Certain condition on the second fundamental form of CR-submanifolds in odd-dimensional unit spheres

M. Asadollahzadeh, E. Abedi, G. Haghighatdoost

Abstract. In this paper, we study $(n+1)$-dimensional real submanifolds M with $(n-1)$-contact CR dimension. On these manifolds there exists an almost contact structure F which is naturally induced from the ambient space. Also, we study the condition $h(FX,Y) - h(X,FY) = g(FX,Y)\varphi, \quad \varphi \in TM^{\perp}$, on the almost contact structure F and on the second fundamental form h of these submanifolds and we characterize certain model spaces in contact odd-dimensional unit sphere.

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1 Introduction

Let $\overline{M}$ be a $(2m+1)$-dimensional Sasakian manifold with Sasakian structure tensors (ϕ, ξ, η, g). The structure tensors satisfy:

$$\phi^2 X = -X + \eta(X)\xi, \ \phi\xi = 0, \ \eta(\xi) = 1, \ \eta(\phi X) = 0, \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \tag{1.2}$$

for any vector fields X and Y on $\overline{M}$ [11]. Let M be a submanifold tangent to the structure vector field ξ isometrically immersed in the Sasakian manifold $\overline{M}$. Then M is called a *contact CR-submanifold* of $\overline{M}$ if there exists a differentiable distribution $D : x \longrightarrow D_x \subset T_x M$ on M satisfying:

- D is invariant with respect to ϕ, i.e., $\phi D_x \subset D_x$
- The complementary orthogonal distribution $D^{\perp} : x \longrightarrow D_x^{\perp} \subset T_x M$ is anti-invariant with respect to ϕ, i.e., $\phi D_x^{\perp} \subset T_x^{\perp} M$

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for $x \in M$. If $dimD = 0$, then the contact CR-submanifold M is called an *anti-invariant submanifold* of $\overline{M}$ tangent to ξ. If $dimD^{\perp} = 0$, then M is an *invariant submanifold* of $\overline{M}$ [12]. Contact CR-submanifold of maximal CR-dimension in an odd-dimensional unit sphere satisfying the condition $h(FX, Y) + h(X, FY) = 0$ has been studied in [7], [8] and [9]. In the present article we study connected $(n+1)$-dimensional real submanifolds of codimension $(p = 2m - n)$ of the odd-dimensional unit sphere S^{2m+1} which are contact CR-submanifolds of contact CR-dimension $(n - 1)$, that is, $dimD^{\perp} = 2$. In Section 2 we collect some basic relations concerning submanifolds, in particular we discuss the notion of contact CR-submanifolds of the Sasakian manifold S^{2m+1}. Section 3 is devoted to the study of contact CR-submanifolds which satisfy the condition $h(FX, Y) - h(X, FY) = g(FX, Y)\varphi,\ \ \varphi \in TM^{\perp}$ on the structure tensor F naturally induced from the almost contact structure ϕ of the ambient manifold and on the second fundamental form h of a submanifold M. M. Djoric studied this relation for complex Euclidean space and complex projective space in [2] and [3]. In Section 4, using the codimension reduction theorem in [5], we obtain codimension reduction result for contact CR-submanifolds of an odd-dimensional unit sphere. Also in [10], Takagi showed that if M is a complete connected hypersurface of S^{2m+1} having 4 constant principal curvatures with the one multiplicity of 1, then M is congruent to $M^{2n}(t)$ for a number t with $0 < t < \frac{\pi}{4}$. And in [6], Nakagawa and Yokote proved:
$Theorem$: For a complete orientable hypersurface with constant principal curvature in S^{2n+1}, we assume that for a (f, g, u, v, λ)- structure on M, there exists a constant ϕ such that $H^i_k f^k_j + f^i_k H^k_j = 2\phi f^i_j$ or equivalently $f^k_j H_{ki} - f^i_k H_{kj} = 2\phi f_{ji}$, where H^i_j denotes the second fundamental tensor in M. Then either of the following two assertions (a) and (b) is true:
(a) M is isometric to one of the following spaces:

- the great sphere $S^{2n+1}(1)$;
- the small sphere $S^{2n}(c)$, where $c = 1 + \phi^2$;
- the product manifold $S^{2n-1}(c_1) \times S^1(c_2)$, where $c_1 = 1 + \phi^2$ and $c_2 = 1 + \frac{1}{\phi^2}$;
- the product manifold $S^n(c_1) \times S^n(c_2)$, where $c_1 = 2(1 + \phi^2 + \phi\sqrt{1 + \phi^2})$ and $c_2 = 2(1 + \phi^2 - \phi\sqrt{1 + \phi^2})$

(b) M has exactly four distinct constant principal curvatures of multiplicities $n - 1, n - 1, 1$ and 1, respectively.
Finally in Section 5 we provide a sufficient condition in order for such a submanifold to be the model space of $S^{2n_1+1}(c_1) \times S^{2n_2+1}(c_2)$, where c_1 and c_2 will be introduced in Section 5.

2 Preliminaries

Let S^{2m+1} be a $(2m+1)$-unit sphere and $Z \in S^{2m+1}$. We put $\xi = JZ$ where J is the complex structure of the complex $(m + 1)$-space $\mathbb{C}^{m+1}$. We consider the orthogonal projection $\pi : T_Z\mathbb{C}^{m+1} \to T_Z S^{2m+1}$, and put $\phi = \pi \circ J$. Then we see that (ϕ, ξ, η, g) is a Sasakian structure on S^{2m+1}, where η is a 1-form dual to ξ and g is the standard metric tensor field on S^{2m+1}. Hence, S^{2m+1} can be regarded as a Sasakian manifold of constant ϕ-sectional curvature 1 [1],[12]. Consider M, an $(n+1)$-dimensional contact CR-submanifold in S^{2m+1} which is tangent to the structure vector field ξ. The subspace D_x is the ϕ-invariant subspace $T_xM \cap \phi T_xM$ of the tangent space T_xM of M at $x \in M$. Then ξ is not in D_x at any x in M. Let $D_x^{\perp}$ denote the complementary orthogonal subspace to D_x in T_xM. For any nonzero vectors U orthogonal to ξ and contained in $D_x^{\perp}$, we have ϕU normal to M which we denote by N, that is,

$$N = \phi U. \tag{2.1}$$

It is clear that $\phi TM \subset TM \oplus span\{N\}$. In the following we assume that $dim D_x = n-1$ and $dim D_x^{\perp} = 2$, at each point x in M. We denote by $\nu(M)$ the complementary orthogonal subbundle of $\phi D^{\perp}$ in the normal bundle $TM^{\perp}$. We have the following orthogonal direct sum decomposition $TM^{\perp} = \phi D^{\perp} \oplus \nu(M)$. It is easy to see that $\nu(M)$ is ϕ-invariant. For vector field X tangent to M and for a local frame $\{N, N_\alpha\}_{\alpha=1,\dots,p-1}$, we have the following decomposition into tangential and normal parts

$$\phi X = FX + u(X)N, \tag{2.2}$$

$$\phi N_\alpha = PN_\alpha, \quad \phi N = -U \qquad \alpha = 1, \dots, p-1, \tag{2.3}$$

where F and P are skew-symmetric linear endomorphisms acting on T_xM and $T_x^{\perp}M$ and u is a 1-form on M. Since ξ is tangent to M, from (1.1), (1.2) and (2.1), we conclude

$$g(X, U) = u(X), \tag{2.4}$$

$$F\xi = 0, \quad u(\xi) = 0, \quad FU = 0, \quad u(U) = 1. \tag{2.5}$$

Using (2.1) again, we get

$$F^2X = -X + \eta(X)\xi + u(X)U, \tag{2.6}$$

also,

$$u(FX) = 0. \tag{2.7}$$

Let us denote by $\overline{\nabla}$ and ∇ the Riemannian connection of S^{2m+1} and M, respectively and by $\nabla^{\perp}$ the normal connection induced from $\overline{\nabla}$ in the normal bundle of M. Then the Gauss and Weingarten formulae for M are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.8}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{2.9}$$

for any vector fields X, Y tangent to M and any vector field N normal to M, where h denotes the second fundamental form and A_N denotes the shape operator (second fundamental tensor) corresponding to N.

Since $\nu(M)$ is ϕ-invariant we can take a local orthonormal frame $\{N, N_\alpha, N_{\alpha^*}\}_{\alpha=1,\dots,q}$ of normal vectors to M, such that $N_{1^*} = \phi N_1, \dots, N_{q^*} = \phi N_q$ then we have

$$\overline{\nabla}_X N = -AX + \sum_{\alpha=1}^{q} \{S_\alpha(X) N_\alpha + S_{\alpha^*}(X) N_{\alpha^*}\}, \tag{2.10}$$

$$\overline{\nabla}_X N_\alpha = -A_\alpha X - S_\alpha(X) N + \sum_{\beta=1}^{q} \{S_{\alpha\beta}(X) N_\beta + S_{\alpha\beta^*}(X) N_{\beta^*}\}, \tag{2.11}$$

$$\overline{\nabla}_X N_{\alpha^*} = -A_{\alpha^*} X - S_{\alpha^*}(X) N + \sum_{\beta=1}^{q} \{S_{\alpha^*\beta}(X) N_\beta + S_{\alpha^*\beta^*}(X) N_{\beta^*}\}, \tag{2.12}$$

where $q = \frac{p-1}{2}$ and S's are the coefficients of the normal connection $\nabla^\perp$ and $A, A_\alpha, A_{\alpha^*}$, are the shape operators corresponding to the normals $N, N_\alpha, N_{\alpha^*}$, respectively. In addition the second fundamental form h and the shape operators $A, A_\alpha, A_{\alpha^*}$ are related by

$$h(X, Y) = g(AX, Y) N + \sum_{\alpha=1}^{q} \{g(A_\alpha X, Y) N_\alpha + g(A_{\alpha^*} X, Y) N_{\alpha^*}\}. \tag{2.13}$$

Differentiating covariantly relations (2.1), (2.2), using (2.10) and comparing the tangential and normal parts, we get

$$(\nabla_Y F)X = -g(X, Y)\xi + \eta(X)Y - g(AY, X)U + u(X)AY, \tag{2.14}$$

$$\nabla_Y U = FAY, \tag{2.15}$$

$$(\nabla_Y u)X = g(FAY, X). \tag{2.16}$$

Since the ambient space is Sasakian, then

$$(\overline{\nabla}_Y \phi)X = -g(X, Y)\xi + \eta(X)Y. \tag{2.17}$$

Moreover,

$$\overline{\nabla}_X \xi = \phi X. \tag{2.18}$$

Using (2.2), the last relation gives

$$\nabla_X \xi = FX, \tag{2.19}$$

and

$$g(A\xi, X) = u(X),$$

that is

$$A\xi = U, \tag{2.20}$$

$$A_\alpha \xi = A_{\alpha^*} \xi = 0, \qquad \alpha = 1, \dots, q. \tag{2.21}$$

3 Contact CR-submanifolds of an odd-dimensional unit sphere

In this section we consider $(n+1)$-dimensional contact CR-submanifold M satisfying in the condition

$$h(FX,Y) - h(X,FY) = g(FX,Y)\varphi, \quad \varphi \in T^{\perp}M \tag{3.1}$$

for all X, Y tangent to M. Using (2.13) and setting

$$\varphi = \rho N + \sum_{\alpha=1}^{q}(\rho^{\alpha}N_{\alpha} + \rho^{\alpha^*}N_{\alpha^*}),$$

we obtain

$$h(FX,Y) - h(X,FY) = g((AF+FA)X,Y)N$$

$$+\sum_{\alpha=1}^{q}\{g((A_{\alpha}F+FA_{\alpha})X,Y)N_{\alpha} + g((A_{\alpha^*}F+FA_{\alpha^*})X,Y)N_{\alpha^*}\}$$

$$= g(FX,Y)(\rho N + \sum_{\alpha=1}^{q}(\rho^{\alpha}N_{\alpha} + \rho^{\alpha^*}N_{\alpha^*})).$$

Then,

$$AFX + FAX = \rho FX, \tag{3.2}$$

$$A_{\alpha}FX + FA_{\alpha}X = \rho^{\alpha}FX, \tag{3.3}$$

$$A_{\alpha^*}FX + FA_{\alpha^*}X = \rho^{\alpha^*}FX, \tag{3.4}$$

for all X tangent to M.
By (2.2), (2.17) and the relation

$$\phi(\overline{\nabla}_X N_{\alpha}) = \overline{\nabla}_X(\phi N_{\alpha}) - (\overline{\nabla}_X\phi)N_{\alpha}$$

we have

$$\phi(\overline{\nabla}_X N_{\alpha}) = \overline{\nabla}_X N_{\alpha^*}.$$

From a direct computation and comparing the tangential and normal parts, we get

$$A_{\alpha^*}X = FA_{\alpha}X - S_{\alpha}(X)U, \tag{3.5}$$

$$S_{\alpha\beta}(X) = S_{\alpha^*\beta^*}(X), \tag{3.6}$$

$$S_{\alpha\beta^*}(X) = -S_{\alpha^*\beta}(X), \tag{3.7}$$

$$S_{\alpha^*}(X) = u(A_{\alpha}X). \tag{3.8}$$

Similarly we obtain

$$A_\alpha X = -FA_{\alpha^*}X + S_{\alpha^*}(X)U, \tag{3.9}$$

$$S_\alpha(X) = -u(A_{\alpha^*}X). \tag{3.10}$$

Also using (3.5), (3.8), (3.9) and (3.10)

$$g((A_\alpha F + FA_\alpha)X, Y) = S_\alpha(X)u(Y) - S_\alpha(Y)u(X), \tag{3.11}$$

$$g((A_{\alpha^*}F + FA_{\alpha^*})X, Y) = S_{\alpha^*}(X)u(Y) - S_{\alpha^*}(Y)u(X). \tag{3.12}$$

Lemma 3.1. *Let M be a $(n+1)$-dimensional contact CR-submanifold of CR-dimension $(n-1)$ of S^{2m+1}. If (3.1) is satisfied then*

$$A_\alpha F + FA_\alpha = 0,$$

$$A_{\alpha^*}F + FA_{\alpha^*} = 0 \quad , that \quad is, \rho^\alpha = \rho^{\alpha^*} = 0.$$

Proof. Since (3.1) is equivalent to (3.3) and (3.4), then using (3.11) and (3.12) we have

$$g(\rho^\alpha FX, Y) = S_\alpha(X)u(Y) - S_\alpha(Y)u(X),$$

$$g(\rho^{\alpha^*} FX, Y) = S_{\alpha^*}(X)u(Y) - S_{\alpha^*}(Y)u(X).$$

Then

$$\rho^\alpha g(FX, Y) = S_\alpha(X)u(Y) - S_\alpha(Y)u(X), \tag{3.13}$$

$$\rho^{\alpha^*} g(FX, Y) = S_{\alpha^*}(X)u(Y) - S_{\alpha^*}(Y)u(X). \tag{3.14}$$

Putting $Y = U$ in (3.13) and (3.14) we have

$$S_\alpha(X) = S_\alpha(U)u(X), \tag{3.15}$$

$$S_{\alpha^*}(X) = S_{\alpha^*}(U)u(X). \tag{3.16}$$

Substituting (3.15) and (3.16) in (3.13) and (3.14) respectively, we obtain

$$\rho^\alpha = \rho^{\alpha^*} = 0.$$

Hence,

$$A_\alpha F + FA_\alpha = 0, \tag{3.17}$$

$$A_{\alpha^*}F + FA_{\alpha^*} = 0. \tag{3.18}$$

□

Further, using (3.2), (2.5) and (2.6) we get

$$AU = \xi + \alpha U, \quad such \ \ that \ \ \alpha = g(AU, U) = u(AU). \tag{3.19}$$

Lemma 3.2. *Let M be a complete $(n+1)$-dimensional contact CR-submanifold of CR-dimension $(n-1)$ of S^{2m+1}. If the condition (3.1) is satisfied, then U is an eigenvector of the shape operator A_α with respect to the normal vector field N_α at any point of M. Also the same result holds for the shape operator A_{α^*}.*

Proof.

$$F^2(A_\alpha U) = F(FA_\alpha U) = F((\rho^\alpha F - A_\alpha F)U).$$

Using (2.5) and (2.6) we have

$$-A_\alpha U + \eta(A_\alpha U)\xi + u(A_\alpha U)U = 0.$$

With (2.21) the last relation reads

$$A_\alpha U = \beta U, \tag{3.20}$$

where $\beta = u(A_\alpha U)$.
Similarly

$$A_{\alpha^*} U = \gamma U, \tag{3.21}$$

where $\gamma = u(A_{\alpha^*} U)$. □

Also, from (3.8) and (3.10) we get

$$A_\alpha U = S_{\alpha^*}(U)U, \tag{3.22}$$

$$A_{\alpha^*} U = -S_\alpha(U)U. \tag{3.23}$$

Since S^{2m+1} is of constant curvature 1,

$$\overline{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$$

From a direct computation and from the equation above, the Codazzi equation implies that

$$(\nabla_X A)Y - (\nabla_Y A)X = \tag{3.24}$$

$$\sum_{\alpha=1}^{q}\{S_\alpha(X)A_\alpha Y - S_\alpha(Y)A_\alpha X + S_{\alpha^*}(X)A_{\alpha^*}Y - S_{\alpha^*}(Y)A_{\alpha^*}X\},$$

$$(\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X = S_\alpha(X)AY - S_\alpha(Y)AX \tag{3.25}$$

$$+\sum_{\beta=1}^{q}\{S_{\alpha\beta}(X)A_\beta Y - S_{\alpha\beta}(Y)A_\beta X + S_{\alpha\beta^*}(X)A_{\beta^*}Y - S_{\alpha\beta^*}(Y)A_{\beta^*}X\},$$

$$(\nabla_X A_{\alpha^*})Y - (\nabla_Y A_{\alpha^*})X = S_{\alpha^*}(X)AY - S_{\alpha^*}(Y)AX \tag{3.26}$$

$$+\sum_{\beta=1}^{q}\{S_{\alpha\beta^*}(X)A_\beta Y - S_{\alpha\beta^*}(Y)A_\beta X + S_{\alpha^*\beta^*}(X)A_{\beta^*}Y - S_{\alpha^*\beta^*}(Y)A_{\beta^*}X\}.$$

In addition from a direct calculation, the Ricci equation is

$$g((A_\alpha A - AA_\alpha)X, Y) + (\nabla_X S_\alpha)Y - (\nabla_Y S_\alpha)X \tag{3.27}$$

$$+\sum_{\beta=1}^{q}\{S_{\alpha\beta}(Y)S_\beta(X) - S_{\alpha\beta}(X)S_\beta(Y)\}$$

$$+\sum_{\beta=1}^{q}\{S_{\alpha\beta^*}(Y)S_{\beta^*}(X) - S_{\alpha\beta^*}(X)S_{\beta^*}(Y)\} = 0.$$

Lemma 3.3. *Let M be an $(n+1)$-dimensional CR-submanifold of CR-dimension $(n-1)$ of S^{2m+1}. If (3.1) is satisfied, then the unit normal vector field N is parallel with respect to the normal connection. Furthermore $A_\alpha = 0 = A_{\alpha^*}$, where A_α, A_{α^*}, are the shape operators for the normals N_α, N_{α^*}, respectively.*

Proof. Differentiating (3.22) and using (2.15) we get

$$g((\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X, U) + g(A_\alpha AFX, Y) - g(A_\alpha AFY, X) = \tag{3.28}$$

$$X(S_{\alpha^*}(U))u(Y) - Y(S_{\alpha^*}(U))u(X) + S_{\alpha^*}(U)g((FA + AF)X, Y).$$

Substituting (3.25) in the above relation and using (3.15), (3.16), (3.19), (3.22), (3.23) we obtain

$$S_\alpha(U)\eta(X)u(Y) - S_\alpha(U)\eta(Y)u(X) + \sum_{\beta=1}^{q}\{S_{\alpha\beta}(Y)S_{\beta^*}(X) - S_{\alpha\beta}(X)S_{\beta^*}(Y)\} \tag{3.29}$$

$$+\sum_{\beta=1}^{q}\{S_{\alpha\beta^*}(Y)S_\beta(X) - S_{\alpha\beta^*}(X)S_\beta(Y)\} + g(A_\alpha AFX, Y) - g(A_\alpha AFY, X) =$$

$$X(S_{\alpha^*}(U))u(Y) - Y(S_{\alpha^*}(U))u(X) + S_{\alpha^*}(U)\rho g(FX, Y).$$

Taking $Y = U$ and $X = U$ in the last relation we obtain

$$-g(FA_\alpha AX, Y) + \rho g(FA_\alpha X, Y) - g(FAA_\alpha X, Y) - S_\alpha(U)\eta(Y)u(X) \tag{3.30}$$

$$= S_{\alpha^*}(U)\rho g(FX, Y).$$

Replacing Y with FY and using (2.6)we get

$$-g((A_\alpha A + AA_\alpha)X, Y) + 2\alpha S_{\alpha^*}(U)u(X)u(Y) \tag{3.31}$$

$$+S_{\alpha^*}(U)\eta(X)u(Y) + S_{\alpha^*}(U)u(X)\eta(Y)$$

$$+\rho\{g(A_\alpha X, Y) - S_{\alpha^*}(U)g(X, Y) + S_{\alpha^*}(U)\eta(X)\eta(Y)\} = 0.$$

Now, differentiating relation (3.15), we have

$$(\nabla_X S_\alpha)(Y) = X(S_\alpha(U))u(Y) + S_\alpha(U)g(FAY, X).$$

Replacing X with Y and subtracting the two equations, we get

$$(\nabla_X S_\alpha)(Y) - (\nabla_Y S_\alpha)(X) = X(S_\alpha(U))u(Y) - Y(S_\alpha(U))u(X) + \rho S_\alpha(U)g(FAX, Y). \tag{3.32}$$

Now from relations (3.27) and (3.32) we conclude

$$g((A_\alpha A - AA_\alpha)X, Y) + X(S_\alpha(U))u(Y) - Y(S_\alpha(U))u(X) + \rho S_\alpha(U)g(FAX, Y) \tag{3.33}$$

$$\sum_{\beta=1}^{q}\{S_{\alpha\beta}(Y)S_\beta(X) - S_{\alpha\beta}(X)S_\beta(Y)\} + \sum_{\beta=1}^{q}\{S_{\alpha\beta^*}(Y)S_{\beta^*}(X) - S_{\alpha\beta^*}(X)S_{\beta^*}(Y)\} = 0.$$

For $Y = U$ and $X = U$, using (3.15) and (3.16) the last relation gives

$$g((AA_\alpha - A_\alpha A)X, Y) = \rho S_\alpha(U)g(FX, Y) - S_{\alpha^*}(U)(\eta(X)u(Y) - \eta(Y)u(X)). \tag{3.34}$$

Now adding equations (3.31) and (3.34) gives

$$\begin{aligned} &-2g(A_\alpha AX, Y) + 2\alpha S_{\alpha^*}(U)u(X)u(Y) + 2S_{\alpha^*}(U)u(X)\eta(Y) + \rho S_\alpha(U)g(FX, Y) \\ &+ \rho\{g(A_\alpha X, Y) - S_{\alpha^*}(U)g(X, Y) + S_{\alpha^*}(U)\eta(X)\eta(Y)\} = 0 \end{aligned}$$

Taking $Y = U$ and (3.19) in the last relation we conclude

$$-2\beta g(X, \xi) + (-2\alpha\beta + 2\alpha S_{\alpha^*}(U) + \rho\beta - \rho S_{\alpha^*}(U))g(X, U) = 0.$$

Finally,

$$-2\beta\xi + (-2\alpha\beta + 2\alpha S_{\alpha^*}(U) + \rho\beta - \rho S_{\alpha^*}(U))U = 0. \tag{3.35}$$

Since with (3.20) and (3.22), $\beta = S_{\alpha^*}(U)$ then the last relation shows

$$\beta = S_{\alpha^*}(U) = 0 \quad \textit{and} \quad A_\alpha(U) = 0. \tag{3.36}$$

By (3.16) we get

$$S_{\alpha^*}(X) = 0, \tag{3.37}$$

Finally,

$$A_\alpha(X) = g(A_\alpha X, U)U = 0. \tag{3.38}$$

In a similar manner

$$S_\alpha(X) = 0, \tag{3.39}$$

$$A_{\alpha^*}(X) = 0. \tag{3.40}$$

□

4 Codimension reduction of contact CR-submanifolds in odd-dimensional unit sphere

In this section, we apply the Erbacher's reduction of codimension theorem to contact CR-submanifold in an odd-dimensional unit sphere. Let M be a connected submanifold in a Riemannian manifold. The first normal space $N_1(x)$ is defined to be the orthogonal complement of the set $N_0(x) = \{\zeta \in T_x^\perp M | A_\zeta = 0\}$ in $T_x^\perp M$ [12]. Erbacher proved the following theorem [5]:

Theorem 4.1. *Let $\psi : M^n \longrightarrow \overline{M}^{n+p}(\widetilde{c})$ be an isometric immersion of a connected n-dimensional Riemannian manifold into an $(n+p)$-dimensional Riemannian manifold $\overline{M}^{n+p}(\widetilde{c})$ of constant sectional curvature $\widetilde{c}$. If $N \supset N_1$ and N is a subbundle of $TM^\perp$ invariant with respect to the normal connection and l is the dimension of N, then there exists a totally geodesic submanifold N^{n+l} of $\overline{M}^{n+p}(\widetilde{c})$ such that $\psi(M^n) \subset N^{n+l}$.*

Let M be a connected contact CR-submanifold of S^{2m+1} whose contact CR-dimension is $(n-1)$, i.e, $dimD^\perp = 2$. For any orthogonal direct sum decomposition $TM^\perp = V_1 \oplus V_2$, it is easy to see that V_1 is invariant with respect to the normal connection if and only if V_2 is invariant with respect to the normal connection. Using the results of the previous section and Theorem 4.1, we have the following result

Theorem 4.2. *Let M be an $(n+1)$-dimensional contact CR-submanifold of contact CR-dimension $(n-1)$ of S^{2m+1}. If the condition (3.1) is satisfied, then there exists a totally geodesic unit sphere of dimension $(n+2)$ of S^{2m+1} such that $M \subset S^{n+2}$.*

Proof. By Lemma 3.3, the first normal space $N_1(x) = \phi D_x^\perp$. Hence, by Theorem 4.1 we can conclude that there exists a $(n+2)$-dimensional totally geodesic unit sphere S^{n+2} such that $M \subset S^{n+2}$. □

Lemma 4.3. *Let M be a $(n+1)$-dimensional contact CR-submanifold of CR-dimension $(n-1)$ of S^{2m+1}. If (3.1) is satisfied, then α defined in relation (3.19) is constant.*

Proof. Differentiating equation (3.19) and using (2.15)

$$(\nabla_X A)U = (X\alpha)U + \alpha FAX + FX - AFAX.$$

Since A is self-adjoint then $\nabla_X A$ is symmetric, so

$$g((\nabla_X A)Y, U) = g((\nabla_X A)U, Y) = (X\alpha)u(Y) + \alpha g(FAX, Y) + g(FX, Y) - g(AFAX, Y).$$

Interchanging X with Y and subtracting the last two equations, we get

$$g((\nabla_X A)Y - (\nabla_Y A)X, U) = (X\alpha)u(Y) - (Y\alpha)u(X) + \alpha\rho g(FX, Y) + 2g(FX, Y) - 2g(AFAX, Y) = 0,$$

from Lemma 3.3, equations (3.2) and (3.24). Taking $Y = U$ we have

$$X\alpha = (U\alpha)u(X). \tag{4.1}$$

Then we conclude that

$$grad\alpha = \lambda U, \quad which \quad \lambda = U\alpha. \tag{4.2}$$

Taking the covariant derivative of the last equation and reversing X and Y and subtracting the two relations we get,

$$(Y\lambda)u(X) + \lambda g(FAY, X) - (X\lambda)u(Y) - \lambda g(FAX, Y) = 0; \tag{4.3}$$

since $g(\nabla_X(grad\alpha), Y) = g(\nabla_Y(grad\alpha), X)$.
With (3.2) the last relation reads:

$$(Y\lambda)u(X) - (X\lambda)u(Y) - \rho\lambda g(FX, Y) = 0. \tag{4.4}$$

Replacing X and Y with U and putting in (4.4) we have:

$$\rho\lambda g(FX, Y) = 0. \tag{4.5}$$

Since $\rho \neq 0$ then, $\lambda = U\alpha = 0$, which means that α is constant. □

5 Model space of contact CR-submanifolds satisfying $h(FX, Y) - h(X, FY) = g(FX, Y)\varphi, \quad \varphi \in T^{\perp}M$

Let X be an eigenvector of the shape operator A corresponding to the eigenvalue β. Since A is symmetric, using (2.20) we have:

$$u(X) = \beta\eta(X). \tag{5.1}$$

Also using (3.19) we get:

$$\beta u(X) = \eta(X) + \alpha u(X). \tag{5.2}$$

Substituting (5.1) in (5.2) we have:

$$\beta^2\eta(X) - \alpha\beta\eta(X) - \eta(X) = 0. \tag{5.3}$$

From the last equation, we have two cases:

- $\beta^2 - \alpha\beta - 1 = 0$;
- $\eta(X) = 0$ which shows that X is orthogonal to ξ and from (5.1) is also orthonormal to U.

Theorem 5.1. *Let M be an $(n+1)$-dimensional contact CR-submanifold of CR-dimension $(n-1)$ of an odd-dimensional unit sphere. Also let X be an eigenvector of the shape operator A corresponding to the eigenvalue β. If X is not orthogonal to U and ξ and (3.1) is satisfied, then A has exactly two distinct principal curvatures.*

Proof. From the first case we have:

$$\beta^2 - \alpha\beta - 1 = 0,$$

so A has exactly two principal curvatures which are;

$$\beta_1 = \frac{\alpha + \sqrt{\alpha^2 + 4}}{2};$$

and

$$\beta_2 = \frac{\alpha - \sqrt{\alpha^2 + 4}}{2}.$$

□

Theorem 5.2. *Let M be an $(n+1)$-dimensional contact CR-submanifold of CR-dimension $(n-1)$ of an odd-dimensional unit sphere. Also let X be an eigenvector of the shape operator A corresponding to the eigenvalue β. If X is orthogonal to U and ξ and (3.1) is satisfied, then A has at most two distinct principal curvatures.*

Proof. Since α is constant, by equation (4.4) we have:

$$\alpha\rho g(FX, Y) + 2g(FX, Y) - 2g(AFAX, Y) = 0; \tag{5.4}$$

or,

$$(\alpha\rho + 2)FX - 2AFAX = 0. \tag{5.5}$$

Applying F to the last relation, using (2.6) and the fact that X is orthogonal to ξ we get:

$$2A^2X - 2\rho AX + (\alpha\rho + 2)AX = 0. \tag{5.6}$$

Since X is an eigenvector then the last relation reads:

$$2\beta^2 - 2\rho\beta + (\alpha\rho + 2) = 0,$$

and consequently A has at most two principal curvatures which are:

$$\beta_1 = \frac{\rho + \sqrt{\rho^2 - 2(\alpha\rho + 2)}}{2};$$

and

$$\beta_2 = \frac{\rho - \sqrt{\rho^2 - 2(\alpha\rho + 2)}}{2}.$$

□

Now let us denote the eigenspace by

$$T_k = \{X \in TM | AX = \beta_k X\}, \quad k = 1, 2.$$

Lemma 5.3. *Let M be an $(n+1)$-dimensional contact CR-submanifold of CR-dimension $(n-1)$ of an odd-dimensional unit sphere. Also let X be an eigenvector of the shape operator A corresponding to the eigenvalues β_1 and β_2. Then,* *1. T_k is parallel ;i.e. for $X, Y \in T_k$ we have $\nabla_X Y \in T_k$;*

2. For $\beta_1 \neq \beta_2$ we have $\nabla_X Y \perp T_k$.

Lemma 5.4. *T_k are involutive for $k = 1, 2$.*

Since T_k are involutive then they are integrable. Let M_k be the integral submanifold of T_k.

Lemma 5.5. *M_k are totally geodesic submanifolds for $k = 1, 2$.*

Now using the above lemmas we can conclude that:

Theorem 5.6. *Let M be a $(n+1)$-dimensional CR-submanifold of CR-dimension $(n-1)$ of an odd-dimensional unit sphere. If (3.1) is satisfied then either of the following two assertions (a) and (b) is true:*
M is isometric to one of the following spaces:

- *the product manifold $S^{2n_1+1}(c_1) \times S^{2n_2+1}(c_2)$, where $n_1 + n_2 = \frac{n-1}{2}$ and $c_1 = \frac{1}{\beta_1}$ and $c_2 = \frac{1}{\beta_2}$ which β_1 and β_2 are defined in Theorem 5.1;*
- *the product manifold $S^{2n_1+1}(c_1) \times S^{2n_2+1}(c_2)$, where $n_1 + n_2 = \frac{n-1}{2}$ and $c_1 = \frac{1}{\beta_1}$ and $c_2 = \frac{1}{\beta_2}$ which β_1 and β_2 are defined in Theorem 5.2.*

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Author's address:

Mehri Asadollahzadeh
Institute of Mathematical Sciences, University of Bonab, Azarbaijan, Iran.
E-mail: m.a.zadeh@bonabu.ac.ir

Esmaiel Abedi
Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.
E-mail: esabedi@azaruniv.edu

Ghorbanali Haghightdoost
Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
E-mail: gorbanali@azaruniv.edu