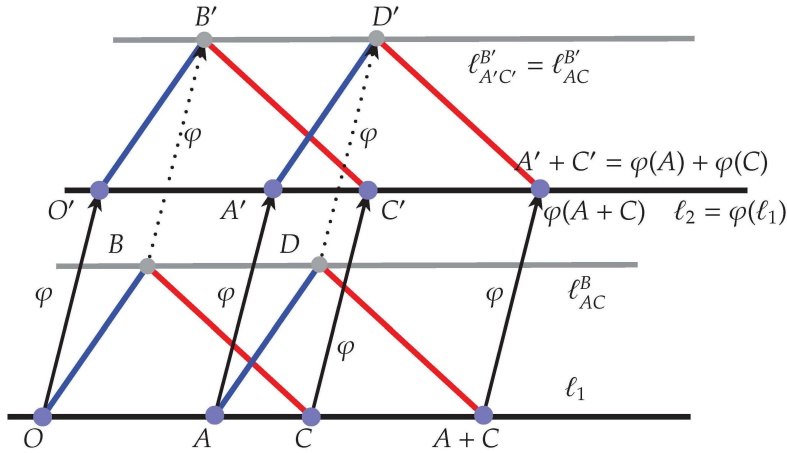$$A' + C' = \varphi(A) + \varphi(C)$$

Figure 5: The addition of two points in a line of Desargues affine plane under Translation: $\varphi(A + C) = \varphi(A) + \varphi(C)$

From the additions of points A and C in line ℓ_1, we have that, the ordered quadruplet (O, A, D, B) and $(C, A + C, D, B)$ are parallelograms (see Fig. 5).

From the translation properties described at [9], we have that

$$OB \parallel \varphi(O)\varphi(B) = O'B', BC \parallel \varphi(B)\varphi(C) = B'C',$$

$$\varphi(O), \varphi(A), \varphi(C), \varphi(A+C) \in \ell_2.$$

For more

$$\varphi(D) \in \ell_{\ell_2}^{B'} = \ell_{\ell_1}^{B'},$$

and have that the following ordered quadruplet are parallelograms:

$$(O, B, B', O'); (O, A, A', O'); (B, C, C', B'); (A, C, C', A');$$

$$(A, D, \varphi(D), A'); (A, A+C, \varphi(A+C), \varphi(D)); (D, A+C, \varphi(A+C), \varphi(D)).$$

Now calculate the addition of points $A' = \varphi(A)$ and $C' = \varphi(C)$. From the addition algorithm (Algorithm 1 in [21] and [11]), we have:

$$\forall A', C' \in \ell_2, \begin{bmatrix} \textbf{Step.1.} \exists B' = \varphi(B) \notin \ell_2 \\[2mm] \textbf{Step.2.} \ell_{\ell_2}^{B'} \cap \ell_{O'B'}^{A'} = D' \\[2mm] \textbf{Step.3.} \ell_{B'C'}^{D'} \cap \ell_2 = E \end{bmatrix} \iff A' + C' = E$$

From the construction of the $A' + C'$ point, we have that:

$$\ell^{O'B'} \parallel \ell^{A'D'}; \ell^{B'C'} \parallel \ell^{D'(A'+C')};$$

and

$$\begin{aligned} D' &\in \ell_{\ell_2}^{B'} = \ell_{\ell_1}^{B'} \quad \text{and} \\ D' &\in \ell_{O'B'}^{A'} = \ell_{OB}^{A'} = \ell_{AD}^{A'} = \ell_{AD}^{\varphi(A)} \\ \Rightarrow D' &= \varphi(D), \end{aligned}$$

and also

$$O', A', C', A' + C' \in \ell_2; B', D' \in \ell_{\ell_2}^{B'} = \ell_{\ell_1}^{B'},$$

therefore

(3.3) $$D'(A' + C') \parallel BC,$$

but $BC \parallel D(A+C)$, then

$$D(A+C) \parallel D'(A'+C').$$

From here have that, the ordered quadruplet, (O', A', D', B') and $(C', A'+C', D', B')$ are parallelograms:

From the Desargues axiom (see [21],[11]) and from the above parallelograms, concretely from the parallelograms:

$$(A, A+C, \varphi(A+C), A') \text{ and } (A, D, D', A'),$$

have that, the ordered quadruplet

$$(D, A + C, \varphi(A + C), D') - \text{is Parallelogram.}$$

From here have that:
$$D(A + C) \parallel D'\varphi(A + C)$$

adding the fact that

$$A' + C' = \varphi(A) + \varphi(C) \in \ell_2$$

and

$$\varphi(A + C) \in \ell_2,$$

we have that

$$\varphi(A + C) = \varphi(A) + \varphi(C).$$

Case2 We see now that stands for multiplication properties:

$$\varphi(A * C) = \varphi(A) * \varphi(C)$$

During the multiplication of points A and C, we choose a point I (necessarily different from the point O that we have assigned in the role of the element 'zero'. We do this freely, as each line in affine plan having at least 2-points) in the role of "**one**" (see Fig. 6), the translation φ gives us the point I in point $\varphi(I) = I'$, according to the multiplication algorithm, for points $A, C \in \ell_1$, we have

$$\forall A, C \in \ell_1, \left\{ \begin{array}{l} \textbf{Step.}1. \exists B \notin \ell_1 \\\\ \textbf{Step.}2. \ell_{IB"}^A \cap OB = D \\\\ \textbf{Step.}3. \ell_{BC}^D \cap \ell_1 = E \end{array} \right\} \iff A * C = E.$$

From this we have parallelisms

$$IB \parallel AD, BC \parallel D(A * C),$$

and $A * C \in \ell_1$ (see Fig. 6).
By translation properties have:

$$O' \in \ell_2, I' \in \ell_2, C' \in \ell_2, A' \in \ell_2, \varphi(A * C) \in \ell_2$$

also for these points we have:

$$OO' \parallel II' \parallel AA' \parallel CC' \parallel (A * C)\varphi(A * C).$$

Calculate now $A' * C'$ according to multiplication Algorithm (see [21],[11]). During the multiplication of points A and C we chose, as auxiliary points, the point $B \notin \ell_1$, here we choose the point $B' = \varphi(B) \notin \ell_2$ (we do this for ease of proof, as in [21], we
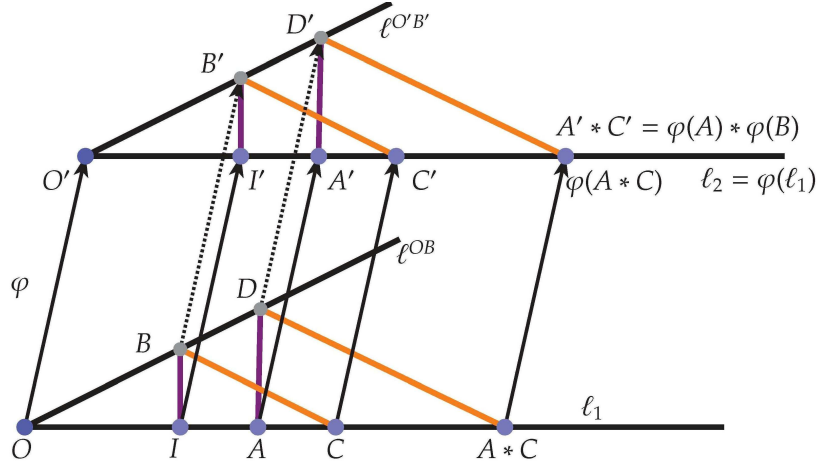
Figure 6: The multiplication of two points in a line on the Desargues affine plane under Translation: $\varphi(A * C) = \varphi(A) * \varphi(C)$

have shown that the choice of auxiliary point B, or in our case $\varphi(B)$ is arbitrary), by applying the multiplication algorithm we have:

$$\forall A' = \varphi(A), C' = \varphi(C) \in \ell_2, \begin{cases} \textbf{Step}.1. \exists B' = \varphi(B) \notin \ell_2 \\ \textbf{Step}.2. \ell_{I'B'}^{A'} \cap O'B' = D' \\ \textbf{Step}.3. \ell_{B'C'}^{D'} \cap \ell_2 = E' \end{cases} \Longleftrightarrow A' * C' = E'.$$

From this we have parallelisms

$$I'B' \parallel A'D', B'C' \parallel D'(A' * C'),$$

but from above, we have that

$$IB \parallel I'B', AD \parallel A'D', BC \parallel B'C',$$

then

$$D' \in \ell_{I'B'}^{A'} = \ell_{IB}^{A'} = \ell_{AD}^{A'},$$

also

$$D' \in O'B' = \varphi(OB) = \ell^{\varphi(O)\varphi(B)}.$$

Then we have the ordered quadruplet

$$(B, D, D', B') - \text{is Parallelogram},$$

but by the translation, we have

$$(B, D, \varphi(D), \varphi(B)) = (B, D, \varphi(D), B') - \text{is a Parallelogram}.$$

So we have two similar parallelogams that have the three same points. Consequently, these two parallelograms will be the same, —em i.e.,

$$(B, D, D', B') = (B, D, \varphi(D), B').$$

Hence,

$$D' = \varphi(D).$$

Thus

$$A' * C' \in \ell_2$$

and

$$A' * C' \in \ell_{B'C'}^{D'} = \ell_{BC}^{D'} = \ell_{D(A*C)}^{\varphi(D)},$$

implies

$$A' * C' = \ell_{D(A*C)}^{\varphi(D)} \cap \ell_2.$$

From the definition and the properties of translation, we also have,

$$A * C \in \ell_1 \Rightarrow \varphi(A * C) \in \ell_2,$$

So, we have that the ordered quadruplet

$$(D, A * C, \varphi(A * C), D') - is \ \ Parallelogram,$$

and we also have (from construction of the points $A' * C'$) the ordered quadruplet

$$(D, A * C, A' * C', D') = (D, A * C, A' * C', \varphi(D)) - is \ \ Parallelogram.$$

and

$$\begin{aligned}
\varphi(A * C) &\in & \ell_{D(A*C)}^{\varphi(D)} \Rightarrow \\
\varphi(A * C) &= & \ell_{D(A*C)}^{\varphi(D)} \cap \ell_2 \\
&= & A' * C' \\
&= & \varphi(A) * \varphi(C).
\end{aligned}$$

Hence, we obtain the desired result, namely, $\varphi(A * C) = \varphi(A) * \varphi(C)$. $\square$

From Lemma 3.2 and Lemma 3.3, we obtain the main result, namely,

Theorem 3.4. *Each dilatation in a finite Desargues affine plane $\mathcal{A}_\mathcal{D} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is an isomorphism's between skew-fields constructed over isomorphic lines $\ell_1, \ell_2 \in \mathcal{L}$, of that plane.*

Theorem 3.5. *Each dilatation in a finite Desargues affine plane $\mathcal{A}_\mathcal{D} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ occurs in a Pappian space.*

Proof. Immediate from Theorem 2.2. $\square$

References

[1] A.D. Aleksandrov, *Non-Euclidean Geometry*, Mathematics. Its Contents, Methods and Meaning, Editor: A.D. Aleksandrov, A.N. Kolmogorov, M.A. Lavrent'ev, Dover Pubs, Inc., Mineola, NY, Trans. from the Russian edition by S.H. Gould, (1999), 97–192.

[2] P.S. Aleksandrov, *Topology*, Mathematics. Its Contents, Methods and Meaning, Editor: A.D. Aleksandrov, A.N. Kolmogorov, M.A. Lavrent'ev, Dover Pubs, Inc., Mineola, NY, Trans. from the Russian edition by S.H. Gould, (1999), 193-226.

[3] D. Hilbert, *The foundations of geometry*, The Open Court Publishing Co., La Salle, Ill., 1959.

[4] Orgest Zaka, *Dilations of line in itself as the automorphism of the skew-field constructed over in the same line in desargues affine plane*, Applied Mathematical Sciences 13, 5, (2019), 231-237.

[5] M. Berger, *Geometry revealed*, Heidelberg, Springer, 2010.

[6] H. S. M. Coxeter, *Introduction to geometry, 2nd ed.*, John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[7] A. Kryftis, *Aconstructive approach to affine and projective planes*, Ph.D. thesis, University of Cambridge, Trinity College and Department of Pure Mathematics and Mathematical Statistics, supervisor: M. Hyland, 2015.

[8] G. Pickert, *Affine Planes: An Example of Research on Geometric Structures*, The Mathematical Gazette, 57, 402, (2004), 278–291.

[9] Orgest Zaka, *A description of collineations-groups of an affine plane*, Libertas Mathematica (N.S.) 37, 2, (2017), 81-96.

[10] M. Prażmowska, *A proof of the projective Desargues axiom in the Desarguesian affine plane*, Demonstratio Mathematica, 37, 4, (2004), 921-924.

[11] K. Filipi, O. Zaka, and A. Jusufi, *The construction of a corp in the set of points in a line of desargues affine plane,* Matematicki Bilten 43, 01, (2019), 27-46.

[12] I.V. Dolgachev and A.P. Shirokov, *Affine space: Encyclopedia of Mathematics*, Editor: M. Hazewinkel, Kluwer, Dordrecht,(1995), 62–63.

[13] H. Tecklenburg, *A proof of the theorem of Pappus in finite Desarguesian affine planes*, Journal of Geometry, 30, (1987), 173–181.

[14] J.H. Maclagan-Wedderburn, *A theorem on finite algebras*, Trans. Amer. Math. Soc., volume 6, no 3, (1905), 349–352.

[15] J.F. Peters, *Computational geometry, topology and physics of digital images with applications. Shape complexes, optical vortex nerves and proximities*, Springer Nature, Cham, Switzerland 2020.

[16] J.F. Peters, *Proximal planar shapes. Correspondence between triangulated shapes and nerve complexes.*, Bull. Allahabad Math. Soc. 33, No. 1, 113-137 (2018).

[17] J.F. Peters, *Two forms of proximal, physical geometry. Axioms, sewing regions together, classes of regions, duality and parallel fibre bundles*, Advan. in Math: Sci. J, 5, 2, (2016), 241–268.

[18] J.F. Peters, *Proximal Vortex Cycles and Vortex Nerve Structures. Non-Concentric, Nesting, Possibly Overlapping Homology Cell Complexes*, Journal of Mathematical Sciences and Modelling, 1, 2, (2018), 56–72.

[19] Orgest Zaka, *Contribution to reports of some algebraic structures with affine plane geometry and applications*, Ph.D. thesis, Polytechnic University of Tirana, Albania, Department of Mathematical Engineering, supervisor: K. Filipi, 2016.

[20] Orgest Zaka, *Three vertex and parallelograms in the affine plane: Similarity and addition abelian groups of similarly n-vertexes in the Desargues affine plane*, Mathematical Modelling and Applications 3, 1, (2018), 9-15.

[21] O. Zaka and K. Filipi, *The transform of a line of Desargues affine plane in an additive group of its points*, Int. J. of Current Research 8, 07, (2016), 34983-34990.

[22] W. Szmielew, *From affine geometry to Euclidean geometry, [An approach through axiomatics]*, (in Polish) Warsaw, Biblioteka Matematyczna [Mathematics Library], 1981.

[23] W. Noll and J.J. Schäffer, *Order-isomorphisms in affine spaces*, Annali di Matematica Pura ed Applicata, 117, 4, (1978), 243-262.

Authors' addresses:

Orgest Zaka
Department of Mathematics-Informatics,
Faculty of Economy and Agribusiness,
Agricultural University of Tirana, Kamza Hill, SH1, Tirana 1000. Albania.
E-mail: ozaka@ubt.edu.al, gertizaka@yahoo.com

James F. Peters
Computational Intelligence Laboratory,
University of Manitoba, WPG, MB, R3T 5V6, Canada

and

Department of Mathematics,
Faculty of Arts and Sciences,
Adiyaman University, 02040 Adiyaman, Turkey.
E-mail: James.Peters3@umanitoba.ca