Adapted linear connections for vector fields

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Abstract. We consider *auto-parallelizable* vector fields (i.e., those vector fields ξ for which there exists a linear connection whose auto-parallel curves are the trajectories of ξ) in order to deal with the geometrization of a vector field on a differentiable manifold. This approach extends our studies [15] and [16], about the geometrization of geodesible vector fields (i.e. auto-parallel vector fields with respect to the Levi-Civita connection of a Riemannian metric).

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1 Introduction

Newtonian Dynamics settled a scientific paradigm which lasted more than 300 hundreds years; the widespread opinion is that it fixes a priori a "geometry" (e.g. the Euclidean one on $\mathbb{R}^3$) and a "force" (e.g. a "gravitational" vector field ξ); one looks for the trajectories of particles, whose "acceleration" (i.e. "covariant derivative") is equal to that "force" (via the Newton's second law). The (*avant la lettre*) geodesics do not appear but in the general statement of the Newton's first law.

The great success of this approach is shaded by some problems of invariance (covariance), its failure in the electromagnetic realm and in the large scale Universe, together with the fact that the solutions do not satisfy (in general) the Geodesic Principle, but only the Fermat Principle.

However, it seems that the simultaneity of the geometric and of the physical hypotheses is a postumous misconception. In Newton's words, only Mechanics comes a priori and Geometry follows, as an a posteriori approach:

"*The description of right lines and circles, upon which geometry is founded, belongs to Mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn.*" ([11])

Unfortunately, in Newton's time, only Euclidean geometry was available as a modelization tool for Mechanics.

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In the Modified Newtonian Dynamics (e.g. [8], [4]), some small changes are made at the level of the Newton's second law, by introducing an interpolating function; the philosophy remains the same, as previously.

Analytical Mechanics replaces the a priori geometrical tools with some a priori (artificial) functions (the Lagrangian or the Hamiltonian), transferring the problem on much bigger spaces.

In the Theory of Relativity, an a priori dynamics is encoded in a (0,2)-tensor field T, and one searches a Lorentzian geometry, ("exact") solution of a complicated system of PDEs (the Einstein's equations). There are no direct connections between the geodesics and an eventual "force" field ξ (encoded in T) and the covariant character of ξ is questioned. The recent debate (cf. [1], [7]. [17]) about the character of the Geodesic Principle (axiom vs. theorem) complicates even more the search for better geometrizations and axiomatizations for the Theory of Relativity.

In a series of papers, we adopted a slightly different viewpoint.In [12]-[16] we give a historical account for and we study the following problem: *given a differentiable manifold M and ξ a (nowhere vanishing) vector field on M, find an adapted Riemannian metric g on M, such that the trajectories of ξ be geodesics of g.* Two main difficulties were pointed out: firstly, there are obstructions to the existence of such adapted metrics; secondly, in case such metrics exist, their analytic (and global) form might be tedious to find. Enlarging the search from Riemannian metrics to semi-Riemannian (indefinite) metrics provides additional difficulties. Several applications were suggested, including the important case when ξ is the Newtonian gravitational vector field.

In this paper, we extend the framework of this generalized "geodesic principle": instead of looking for adapted metrics on M, we look for linear connections ∇ such that ξ be auto-parallel with respect to ∇ (i.e. the trajectories of ξ be auto-parallel curves with respect to ∇). We prove that such connections always exist (§2). The approach extends the geometrization made by E.Cartan in his two ample memoirs [2] and [3], for the Newtonian vector field, and known as the Newton-Cartan theory (see [10],[7] for details and further references).

A second geometrization for a vector field ξ will be provided by all the connections with respect to which ξ is "invariant", that is ξ is an afine collineation. This is a stronger condition and we find obstructions to this property.

On Lie groups, we consider the case when ξ and/or the connections are left invariant (§3). We also include some results for the Newtonian gravitational vector field (§4); more details and properties for this important example will be studied in a forthcoming paper.

2 Adapted connections for vector fields

Let M be an n-dimensional differentiable manifold and ξ a vector field on M. We say that ξ is *auto-parallelizable* if there exists a connection ∇ on M, such that the trajectories of ξ be auto-parallel curves of ∇, i.e.,

$$\nabla_\xi \xi = 0 \tag{2.1}$$

The set of all these connections will be denoted by $C(M,\xi)$ and is a kind of (differential affine) moduli space adapted (associated) to ξ. Denote $C_s(M,\xi)$ and $C_d(M,\xi)$, $C_{d+}(M,\xi)$ the subsets of the symmetric, divergence-free (i.e. the divergence of ξ w.r.t. these connections vanishes), respectively with non-negative divergence adapted connections (i.e. the divergence of ξ w.r.t. these connections is non-negative). For a fixed $\nabla \in C(M,\xi)$, the vector field ξ is called *auto-parallel* with respect to ∇, or ∇-*autoparallel*. We highlight the following problems:

Problem 2.1. *Given a manifold M and a (fixed) linear connection $\nabla \in \mathcal{C}(M)$, find/characterize the vector fields ξ such that ∇ belongs to $C(M,\xi)$, and, eventually, to $C_s(M,\xi)$, $C_d(M,\xi)$, $C_{d+}(M,\xi)$.*

Problem 2.2. *Given a manifold M and a (fixed) vector field ξ, characterize the sets $C(M,\xi)$, $C_s(M,\xi)$, $C_d(M,\xi)$, $C_{d+}(M,\xi)$.*

Problem 2.3. *Given a manifold M, does there exist a (nowhere vanishing) vector field ξ with non-void $\mathcal{C}(M,\xi)$ (and, eventually, non-void $C_s(M,\xi)$, $C_d(M,\xi)$ and/or $C_{d+}(M,\xi)$) ?*

Remark 2.4. (i) The sets $C(M,\xi)$, $C_s(M,\xi)$ and $C_d(M,\xi)$ are differentiable invariants and affine modules. The sets $C(M,\xi)$ and $C_s(M,\xi)$ are closed w.r.t. the operations of transposition and symmetrization.

The set $C_{d+}(M,\xi)$ is not closed w.r.t. transposition and symmetrization. Moreover, it may be void. Its "border" is $C_d(M,\xi)$.

The set $C_{d+}(M,\xi)$ is a differentiable invariant, it is convex but it is not an affine module.

(ii) Denote $C_{LC}(M,\xi)$ the set of Levi-Civita connections of the Riemannian metrics on M, which are in $C_s(M,\xi)$. Unlike $C_s(M,\xi)$ (cf. Prop.2.5.), the set $C_{LC}(M,\xi)$ may be void (see [12]-[16]).

(iii) If ξ is a parallel vector field with respect to some linear connection ∇ on M, then it is also ∇-auto-parallel. For the existence of (complete) parallel vector fields, there exist however strong topological obstructions (see for example [18] and references therein).

(iv) If a vector field ξ on M has singularities, then the set $C_s(M,\xi)$ might be void. For example, consider $\xi = x\partial_x$ in $\mathbb{R}^2$.

As some manifolds do not admit non-singular vector fields (due to topological restrictions), it follows that such manifolds do not have any auto-parallelizable vector field.

(v) Suppose M is a parallelizable manifold and ξ is a nowhere vanishing vector field on M. Then $C_s(M,\xi)$ is nonvoid.

Indeed, denote E_1,..., E_n a parallelization of M, where $E_1 = \xi$. We know there exist three linear connections ∇^-, ∇^+, ∇^0, uniquely defined by $\nabla^-_{E_i}E_j = 0$, $\nabla^+_{E_i}E_j = [E_i,E_j]$, $\nabla^+_{E_i}E_j = \frac{1}{2}[E_i,E_j]$,for every $i,j=\overline{1,n}$. (Here ∇^-, ∇^+, ∇^0 are the Cartan-Schouten connections on M). It follows that each such connection parallelizes ξ. In particular, ∇^0 is symmetric.

Proposition 2.5. *Suppose ξ be a nowhere vanishing vector field on a differentiable manifold M. Then there exists a connection $C_s(M,\xi)\bigcap C_d(M,\xi)$.*

Proof. Locally, a coordinates system $(x^1, ..., x^n)$ may be chosen such that $\xi = \partial_1$. The equation (1) is then, locally, solvable, due to the Remark 2.4.,(iv). The same line of reasoning proves that, locally, there exists an adapted linear connection for ξ, which is symmetric and makes ξ divergence-free. On another hand, $C_s(M, \xi)$ and $C_d(M, \xi)$ are closed w.r.t. affine combinations. An argument using the partition of unity ends the proof. □

Remark 2.6. We study now another kind of moduli spaces, associated to vector fields, via some other kind of invariance.

Let ξ be a (non-null) vector field on M and $\nabla \in \mathcal{C}(M)$. We say ∇ is L_ξ-invariant (or ξ is an affine collineation w.r.t. ∇, or ξ is an affine vector field w.r.t. ∇) if

$$L_\xi \nabla = 0 \tag{2.2}$$

Denote by $\mathcal{C}_2(M, \xi)$ the set of all L_ξ-invariant connections.

In particular, when ∇ is the Levi-Civita connection of a Riemannian metric g on M, and if ξ is a Killing vector field on (M, g), then ξ is also an affine vector field w.r.t. ∇.

Remark 2.7. (i) Obviously, the null vector field would invariate any linear connection. The operator $L_\xi \nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ is a tensor field of type (1,2) and (2) may be also written as

$$[\xi, \nabla_X Y] - \nabla_{[\xi, X]} Y - \nabla_X [\xi, Y] = 0$$

(ii) In general: given a vector field ξ, there does not exist a L_ξ-invariant connection ∇; given a linear connection ∇, there does not exist a (non-null) affine vector field ξ w.r.t. it. So, we have the following natural problems:

Problem 2.8. *Given a manifold M and a linear connection $\nabla \in \mathcal{C}(M)$, characterize the vector fields which invariate ∇.* (This is a classical problem, with many known results concerning the affine collineations of linear connections.)

Problem 2.9. *Given a manifold M and a vector field ξ, characterize the set $\mathcal{C}_2(M, \xi)$.*

Problem 2.10. *Given a manifold M, does there exist a non-trivial vector field ξ with non-void $\mathcal{C}_2(M, \xi)$?*

(Similar questions may be put: for invariant *symmetric* connections; for connections with respect to which ξ has null divergence or non-negative divergence, etc).

Remark 2.11. (i) Fix a vector field ξ. The condition (2.2) is not closed to transposition, symmetrization or affine combinations of connections.

(ii) Let's fix a nowhere vanishing vector field ξ. Locally, a coordinates system $(x^1, ..., x^n)$ may be chosen such that $\xi = \partial_1$. The equation (2.2) (with unknowns the coefficients of the connection) is solvable in this coordinates system. So, the difficulty in finding the L_ξ-invariant connections has global reasons, not local.

(iii) Consider ∇ the canonical connection on $M := \mathbb{R}^2$. The vector field ∂_1 preserves ∇, but $(x^1)^2 \partial_1$ does not.

Definition 2.12. (Generalization) Let ξ be a nowhere vanishing vector field on a differentiable manifold M and k a positive integer. Denote by $\mathcal{C}_3(M,\xi,k)$ the set of all the linear connections, such that

$$(L_\xi)^k(\nabla_\xi \xi) = 0 \tag{2.3}$$

and by $\mathcal{C}_4(M,\xi,k)$ the set of all the linear connections, such that

$$\nabla_\xi^k \xi = 0. \tag{2.4}$$

For $k \geq 2$, the study of these linear connections and extended moduli spaces $\mathcal{C}_3(M,\xi,k)$ and $\mathcal{C}_4(M,\xi,k)$ is beyond the goal of the present paper; they will appear, briefly, only in §4, in a remark concerning the Newtonian gravitational vector field.

3 Adapted invariant connections for invariant vector fields on Lie groups

Let G be a n-dimensional Lie group and ξ a vector field on G. One knows that if ξ is nowhere vanishing, then $C_s(M,\xi) \neq \varnothing$. We have then the following new problems.

Problem 3.1. *Does there exist a left-invariant connection* $\nabla \in C_s(M,\xi)$?

In general, the answer is negative. Take, for example, the Lie group $\mathbb{R}^2$ and $\xi = (x^2+1)\partial_x$ in Cartesian coordinates. Then any connection $\nabla \in C(G,\xi)$ must satisfy $\nabla_{\partial_x}\partial_x = -\frac{2x}{x^2+1}\partial_x$. Thus, ∇ cannot be left-invariant.

Problem 3.2. *If* ξ *is left-invariant, does there exist a left-invariant (or a bi-invariant) connection* $\nabla \in C_s(G,\xi)$?

The answer is affirmative. If ξ is the null vector field, the proof is obvious. Suppose ξ never vanishes. Since G is parallelizable, we may chose a parallelization given by a basis { E_1,..., E_n } of the Lie algebra $L(G)$, where $E_1 = \xi$. Then, Remark 2.4., (v) provides the Cartan-Schouten connection ∇^0 in $C_s(G,\xi)$. Moreover, this connection is bi-invariant.

A converse statement is false: take, for example the vector field $\xi := x\partial_y$, which is not left invariant on $\mathbb{R}^2$, but is auto-parallel with respect to the canonical bi-invariant connection.

Remark 3.3. (i) In [13], [14] and [15] we defined some new invariants associated to Lie groups, via the left invariant pseudo-Riemannian metrics adapted to left invariant vector fields (the geodesic heights, the geodesic "fingerprint"). In the differential affine framework, for left invariant adapted connections, these invariants are redundant, so we have to find new ones, from different arguments.

(ii) The sets of left invariant connections in $C(G,\xi)\bigcap C_d(G,\xi)$ and $C(G,\xi)$ admit structures of real vector spaces, of dimension n^3-n-1 and n^3-n, respectively.

The set of bi-invariant connections in $C(G,\xi)$ admits a structure of real vector space, of dimension at least 1. Its maximal dimension (with respect to all left invariant vector fields) will be denoted by $m_1(G)$. On compact Lie groups, the spaces of bi-invariant connections were classified by Laquer ([5], [6]). Then one may classify the compact Lie groups by using this new invariant m_1, via the following

Theorem 3.4. *Let G be a compact Lie group with $L(G) = \zeta \oplus g_1 \oplus ... \oplus g_q$, where its center ζ has dimension p, and where g_i are simple ideals in $L(G)$; suppose there are exactly r ideals $su(n)$, ($n \geq 3$), among them.*

Then the maximal dimension m_1 of a space of bi-invariant connections in $C(G, \xi)$, for non-null $\xi \in L(G)$, satisfies the inequalities $1 \leq m_1 \leq p^3 + 3pq + q + r$.

Remark 3.5. (i) In order to find other new invariants from left invariant vector fields, we start with the left (and bi-) invariant connections which are invariant w.r.t. left invariant vector fields (cf. Remarks 2.6. and 2.7).

(ii) Let ∇ be a bi-invariant connection on G. Then ∇ is ξ-invariant, with respect to any $\xi \in L(G)$. In particular, this happens for the Cartan-Schouten connections ∇^-, ∇^+, ∇^0.

We remark that (2.2) is a refinement of the property of a linear connection on G to be bi-invariant.

(iii) Let ξ be a left invariant vector field. The set of ξ-invariant left-invariant connections is a vector subspace of dimension at most n^3. We denote $d(\xi)$ its dimension. Obviously, for $a \neq 0$, $d(\xi) = d(a\xi)$.

Denote by $m_2(G) = max\{d(\xi) \mid \xi \in L(G)\}$. Then one may classify the Lie groups following this new invariant m_2.

(iv) Denote by $m_3(G)$ the maximal number of linearly independent vector fields $\xi_1, ..., \xi_{m_3} \in L(G)$, such that there exists a left invariant metric connection, simultaneously ξ_i-invariant, for all $i = \overline{1, m_3(G)}$. Then one may classify the Lie groups following this new invariant m_3.

If G admits bi-invariant metrics, then $m_3(G) = n$. (This happens if, and only if, G is a direct product of a compact group with some $\mathbb{R}^k$, cf. [9]).

4 Adapted connections for the 2-bodies problem: the Newtonian gravitational field

Let $M = \mathbb{R}^2 \backslash \{0\}$ and m be a positive constant (with signification of mass); denote by (r, φ) the polar coordinates on M and by $\xi = -mr^{-2}\partial r$ the "Newtonian gravitational vector field" on M. (We restrict ourselfs to gravitational interpretations, but similar considerations may be made for the Coulomb vector fields).

In [15] and [16] we studied $C_{LC}(M, \xi)$. In what follows, we extend the study to $C_s(M, \xi)$.

Remark 4.1. (i) ([15]) The Euclidean metric h on M has the well-known components: $h_{11} = 1$, $h_{12} = 0$, $h_{22} = r^2$. The (only non-vanishing) Christoffel coefficients (of the Levi-Civita connection) are: $\Gamma^1_{22} = -r$, $\Gamma^2_{12} = \Gamma^2_{21} = r^{-1}$. The canonically parametrized geodesics are ("lines") of the form $\gamma(s) = (r(s), \varphi(s))$, with $r^2(s) = s^2 + a^2$, $\varphi(s) = b + arctg\frac{s}{a}$, where a and b are arbitrary real constants.

(ii) More generally, an arbitrary left-invariant connection ∇ on M has the following coefficients:

(ii)$_1$ in Cartesian coordinates (x^1, x^2)= (x, y), all $|^{i}_{jk}|$, for $i, j, k = \overline{1, 2}$, are arbitrary real numbers;

(ii)$_2$ in polar coordinates $(r,\varphi)=(y^1,y^2)$,all $|\tilde{^i_{jk}}|$, for $i,j,k=\overline{1,2}$, are linear combinations (with real coefficients) of $cos^3\varphi$, $sin^3\varphi$, $cos^2\varphi sin\varphi$, $cos\varphi sin^2\varphi$, modulo an eventual multiplication with $\frac{1}{r}$, $\frac{1}{r^2}$, r^2 or r. We have

$$|\tilde{^1_{11}}| = cos^3\varphi\ |^1_{11}| + cos^2\varphi sin\varphi\ |^2_{11}| + cos\varphi sin^2\varphi\ |^1_{22}| + sin^3\varphi\ |^2_{22}| + $$
$$+ cos^2\varphi sin\varphi(|^1_{12}| + |^1_{21}|) + cos\varphi sin^2\varphi(|^2_{12}| + |^2_{21}|)$$
$$|\tilde{^2_{11}}| = \frac{1}{r}\{cos^3\varphi\ |^2_{11}| - cos^2\varphi sin\varphi\ |^1_{11}| + cos\varphi sin^2\varphi\ |^2_{22}| - sin^3\varphi\ |^1_{22}| +$$
$$+ cos^2\varphi sin\varphi(|^2_{12}| + |^2_{21}|) - cos\varphi sin^2\varphi(|^1_{12}| + |^1_{21}|)\}$$
$$|\tilde{^1_{22}}| = r^2\{cos\varphi(sin^2\varphi\ |^1_{11}| + cos^2\varphi\ |^2_{11}| - cos\varphi sin\varphi\ |^1_{12}| - sin\varphi cos\varphi\ |^1_{21}|) +$$
$$+ sin\varphi(sin^2\varphi\ |^2_{11}| + cos^2\varphi\ |^2_{22}| - cos\varphi sin\varphi\ |^2_{12}| - cos\varphi sin\varphi\ |^2_{21}|)\} - r$$
$$|\tilde{^2_{22}}| = r\{-sin\varphi(sin^2\varphi\ |^1_{11}| + cos^2\varphi\ |^2_{11}| - cos\varphi sin\varphi\ |^1_{12}| - sin\varphi cos\varphi\ |^1_{21}|) +$$
$$+ cos\varphi(sin^2\varphi\ |^2_{11}| + cos^2\varphi\ |^2_{22}| - cos\varphi sin\varphi\ |^2_{12}| - cos\varphi sin\varphi\ |^2_{21}|)\}$$
$$|\tilde{^1_{12}}| = r\{-sin\varphi cos^2\varphi\ |^1_{11}| - sin^2\varphi cos\varphi\ |^2_{11}| + cos^3\varphi\ |^1_{12}| + cos^2\varphi sin\varphi\ |^2_{12}| -$$
$$- sin^2\varphi cos\varphi\ |^1_{21}| - sin^3\varphi\ |^2_{21}| + sin\varphi cos^2\varphi\ |^1_{22}| + sin^2\varphi cos\varphi\ |^2_{22}|\}$$
$$|\tilde{^2_{12}}| = sin^2\varphi cos\varphi\ |^1_{11}| - sin\varphi cos^2\varphi\ |^2_{11}| - cos^2\varphi sin\varphi\ |^1_{12}| + cos^3\varphi\ |^2_{12}| +$$
$$+ sin^3\varphi\ |^1_{21}| - sin^2\varphi cos\varphi\ |^2_{21}| - sin^2\varphi cos\varphi\ |^1_{22}| + sin\varphi cos^2\varphi\ |^2_{22}|$$
$$|\tilde{^1_{21}}| = r\{sin\varphi cos^2\varphi(-\ |^1_{11}| + |^1_{22}| + |^2_{21}|) + sin^2\varphi cos\varphi(-\ |^1_{12}| - |^2_{11}| + |^2_{22}|) +$$
$$+ cos^3\varphi\ |^1_{12}| - sin^3\varphi\ |^2_{12}|\}$$
$$|\tilde{^2_{21}}| = -sin\varphi cos^2\varphi(|^1_{21}| + |^2_{11}| - |^2_{22}|) + sin^2\varphi cos\varphi(|^1_{11}| - |^2_{12}| - |^1_{22}|) +$$
$$+ cos^3\varphi\ |^2_{21}| + sin^3\varphi\ |^1_{12}| + r^{-1}$$

In particular, the canonical linear connection on M, given in (i), is left invariant and

$$\nabla_{\partial_r}\partial_r = 0\ \ ,\ \ \nabla_{\partial_\varphi}\partial_r = \nabla_{\partial_r}\partial_\varphi = \frac{1}{r}\partial_\varphi\ \ ,\ \ \nabla_{\partial_\varphi}\partial_\varphi = -r\partial_r$$

The previous calculations are based on the following obvious formulae:

$$r = \sqrt{x^2+y^2}\ \ ,\ \ x = rcos\varphi\ \ ,\ \ y = rsin\varphi$$

and

$$\partial_x = cos\varphi\partial_r - r^{-1}sin\varphi\partial_\varphi\ \ ,\ \ \partial_y = sin\varphi\partial_r + r^{-1}cos\varphi\partial_\varphi$$
$$\partial_r = \frac{x}{\sqrt{x^2+y^2}}\partial_x + \frac{y}{\sqrt{x^2+y^2}}\partial_y\ \ ,\ \ \partial_\varphi = -y\partial_x + x\partial_y$$
$$\partial_r = cos\varphi\partial_x + sin\varphi\partial_y\ \ ,\ \ \partial_\varphi = -rsin\varphi\partial_x + rcos\varphi\partial_y$$

(iii) For the Newtonian vector field ξ on M, we calculate the (classical) divergence $div\xi = mr^{-3}$; we remark that the sign is positive. This fact is specific to the dimension 2. The Newtonian vector field in $\mathbb{R}^3$ is divergence-free; in $\mathbb{R}^n$, with $n \geq 4$, its divergence function is negative.

In what follows, we shall consider adapted connections for ξ and we shall compare them with the previous ones.

Remark 4.2. (i) Denote $|^{i}_{jk}|$ the coefficients,in polar coordinates, for an arbitrary linear connection $\nabla \in C_s(M, \xi)$. From (2.1) we deduce, as only constraints, that

$$|^{1}_{11}| = 2r^{-1} \ , \ |^{2}_{11}| = 0 \tag{4.1}$$

The coefficients $|^{1}_{12}|$, $|^{2}_{12}|$, $|^{2}_{22}|$, $|^{1}_{22}|$ are arbitrary.

(ii) If, moreover, $\nabla \in C_{LC}(M, \xi)$, then we have additional constraints ([16]): there exists $\gamma = \gamma(r, \varphi)$, a nowhere vanishing differentiable function and a differentiable function $\beta = \beta(\varphi)$, such that

$$|^{1}_{12}| = -2m\beta r^{-3} + m\beta\gamma^{-1}r^{-2}\partial_r\gamma$$

$$|^{2}_{12}| = \gamma^{-1}\partial_r\gamma - 2r^{-1} \quad , \quad |^{2}_{22}| = \gamma^{-1}\partial_\varphi\gamma - m\beta\gamma^{-1}r^{-2}\partial_r\gamma + 2m\beta r^{-3}$$

$$|^{1}_{22}| = m\beta\gamma^{-1}r^{-2}\partial_\phi\gamma - m\beta' r^{-2} + 2m^2\beta^2 r^{-5} +$$
$$+2m^4\gamma^2 r^{-9} - m^2\beta^2\gamma^{-1}r^{-4}\partial_r\gamma - m^4\gamma r^{-8}\partial_r\gamma$$

We remark that $r^2 \ |^{1}_{12}| = -m\beta \ |^{2}_{12}|$. Thus, we can construct symmetric linear connections in $C(M, \xi)$, which are not in $C_{LC}(M, \xi)$, by taking:

$$|^{1}_{11}| = 2r^{-1} \ , \ |^{2}_{11}| = |^{2}_{12}| = |^{2}_{21}| = 0 \ , \ |^{1}_{12}| = |^{1}_{21}| \neq 0$$

with arbitrary $|^{1}_{22}|$ and $|^{2}_{22}|$.

(iii) Suppose (4.1) and $|^{1}_{12}| = |^{2}_{12}| = |^{2}_{22}| = |^{1}_{22}| = 0$. Then, the auto-parallel curves of ∇ are given by

$$[r(t)]^3 = at + b \quad , \quad \varphi(t) = ct + d$$

with arbitrary $a, b, c, d \in \mathbb{R}$.

Consider only non-degenerated auto-parallel curves, i.e. with $a^2 + c^2 > 0$. The first family of curves contains the (segments of) radial curves: $(c = 0)$, i.e.,

$$[r(t)]^3 = at + b \quad , \quad \varphi(t) = d$$

The second family contains the (arcs of) circles: $(a = 0)$, i.e.

$$[r(t)]^3 = b \quad , \quad \varphi(t) = ct + d$$

The third family of generic curves ($a \neq 0$, $c \neq 0$) contains bounded "spirals", given by implicit equations of the form $r^3 = A\varphi + B$, with arbitrary constants $A \neq 0$, B.

By direct computation, we have the following results.

Proposition 4.3. *Let $\xi = f(r)\partial r$ a "Newtonian-like" vector field on M, for an arbitrary derivable real valuated function f. Then, there do not exist left invariant connections in $C(M, \xi)$.*

Proposition 4.4. *Let $\xi = -r^{-2}\partial r$ a Newtonian vector field on M. Then, the linear connections w.r.t. which ξ is parallel are exactly those whose components satisfy:*

$$|^{1}_{11}| = \frac{2}{r} \quad , \quad |^{1}_{21}| = |^{2}_{21}| = 0$$

(*the remaining components being arbitrary*).

Remark 4.5. For M and ξ defined previously, the coefficients of a linear connection $\nabla \in \mathcal{C}_4(M, \xi, 2)$ must satisfy the following ODE system

$$r^2(\partial_r(|^1_{11}|) + (|^1_{11}|)^2 + |^1_{12}||^2_{11}|) - 6r\ |^1_{11}|\ +10 = 0$$

$$r\partial_r(|^2_{11}|) - 6\ |^2_{11}|\ +r\ |^2_{11}|\ (|^1_{11}| + |^2_{12}|) = 0$$

In general, consider $\nabla \in \mathcal{C}_4(M, \xi, n)$, with $n \geq 2$. The coefficients $|^i_{jk}|$ (with $i, j, k = \overline{1, 2}$) are functions of (r, φ) and must be determined as solutions of a system of two differential equations of degree $n - 1$, all the derivatives being done w.r.t the first variable r. The system is (obviously) compatible, having (4.1) as a particular solution.

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