Multitime KdV solitons

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Abstract. The term of multitime soliton has recently been coined in our research school to describe a multitime pulselike nonlinear wave (multitime solitary wave) which emerges from a collision with a similar pulse having unchanged shape and speed vector. To introduce this notion, it was necessary to introduce multitime PDEs that extend solitonic single-time PDEs via geometrical ingredients. This paper covers the status for multitime soliton research based on multitime dual power law nonlinear KdV PDE, paying particular attention to methods whereby an initial value problem for such a PDE can be solved exactly through a succession of calculations. Discussion of the interaction between two bi-temporal solitons show their own physical sense.

M.S.C. 2010: 37K40, 35C08, 35Q51.
Key words: multitime solitons; bi-temporal solitons interaction; multitime KdV PDE; higher order Riccati equation.

1 Introduction

A lot of nonlinear phenomena and physical systems has nonlinear PDEs as models of representation. For this reason, the getting of the exact solutions for this type of equations represents a significant problem in the nonlinear science. Sine-Gordon PDE, with some versions or related equations, Korteweg-de Vries (KdV) or Rayleigh equations, with related equations too, Boussinesq equations and many others are PDEs which admit soliton solutions (see [1]-[7], [11], [14]-[16], [18]).

Because in engineering, physics, meteorology or computer science, the multitime modeling is an area for solving physical problems which have important properties for multidimensional parameters of evolution (multitimes), we shall pass from single-time to multitime using geometrical ingredients (derivations, trace, etc.), which extend the initial PDEs. In other words we re-write the initial PDEs using multitime variables.

This paper is based on so-called multitime (or field) formalism. The matter is that sometimes the parameter of evolution is m-dimensional (like in deformation theory). This view is related to timescale theory which requires $t^0 = t, t^1 = \epsilon t, t^2 = \epsilon^2 t, ...$ as new variables and a new function $u(x, (t^0, t^1, t^2, ...))$ representing the true phenomena.

Balkan Journal of Geometry and Its Applications, Vol.22, No.2, 2017, pp. 63-74.

2 Single-time dual power law nonlinear KdV PDE

The *KdV equation* represents a mathematical model for the waves at the surface of the shallow waters. It has many connections with physical problems, describing with approximation the evolution of the longue uni-dimensional waves, in different physical applications: the shallow water waves with weakly nonlinear restoring forces, the internal longue waves from density-stratified ocean, the ion-acoustic waves from plasma and the acoustic waves in a crystal lattice.

The content of this paper was suggested by the *single-time dual power law nonlinear KdV PDE* (see [1], [3], [5]-[6])

$$\frac{\partial u}{\partial t}(x,t) = (au^p(x,t) + bu^{2p}(x,t))\,\frac{\partial u}{\partial x}(x,t) + c\,\frac{\partial^3 u}{\partial x^3}(x,t), \tag{2.1}$$

where p is a positive integer and a, b, c are real parameters. The basic idea is to produce motivations for the multitime soliton physics.

Using some geometrical concepts from Differential Geometry, Riemannian Geometry or Bundle Theory (differential operator along certain direction, metric, connection, jet bundles, etc.), we shall succeed to make the extension of this single-time PDE to a multiple time variable PDE.

3 Multitime dual power law nonlinear KdV PDE

To generate multitime PDEs, we recall geometrical objects from the first order jet bundle (e.g., metric, connection, vector fields, tensor fields), creating multitime extensions for significant PDEs from geometry or mathematical physics.

Let $\mathbb{R}^m$ endowed with the *product order*, $t = (t^1, \ldots, t^m) \in \mathbb{R}^m$ be a generic point, as an m-dimensional evolution parameter, called *multitime* and $H = (h^\alpha)$, where $h^\alpha = h^\alpha(x,t)$, $\alpha = 1, \ldots, m$ on $\mathbb{R}^m$ be a distinguished vector field borrowed from the geometry of the jet bundle of order one $J^1(\mathbb{R} \times \mathbb{R}^m, \mathbb{R})$, associated to a $\mathcal{C}^2$ function $u : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$. The distinguished vector field $H = (h^\alpha)$ defines the following *multitime differential operator (along the direction H)* (see [17]),

$$\mathrm{D}_H u = h^\alpha \frac{\partial u}{\partial t^\alpha} \tag{3.1}$$

and using it, we define the multitime PDE

$$\mathrm{D}_H u = (au^p + bu^{2p})\frac{\partial u}{\partial x} + c\frac{\partial^3 u}{\partial x^3},$$

that is,

$$h^\alpha(x,t)\frac{\partial u}{\partial t^\alpha}(x,t) = (au^p(x,t) + bu^{2p}(x,t))\,\frac{\partial u}{\partial x}(x,t) + c\,\frac{\partial^3 u}{\partial x^3}(x,t), \tag{3.2}$$

where $x \in \mathbb{R}$ and $t = (t^1, \ldots, t^m) \in \mathbb{R}^m$, which will be called the *multitime KdV PDE with dual power law nonlinearity in the direction H*. A solution for this PDE must be understood in the weak sense, since it no longer makes sense to require the PDE to be literally satisfied at a point of discontinuity.

We look for those multitime PDEs that possess infinitely many exact analytic solutions, particularly solitons. The stationary parts of a single-time PDE or an associated multitime PDE coincides.

The existence of the solutions for the multitime PDE (3.2) is given by the Theorem 3.1. from [17], that we recall here.

Theorem 3.1. There exists an infinity of distinguished vector fields $H = (h^\alpha)$ on $\mathbb{R}^m$ such that o solution of PDE (2.1) is also a solution of the multitime PDE (3.2).

Remark 3.1. The proof of this theorem can be also find in [17].

Due to the previous theorem we can use the terminology of *multitime geometrical prolongation* for the KdV PDE with dual power law nonlinearity (2.1). To solve certain problems, we can choose and fix various relations that can be satisfied by the distinguished vector field $H = (h^\alpha)$.

4 Why we need multitime KdV solitons?

The evolution PDEs may have multi-temporal behavior if the dynamical system, whose conduct is described, supports linear or nonlinear perturbations or if the PDEs contain nonlinearities due to friction, spoiling, defective products or the presence of constituents made from intelligent materials. In these cases, the dynamics stretches oneself on more temporal scales evolving from slow to fast, that can be described by more temporal variables. The multi-temporal modeling is particularly important in engineering since it allows the evaluation of the properties of some materials or the behavior of some systems having as basis the knowledge of the associated geometry. In this sense, we recall that the evaluation of the dumping and of the dissipated energy realize using single-time soliton theory, for materials of small dimension (<1 *micron*), by the Rogers and Schief (1997) method of pseudo-spherical reduction of Euler-Lagrange PDEs of the deformation problem to a pseudospherical surface (Tzitzeica surface). As for example, the KdV equation with hysteresis, which describes the propagation of the waves generated by an obstacle in a rectangular channel of small depth, in which the viscous resistance at interface between the structure and fluid and the tension of resistance produced by turbulence effects are taken into consideration, can be solved via a system of differential inclusions which requires more temporal variables.

Let $u_1(x,t) = q_{v_1}(x - v_1 t)$ and $u_2(x,t) = q_{v_2}(x - v_2 t)$, $0 < v_2 < v_1$, be two single-time soliton solutions of the PDE (2.1). An m-soliton looks superficially like a linear combination of several 1-solitons. Also, there exist solutions which look asymptotically like linear combinations of two or more solitons for large $|t|$.

The single-time collision between the solitons $q_{v_1}(x-v_1 t)$ and $q_{v_2}(x-v_2 t)$ is better described by a two-time collision between $q_{v_1}(x - v_1 t^1)$ and $q_{v_2}(x - v_2 t^2)$ since the two-time variables separate time scales (for example voltage as function of signal time scale and clock time scale). Also, instead of one single-time doublet solution (i.e., two interacting solitons), we can use a "two-time soliton".

The development of the multitime PDE concepts is now in vogue in electronics (widely-separated time scales, difference-frequency time scales, etc.) (see [12]) and in mathematics (see [8]-[10], [17]). Indeed, to handle the frequency-modulation effectively, the paper [12] use of a novel concept, *warped time*, within a multitime partial differential equation framework. Generally, the purpose of a multidimensional model is to represent efficiently phenomena including widely separated time scales (for example, control of composite systems via the multi time-scale approach). Of course, the multitime simulation requires special integrators.

5 Multitime KdV solitons

The first aim is to prove that the *multitime dual power law nonlinear KdV PDE* (3.2) admits special solutions, called *multitime soliton solutions*. We are looking for the multitime soliton solutions for the multitime PDE (3.2) in the form (*the method of multitime traveling waves*)

$$u(x,t) = \Phi(x - v_\alpha t^\alpha) = \Phi(z),\ u(x,0) = \Phi(x), \tag{5.1}$$

where $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^3$ unknown *profile function*, $v = (v_\alpha)$ is a constant *speed vector* and $z = x - v_\alpha t^\alpha$. In other words, we wish to look for a solution that is transported, i.e., the solution is simply a multi-temporal shift of the initial condition via the inner product $\langle v, t\rangle = v_\alpha t^\alpha$.

Replacing the partial derivatives of the unknown function $u(x,t)$,

$$\frac{\partial u}{\partial x} = \Phi'(z), \quad \frac{\partial^2 u}{\partial x^2} = \Phi''(z), \quad \frac{\partial^3 u}{\partial x^3} = \Phi'''(z), \quad \frac{\partial u}{\partial t^\alpha} = \Phi'(z)(-v_\alpha),$$

into multitime PDE (3.2), we obtain the *profile* ODE

$$-h^\alpha v_\alpha \Phi'(z) = (a\Phi^p(z) + b\Phi^{2p}(z))\Phi'(z) + c\Phi'''(z)$$

and, imposing the variable direction H to satisfy

$$h^\alpha v_\alpha = d(z),$$

we reduce the problem to solving of a third order profile ODE

$$c\Phi'''(z) + d(z)\Phi'(z) + (a\Phi^p(z) + b\Phi^{2p}(z))\Phi'(z) = 0, \tag{5.2}$$

where a, b, c are real constants.

Remark 5.1. Some physical problems require a multitime soliton solution for which $u; u_x; u_{xx} \to 0$ as $|x| \to \infty$ or $u; u_{t^\alpha}; u_{t^\alpha t^\beta} \to 0$ as $||t|| \to \infty$ (for single-time case, see [1]-[7], [11], [14]-[16], [18]).

5.1 Autonomous profile ODE

In order to find some multitime soliton solutions of the multitime KdV PDE (3.2), we put under discussion the following cases: the particular case $d(z) = \Phi^p(z)$, the case of the constant direction H, i.e. $d(z) = constant = k \neq 0$ and the case of a variable direction H othogonal to the speed vector, i.e. $d(z) = 0$. Even these cases may seem to reduce the multitime approach to the single-time theory, the obtaining of the solutions for the autonomous profile ODEs is not obvious at all and leads to multitime soliton solutions of our multitime PDE.

In the following, we shall consider the first case, the other two being based on the application of the same method of solving the profile ODEs and leading to similar successive calculus and, in the end, to identical expressions of the multitime soliton solutions of the multitime KdV PDE (3.2).

Thus, for the direction H verifying $d(z) = \Phi^p(z)$, the ODE (5.2) becomes a profile equation with constant coefficients,

$$c\Phi'''(z) + \big((a+1)\Phi^p(z) + b\Phi^{2p}(z)\big)\,\Phi'(z) = 0.$$

By integration and taking the constant of integration 0, we obtain the second order Riccati ODE

$$c(p+1)(2p+1)\Phi'' + (a+1)(2p+1)\Phi^{p+1} + b(p+1)\Phi^{2p+1} = 0$$

and, introducing the dual function (change of function)

$$\Phi = \Psi^{1/p}, \tag{5.3}$$

we get

$$M\Psi\Psi'' + N{\Psi'}^2 + R\Psi^3 + T\Psi^4 = 0, \tag{5.4}$$

where $M = cp(1+p)(1+2p)$, $N = c(1-p^2)(1+2p)$, $R = (a+1)p^2(1+2p)$, $T = bp^2(1+p)$.

In order to find exact solutions for this last equation, we shall use the extended trial equation method (see [13]). In other words, we look for special solutions of the form

$$\Psi = \sum_{i=0}^{\theta} c_i G^i,$$

where each G satisfies the trial ODE

$${G'}^2 = \frac{P(G)}{Q(G)} = \frac{a_0 + a_1 G + \ldots + a_\beta G^\beta}{b_0 + b_1 G + \ldots + b_\gamma G^\gamma}, \tag{5.5}$$

In these conditions, the degrees of functions Ψ, ${\Psi'}^2$ and Ψ'', expressed by polynomials in G, are $\deg(\Psi) = \theta$, $\deg({\Psi'}^2) = \beta - \gamma + 2(\theta - 1)$, $\deg(\Psi'') = \beta - \gamma + \theta - 2$ and, using the homogeneous balance principle for the ODE (5.4), we get $\beta = \gamma + 2\theta + 2$, thus we may choose $\theta = 1$, $\beta = 4$ and $\gamma = 0$. In this way

$$\Psi = c_0 + c_1 G, \quad {\Psi'}^2 = c_1^2 \frac{P(G)}{Q(G)},$$

that is

$${\Psi'}^2 = \frac{c_1^2}{b_0}(a_0 + a_1 G + a_2 G^2 + a_3 G^3 + a_4 G^4),$$

and

$$\Psi'' = c_1 \frac{P'(G)Q(G) - P(G)Q'(G)}{2Q^2(G)},$$

i.e.,

$$\Psi'' = \frac{c_1}{2b_0}(a_1 + 2a_2 G + 3a_3 G^2 + 4a_4 G^3).$$

In conclusion, we look for the solutions of the ODE (5.4) in the form

$$\Psi = c_0 + c_1 G, \quad c_1 \neq 0$$

where

$$G'^2 = \frac{1}{b_0}(a_0 + a_1 G + a_2 G^2 + a_3 G^3 + a_4 G^4), \quad a_4 \neq 0, \quad b_0 \neq 0 \tag{5.6}$$

and then

$$\Psi'^2 = \frac{c_1^2}{b_0}(a_0 + a_1 G + a_2 G^2 + a_3 G^3 + a_4 G^4),$$

and

$$\Psi'' = \frac{c_1}{2b_0}(a_1 + 2a_2 G + 3a_3 G^2 + 4a_4 G^3).$$

Replacing Ψ, Ψ'^2 and Ψ'', given by the above relations, into the ODE (5.4), we obtain a null polynomial (identity) in G. It follows a system of algebraic equations, in the unknowns a_0, a_1, a_2, a_3, a_4; b_0 and c_0, c_1:

$$\begin{cases} Ma_1c_0c_1 + 2Na_0c_1^2 + 2Rb_0c_0^3 + 2Tb_0c_0^4 = 0 \\ 2Ma_2c_0 + Ma_1c_1 + 2Na_1c_1 + 6Rb_0c_0^2 + 8Tb_0c_0^3 = 0 \\ 3Ma_3c_0 + 2Ma_2c_1 + 2Na_2c_1 + 6Rb_0c_0c_1 + 12Tb_0c_0^2c_1 = 0 \\ 4Ma_4c_0 + 3Ma_3c_1 + 2Na_3c_1 + 2Rb_0c_1^2 + 8Tb_0c_0c_1^2 = 0 \\ 2Ma_4 + Na_4 + Tb_0c_1^2 = 0, \end{cases}$$

which has the general solution

$$a_0 = \frac{a_4c_0^3}{c_1^4}\left[c_0 + \frac{2R(2M+N)}{T(3M+2N)}\right], \quad a_1 = \frac{2a_4c_0^2}{c_1^3}\left[2c_0 + \frac{3R(2M+N)}{T(3M+2N)}\right],$$

$$a_2 = \frac{6a_4c_0}{c_1^2}\left[c_0 + \frac{R(2M+N)}{T(3M+2N)}\right], \quad a_3 = \frac{2a_4}{c_1}\left[2c_0 + \frac{R(2M+N)}{T(3M+2N)}\right],$$

$$a_4 = a_4,\ b_0 = -\frac{a_4(2M+N)}{c_1^2T},\ c_0 = c_0,\ c_1 = c_1.$$

Replacing these results into the equation (5.6), we obtain

$$G'^2 = -\frac{c_1^2T}{2M+N}\left[\frac{c_0^3}{c_1^4}\left(c_0 + \frac{2R(2M+N)}{T(3M+2N)}\right) + \frac{2c_0^2}{c_1^3}\left(2c_0 + \frac{3R(2M+N)}{T(3M+2N)}\right)G\right.$$

$$\left. + \frac{6c_0}{c_1^2}\left(c_0 + \frac{R(2M+N)}{T(3M+2N)}\right)G^2 + \frac{2}{c_1}\left(2c_0 + \frac{R(2M+N)}{T(3M+2N)}\right)G^3 + G^4\right].$$

Consequently,

$$G' = \pm\sqrt{-\frac{c_1^2 T}{2M+N}\Lambda(G)} \tag{5.7}$$

where

$$\Lambda(G) = \frac{c_0^3}{c_1^4}\left(c_0 + \frac{2R(2M+N)}{T(3M+2N)}\right) + \frac{2c_0^2}{c_1^3}\left(2c_0 + \frac{3R(2M+N)}{T(3M+2N)}\right)G$$

$$+\frac{6c_0}{c_1^2}\left(c_0 + \frac{R(2M+N)}{T(3M+2N)}\right)G^2 + \frac{2}{c_1}\left(2c_0 + \frac{R(2M+N)}{T(3M+2N)}\right)G^3 + G^4.$$

If we denote

$$A = \sqrt{-\frac{2M+N}{c_1^2 T}},$$

then the ODE (5.7) is reduced to the elementary integral form

$$\pm(z+C) = A\int \frac{dG}{\sqrt{\Lambda(G)}}, \quad C \in \mathbb{R}. \tag{5.8}$$

Using the complete discrimination system for the polynomial $\Lambda(G)$, we shall classify the roots g_1, g_2, g_3, g_4 of this polynomial and then we will compute the above integral. Thus, the relation (5.8) transcribes:

$$\pm(z+C) = -\frac{A}{G-g_1}; \tag{5.9}$$

$$\pm(z+C) = \frac{2A}{g_2-g_1}\sqrt{\frac{G-g_2}{G-g_1}}, \quad g_1 > g_2; \tag{5.10}$$

$$\pm(z+C) = \frac{A}{g_1-g_2}\ln\left|\frac{G-g_1}{G-g_2}\right|, \quad g_1 > g_2; \tag{5.11}$$

$$\pm(z+C) = \frac{A}{\sqrt{(g_1-g_2)(g_1-g_3)}}\ln\left|\frac{\sqrt{(G-g_2)(g_1-g_3)}-\sqrt{(G-g_3)(g_1-g_2)}}{\sqrt{(G-g_2)(g_1-g_3)}+\sqrt{(G-g_3)(g_1-g_2)}}\right|,$$

$$g_1 > g_2 > g_3; \tag{5.12}$$

$$\pm(z+C) = \frac{2A}{\sqrt{(g_1-g_3)(g_2-g_4)}}F(\varphi, l), \quad g_1 > g_2 > g_3 > g_4;$$

where

$$F(\varphi, l) = \int_0^{\varphi} \frac{d\xi}{1 - l^2\sin^2\xi},$$

and

$$\varphi = \arcsin\sqrt{\frac{(G-g_1)(g_2-g_4)}{(G-g_2)(g_1-g_4)}}, \quad l^2 = \frac{(g_2-g_3)(g_1-g_4)}{(g_1-g_3)(g_2-g_4)}.$$

The relations (5.9)-(5.12) yield the expressions of the function G, which, replaced into (5.5), give five different solutions (profile functions) of the ODE (5.4),

$$\Psi = c_0 + c_1 g_1 \pm \frac{c_1 A}{z+C},$$

$$\Psi = c_0 + c_1 g_1 + \frac{4A^2(g_2-g_1)c_1}{4A^2-(g_2-g_1)^2(z+C)^2},$$

$$\Psi = c_0 + c_1 g_1 + \frac{(g_1-g_2)c_1}{\exp\left(\frac{g_1-g_2}{A}(z+C)\right)-1},$$

$$\Psi = c_0 + c_1 g_2 + \frac{(g_2-g_1)c_1}{\exp\left(\frac{g_1-g_2}{A}(z+C)\right)-1},$$

$$\Psi = c_0 + c_1 g_1 - \frac{2(g_1-g_2)(g_1-g_3)}{2g_1-g_2-g_3+(g_2-g_3)\cosh\left[\frac{\sqrt{(g_1-g_2)(g_1-g_3)}}{A}(z+C)\right]}.$$

The above formulas and the substitutions (5.3) and (5.1) provide the expressions of the multitime soliton solutions for the multitime PDE (3.2), under the assumption of the particular case under discussion.

Theorem 5.1. *Under the foregoing assumptions, the multitime PDE* (3.2) *has the following families of multitime soliton solutions*

$$u(x,t) = \left\{c_0 + c_1 g_1 \pm \frac{c_1 A}{x - v_\alpha t^\alpha + C}\right\}^{1/p},$$

$$u(x,t) = \left\{c_0 + c_1 g_1 + \frac{4A^2(g_2-g_1)c_1}{4A^2-(g_2-g_1)^2(x-v_\alpha t^\alpha+C)^2}\right\}^{1/p},$$

$$u(x,t) = \left\{c_0 + c_1 g_1 + \frac{(g_1-g_2)c_1}{\exp\left(\frac{g_1-g_2}{A}(x-v_\alpha t^\alpha+C)\right)-1}\right\}^{1/p},$$

$$u(x,t) = \left\{c_0 + c_1 g_2 + \frac{(g_2-g_1)c_1}{\exp\left(\frac{g_1-g_2}{A}(x-v_\alpha t^\alpha+C)\right)-1}\right\}^{1/p},$$

$$u(x,t) = \left\{c_0 + c_1 g_1 - \frac{2(g_1-g_2)(g_1-g_3)}{2g_1-g_2-g_3+(g_2-g_3)F(x,t)}\right\}^{1/p},$$

where

$$F(x,t) = \cosh\left[\frac{\sqrt{(g_1-g_2)(g_1-g_3)}}{A}(x - v_\alpha t^\alpha + C)\right],\ C \in \mathbb{R}.$$

Remark 5.2. If we suppose $c_0 = -c_1 g_1$ and we take $C = 0$, then the above multitime solutions can be reduced to the following types:

- multitime algebraic functions solutions

$$u(x,t) = \left\{\pm\frac{c_1 A}{x - v_\alpha t^\alpha}\right\}^{1/p},$$

$$u(x,t) = \left\{\frac{4A^2(g_2-g_1)c_1}{4A^2 - (g_2-g_1)^2(x - v_\alpha t^\alpha)^2}\right\}^{1/p},$$

- multitime traveling wave solutions

$$u(x,t) = \left\{\frac{c_1(g_2-g_1)}{2}\left[1 \mp \coth\left(\frac{g_1-g_2}{2A}(x - v_\alpha t^\alpha)\right)\right]\right\}^{1/p},$$

- multitime soliton solutions

$$u(x,t) = \frac{\mathcal{A}}{\{B + \cosh\left[D(x - v_\alpha t^\alpha)\right]\}^{1/p}},$$

where $\mathcal{A} = \left[\frac{2c_1(g_1-g_2)(g_1-g_3)}{g_3-g_2}\right]^{1/p}$ is the amplitude of the multitime soliton, $B = \frac{2g_1-g_2-g_3}{g_2-g_3}$, while $D = \frac{\sqrt{(g_1-g_2)(g_1-g_3)}}{A}$ is the inverse width of the multitime soliton.

5.2 Non-autonomous profile ODE

In the cases $d(z) = \Phi^p(z)$, $d(z) = k \in \mathbb{R} - \{0\}$, $d(z) = 0$, the profile ODEs are autonomous. The following cases refer to the non-autonomous multitime PDE.
Case 1 In the case $d(z) = z$, the profile ODE is

$$c\Phi'''(z) + (z + a\Phi^p(z) + b\Phi^{2p}(z))\Phi'(z) = 0$$

and for the particular profile ODE

$$\Phi'''(z) + (z + a\Phi(z) + b\Phi^2(z))\Phi'(z) = 0,$$

with the initial conditions $\Phi(0) = 3$, $\Phi'(0) = 0$, $\Phi''(0) = 1$, we have a solution as a formal series

$$\Phi(z) = 3 + \frac{1}{2}z^2 + (-\frac{1}{8}a - \frac{3}{8}b)z^4 - \frac{1}{60}z^5 + \mathcal{O}(z^6).$$

Case 2 Generally, in the case of non-autonomous PDE, the profile ODE is intractable in the sense of exact analytic solutions, particularly solitons. But we can show that

a function of soliton type is a solution of the multitime nonlinear PDE, with adapted direction. E.g.,
(i) for $\Phi(z) = \lambda\,\mathrm{sech}^2(\mu z)$, we find

$$d(z) = 12c\mu^2\mathrm{sech}^2(\mu z) - 4c\mu^2 - \left[a\lambda^p\mathrm{sech}^{2p}(\mu z) + b\lambda^{2p}\mathrm{sech}^{4p}(\mu z)\right];$$

(ii) for $\Phi(z) = \lambda \mathrm{e}^{-\mu z^2}$, it follows

$$d(z) = 2c\mu(3 - 2\mu z^2) - \left(a\lambda^p\mathrm{e}^{-p\mu z^2} + b\lambda^{2p}\mathrm{e}^{-2p\mu z^2}\right).$$

6 Interaction of two bi-temporal solitons

The function $z \to \phi(z)$ is called *profile function*. The graph of the function $z \to \phi(z)$ is called *profile of the soliton*. The function $z = x - v\tau$ is called *phase* of a single-time soliton; the function $z = x - v_\alpha t^\alpha$ is called *phase* of a multitime soliton.

Dynamically, the profile of a single-time soliton creates a family of graphs moving after the parameter time τ along the axis Oz. Similarly, the profile of a multi-time soliton creates a family of graphs moving after the parameter m-time $t = (t^1, ..., t^m)$ along the axis Oz. The movement depending on m parameters is not so simple as those depending on 1 parameter.

Let $u(x, \tau)$ be a single-time soliton. The graph of this soliton is a cylindrical surface in R^3 with the generators parallel to the straight line $x - v\tau = 0, u = u_0$. The director curve or the profile of the soliton is the section of the cylindrical surface by the plane $t = 0$.

Let $u(x, t = (t^1, t^2))$ be a bi-temporal soliton. The graph of this soliton is a cylindrical hypersurface in R^4 with the generators (planes) parallel to the plane $x - v_1t^1 - v_2t^2 = 0, u = u_0$. The director curve or the profile of the soliton is the section of the cylindrical surface by the plane $t^1 = 0$, $t^2 = 0$.

Geometrically, the profile of a multitime soliton can be the same as those of a single-time soliton. This statement is confirmed by the commutative diagram

$$\begin{array}{ccc} \text{singletime } PDE & \overset{\longleftarrow}{\longrightarrow} & \text{profile } ODE_1 \\ \text{rules} \downarrow & & \uparrow \text{hypotheses} \\ \text{multitime } PDE & \overset{\longleftarrow}{\longrightarrow} & \text{profile } ODE_m \end{array}$$

Analytically, a multitime soliton can have the same profile as a single-time soliton iff they have the same amplitude, same width coefficient, and the single-time τ admits the decomposition $v\tau = v_1t^1 + v_2t^2$, and t^1, t^2 vary both ascending. Otherwise, the parameter τ is not ascending.

Precisely, from a single-time soliton we cannot create directly a multitime PDE and consequently, no multitime soliton. Indeed, taking into account the single-time PDE and the substitution $\tau = v_\alpha t^\alpha$, we find a system of PDEs, but not the multitime nonlinear PDE stated in this paper.

The interaction of two uni-temporal or bi-temporal solitons is reduced to the interaction of the corresponding profiles via progressive graphs with respect to uni-time or bi-time.

Let us consider an example of bi-time soliton, described by the speed vector $v = (v_1, v_2)$,

$$u_1(x,t) = 3\,||v||\ \mathrm{sech}^2\ (a(x - v_\alpha t^\alpha))\,.$$

There are three typical sizes in a bi-time soliton, amplitude $3\,||v||$, width coefficient a and speed vector $v = (v_1, v_2)$ related to each other, and this shows the following properties of bi-temporal soliton: (i) amplitude is proportional to the length of speed vector; (ii) amplitude is inversely proportional to the square of the width coefficient of the bi-wave; (iii) bi-temporal soliton moving back and forth on Ox_+-axis (physically, solution makes sense only for positive components of the speed vector; we imagine the cartoon "Tom and Jerry", when Jerry, the mouse, is running under a carpet).

The simplest interaction of two bi-temporal solitons is defined by the initial condition $u(x, t = 0)$. For example, let us take two bi-temporal solitons, described by the speed vectors $v_1 = (v_{11}, v_{12})$, $v_2 = (v_{21}, v_{22})$, i.e.,

$$u_1(x,t) = 3\,||v_1||\ \mathrm{sech}^2\ \left(a_1(x - v_{11}t^1 - v_{12}t^2)\right),$$

$$u_2(x,t) = 3\,||v_2||\ \mathrm{sech}^2\ \left(a_2(x - v_{21}t^1 - v_{22}t^2)\right).$$

Physically, the simplest interaction is described by the solution $u(x, t)$ with the profile

$$u(x, t^1 = 0, t^2 = 0) = 3\,||v_1||\ \mathrm{sech}^2\ (a_1(x - x_1)) + 3\,||v_2||\ \mathrm{sech}^2\ (a_2(x - x_2))\,.$$

To fix the interaction of two bi-temporal solitons we must give the speed vectors (v_{11}, v_{12}), (v_{21}, v_{22}), and the centers x_1, x_2, at the initial bi-time. Here is interesting the moment of collision, when nonlinearity is very expressed, and there is a shift in the phase, which is reflected in a "saddle". One observes that this "saddle" at no bi-moment does not disappear, which is a confirmation of the phase shift.

Let us underline that the sum of two solitons, $u_1(x, t) + u_2(x, t)$, is not a solution of the multitime nonlinear PDE.

If we accept the substitution $v\tau = v_1 t^1 + v_2 t^2$ and order after the parameter τ, at arbitrary initial conditions, after solitons traveling to the right, as solitary waves, appear traveling waves that spread in the opposite direction. Otherwise, since the pairs (t^1, t^2) are not ordered, the relative movement (back and forth) of the two bi-solitons is more complicated. If the pairs (t^1, t^2) evolve only in increasing sense, then, more slowly or more rapidly, we mimic the uni-temporal movement.

Acknowledgements. This work was partially supported by University Politehnica of Bucharest and by Academy of Romanian Scientists - Bucharest, Romania. We would also like to thank Prof. Dr. Ionel Ţevy for checking the equations and improving the theoretical part.

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