

Induced and intrinsic Hashiguchi connections on Finsler submanifolds

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Abstract. We investigate the geometry of Finsler submanifolds using the pull-back approach. We define the Finsler normal pull-back bundle and obtain the induced geometric objects, namely, induced pull-back Finsler connections, normal pull-back Finsler connections, second fundamental form and shape operator. Some characteristic theorems on induced and intrinsic Hashiguchi connections are obtained. Under a certain condition, we prove that induced and intrinsic Hashiguchi connections coincide on the pull-back bundle of Finsler submanifolds.

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Key words: Hashiguchi connection; Finslerian submanifold; pull-back Finsler connection.

1 Introduction

Let $(\widetilde{M}, \widetilde{F})$ be a Finsler manifold and $T\widetilde{M}_0$ be its slit tangent bundle. There exist in the literature, several frameworks for the study of Finsler geometry. For example an approach through the double tangent bundle $TT\widetilde{M}_0$ (Grifone's approach, [5]), an approach via the vertical subbundle of $TT\widetilde{M}_0$ (see Bejancu-Farran [3]) and the pull-back bundle approach (see Bao-Chern-Shen [2]). The latter is for us the most natural approach, because it facilitates the analogy with the Riemannian geometry. In [3] Bejancu and Farran described the theory of Finsler submanifold via the vertical bundle and applied their study to some induced geometric objects as connections and curvatures. The study was applied for the induced and intrinsic Cartan, Chern and Berwald connections, but in Finsler geometry there is also another connection, namely, Hashiguchi connection which also deserves to be studied. The purpose of this paper is to suggest under the pull-back approach in Finsler submanifolds, a comparison between the induced and the intrinsic Hashiguchi connections on Finsler submanifolds.

The paper is organized as follows. In Section 2, we recall some basic definitions and concepts that are used throughout the paper (see [3], [6] and [7] for more details). In Section 3, we construct the Finsler normal and tangential pull-back bundle

and obtain some induced geometric objects, namely, induced and intrinsic Finsler-Ehresmann connections, pull-back Finsler connections, second fundamental form and shape operator. Finally, the Section 4 is devoted to the comparison between the induced and the intrinsic Hashiguchi connections on Finsler submanifolds.

2 Preliminaries

Let $\pi : TM \rightarrow M$ be a tangent bundle of a connected smooth Finsler manifold M of dimension m . We denote by $v = (x, y)$ the point in TM if $y \in \pi^{-1}(x) = T_x M$. We denote by $O(M)$ the zero section of TM , and by TM_0 the slit tangent bundle $TM \setminus O(M)$. We introduce a coordinate system on TM as follows. Let $U \subset M$ be an open set with local coordinate (x^1, \dots, x^m) . By setting $v = y^i \frac{\partial}{\partial x^i}$, for every $v \in \pi^{-1}(U)$, we introduce a local coordinate $(x, y) = (x^1, \dots, x^m, y^1, \dots, y^m)$ on $\pi^{-1}(U)$.

Definition 2.1. A function $F : TM \rightarrow [0, +\infty)$ is called a Finsler structure or Finsler metric on M if:

- (i) $F \in C^\infty(TM_0)$,
- (ii) $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$,
- (iii) The $m \times m$ Hessian matrix (g_{ij}) , where

$$(2.1) \quad g_{ij} := \frac{1}{2}(F^2)_{y^i y^j},$$

is positive-definite at all (x, y) of TM_0 .

The pair (M, F) is called *Finsler manifold*. The pull-back bundle π^*TM is a vector bundle over the slit tangent bundle TM_0 defined by

$$(2.2) \quad \pi^*TM := \{(x, y, v) \in TM_0 \times TM : v \in T_{\pi(x, y)}M\}.$$

By (2.1), the pull-back vector bundle π^*TM admits a natural Riemannian metric

$$(2.3) \quad g := g_{ij} dx^i \otimes dx^j.$$

This is, in general, called the *fundamental tensor* (see [2] for more details). Likewise, there are some Finslerian tensors which play important roles in the Finslerian geometry, namely, the distinguished section

$$(2.4) \quad l = \frac{y^i}{F} \frac{\partial}{\partial x^i},$$

and the Cartan tensor given by

$$(2.5) \quad A = A_{ijk} dx^i \otimes dx^j \otimes dx^k,$$

where

$$(2.6) \quad A_{ijk} := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

Note that, with a slight abuse of notation, $\frac{\partial}{\partial x^i}$ and dx^i are regarded as sections of π^*TM and π^*T^*M , respectively.

Now, for the differential π_* of the submersion $\pi : TM_0 \rightarrow M$, the vertical subbundle \mathcal{V} of TTM_0 is defined by $\mathcal{V} = \ker \pi_*$, and \mathcal{V} is locally spanned by $\{F \frac{\partial}{\partial y^1}, \dots, F \frac{\partial}{\partial y^n}\}$ on each $\pi^{-1}(U)$. Then, it induces the exact sequence

$$(2.7) \quad 0 \longrightarrow \mathcal{V} \xrightarrow{i} TTM_0 \xrightarrow{\pi_*} \pi^*TM \longrightarrow 0.$$

The horizontal subbundle \mathcal{H} is defined by a subbundle $\mathcal{H} \subset TTM_0$, which is complementary to \mathcal{V} . These subbundles give a smooth splitting

$$(2.8) \quad TTM_0 = \mathcal{H} \oplus \mathcal{V}.$$

Although the vertical subbundle \mathcal{V} is uniquely determined, the horizontal subbundle is not canonically determined. An Ehresmann connection of the submersion $\pi : TM_0 \rightarrow M$ depends on a choice of horizontal subbundles.

In this paper, we shall consider the choice of Ehresmann connection which arises from the Finsler structure F , constructed as follows. Recall that [9] every Finslerian structure F induces a spray

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

in which the spray coefficients G^i are defined by

$$(2.9) \quad G^i(x, y) := \frac{1}{4} g^{il} \left[2 \frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y) \right] y^j y^k,$$

where the matrix (g^{ij}) means the inverse of (g_{ij}) .

Define a π^*TM -valued smooth form on TM_0 by

$$(2.10) \quad \theta = \frac{\partial}{\partial x^i} \otimes \frac{1}{F} (dy^i + N_j^i dx^j),$$

where functions $N_j^i(x, y)$ are given by

$$N_j^i(x, y) := \frac{\partial G^i}{\partial y^j}(x, y).$$

This π^*TM -valued smooth form θ is globally well defined on TM_0 [4].

From the form θ defined in (2.10) which is called *Finsler-Ehresmann form*, we define the *Finsler-Ehresmann connection* as follows.

Definition 2.2. A Finsler-Ehresmann connection of the submersion $\pi : TM_0 \rightarrow M$ is the subbundle \mathcal{H} of TTM_0 given by $\mathcal{H} = \ker \theta$, where $\theta : TTM_0 \rightarrow \pi^*TM$ is the bundle morphism defined in (2.10), and which is complementary to the vertical subbundle \mathcal{V} .

It is well known that, π^*TM can be naturally identified with the horizontal subbundle \mathcal{H} and the vertical one \mathcal{V} [1]. Thus, any section \bar{X} of π^*TM is considered as

a section of \mathcal{H} or a section of \mathcal{V} . We denote by \overline{X}^H and \overline{X}^V , respectively, the section of \mathcal{H} and the section of \mathcal{V} corresponding to $\overline{X} \in \Gamma(\pi^*TM)$:

$$(2.11) \quad \overline{X} = \frac{\partial}{\partial x^i} \otimes \overline{X}^i \in \pi^*TM \iff \overline{X}^H = \frac{\delta}{\delta x^i} \otimes \overline{X}^i \in \Gamma(\mathcal{H}),$$

and

$$(2.12) \quad \overline{X} = \frac{\partial}{\partial x^i} \otimes \overline{X}^i \in \pi^*TM \iff \overline{X}^V = F \frac{\partial}{\partial y^i} \otimes \overline{X}^i \in \Gamma(\mathcal{V}),$$

where

$$\left\{ F \frac{\partial}{\partial y^i} := \left(\frac{\partial}{\partial x^i} \right)^V \right\}_{i=1, \dots, m} \quad \text{and} \quad \left\{ \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_j^i \frac{\partial}{\partial y^j} = \left(\frac{\partial}{\partial x^i} \right)^H \right\}_{i=1, \dots, m},$$

are the vertical and horizontal lifts of natural local frame field $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m} \right\}$ with respect to the Finsler-Ehresmann connection \mathcal{H} , respectively.

Let $\{dx^1, \dots, dx^m\}$ be the basis of the dual space \mathcal{H}^* , and $\left\{ \frac{\delta y^i}{F} := \frac{1}{F}(dy^i + N_j^i dx^j) \right\}_{i=1, \dots, m}$ the basis of \mathcal{V}^* . For two bundle morphisms π_* and θ from TTM_0 onto π^*TM , we have the following.

Proposition 2.1. [1] *The bundle morphism π_* and θ satisfy*

$$\pi_*(\overline{X}^H) = \overline{X}, \quad \pi_*(\overline{X}^V) = 0, \quad \theta(\overline{X}^H) = 0, \quad \theta(\overline{X}^V) = \overline{X},$$

for every $\overline{X} \in \Gamma(\pi^*TM)$.

The Proposition 2.1 means that $\mathcal{H}TM_0$, as well as $\mathcal{V}TM_0$, can be naturally identified with the bundle π^*TM , that is,

$$(2.13) \quad \mathcal{H}TM_0 \cong \pi^*TM \quad \text{and} \quad \mathcal{V}TM_0 \cong \pi^*TM.$$

3 Pulled-back bundle Finsler connections

Now, by the Finsler-Ehresmann connection and any linear connection on the pull-back bundle π^*TM , we introduce the concept of Pulled-back bundle Finsler connection.

Definition 3.1. Let (M, F) be a Finsler manifold and π^*TM the pull-back bundle over TM_0 . Suppose that there exist a linear connection ∇ on π^*TM and Finsler-Ehresmann form θ on TM_0 . Then the pull-back bundle Finsler connection on π^*TM is the pair $(\ker \theta, \nabla)$.

Note that all outstanding connections of the Finsler geometry, namely [6]: Cartan, Berwald, Chern and Hashiguchi connections on π^*TM are the pull-back bundle Finsler connections.

3.1 Normal and tangential Finsler Pulled-Back Bundles

Let $(\widetilde{M}, \widetilde{F})$ be a real $(m+n)$ -dimensional Finsler manifold. By (2.3), there exist a Riemannian metric \widetilde{g} on the pull-back bundle $\pi^*T\widetilde{M}$, whose local components are given by

$$(3.1) \quad \widetilde{g}_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \widetilde{F}^2}{\partial y^i \partial y^j}.$$

The local coordinates on $T\widetilde{M}_0$ will be (x^i, y^i) , $i \in \{1, \dots, m+n\}$, where (x^i) are the local coordinates on \widetilde{M} .

In the following, we use the ranges for indices: $i, j, k, \dots \in \{1, \dots, m+n\}$; $\alpha, \beta, \gamma, \dots \in \{1, \dots, m\}$; $a, b, c, \dots \in \{m+1, \dots, m+n\}$.

Suppose that M is a real m -dimensional submanifold of \widetilde{M} defined by the equations

$$(3.2) \quad x^i = x^i(u^1, \dots, u^m); \quad \text{rank}[B_\alpha^i] = m; \quad B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}.$$

Denote by ι the immersion of M in \widetilde{M} and consider the tangent map ι_* of TM_0 in $T\widetilde{M}_0$. Locally, a point of TM_0 with coordinates (u^α, v^α) is carried by ι_* into a point of $T\widetilde{M}_0$ with coordinates $(x^i(u), y^i(u, v))$, where x^i are function in (3.2) and

$$(3.3) \quad y^i(u, v) = B_\alpha^i v^\alpha.$$

Recall that the sections $\frac{\partial}{\partial x^i}$ and dx^i of $T\widetilde{M}$ and his dual $T^*\widetilde{M}$ give rise to sections of the pull-back bundles [2]. In order to keep the notation simple, we also use the symbols $\frac{\partial}{\partial x^i}$ and dx^i to denote the basis sections of $\pi^*T\widetilde{M}$ and $\pi^*T^*\widetilde{M}$.

Note that, the pull-back bundle π^*TM is a vector subbundle of $\pi^*T\widetilde{M}|_{TM_0}$. In fact for all $(u, v) \in TM_0$, $\pi^*TM|_{(u, v)}$ is a vector subspace of $\pi^*T\widetilde{M}|_{(u, v)}$. Furthermore, the relation

$$(3.4) \quad \frac{\partial}{\partial u^\alpha} = B_\alpha^i \frac{\partial}{\partial x^i},$$

between the basis of $T_u M$ and that of $T_u \widetilde{M}$ induces by natural lift, a similar relation between the basis sections $\frac{\partial}{\partial u^\alpha}$ and $\frac{\partial}{\partial x^i}$ of π^*TM and $\pi^*T\widetilde{M}|_{TM_0}$, respectively. Hence the Riemannian metric \widetilde{g} induces a Riemannian metric g on π^*TM . More precisely, g is locally defined by

$$(3.5) \quad g_{\alpha\beta}(u, v) = B_\alpha^i B_\beta^j \widetilde{g}_{ij}(x(u), y(u, v)).$$

On the other hand, the Finsler structure \widetilde{F} of \widetilde{M} induces on TM_0 the function F locally given by

$$F(u, v) = \widetilde{F}(x(u), y(u, v)).$$

Then by straightforward calculations it follows that (M, F) is a Finsler manifold and the fundamental tensor of F is given by

$$(3.6) \quad g_{\alpha\beta} = \frac{1}{2} \frac{\partial^2 F^2}{\partial v^\alpha \partial v^\beta}.$$

Let denote by π^*TM^\perp the orthogonal complementary pull-back bundle of π^*TM in $\pi^*\widetilde{TM}|_{TM_0}$ with respect to \widetilde{g} . The bundle π^*TM^\perp is called normal pull-back bundle of Finsler submanifold (M, F) in $(\widetilde{M}, \widetilde{F})$. So we have the orthogonal decomposition

$$(3.7) \quad \pi^*\widetilde{TM}|_{TM_0} = \pi^*TM \oplus \pi^*TM^\perp.$$

Now, we consider local sections of orthonormal basis $\{\mathfrak{N}_a = \mathfrak{N}_a^i \frac{\partial}{\partial x^i}\}$ of π^*TM^\perp with respect to \widetilde{g} , where \mathfrak{N}_a^i are the functions on TM_0 , satisfying the following relations:

$$(3.8) \quad \widetilde{g}_{ij} B_\alpha^i \mathfrak{N}_a^j = 0; \quad \forall \alpha \in \{1, \dots, m\}, \quad a \in \{m+1, \dots, m+n\},$$

and

$$(3.9) \quad \widetilde{g}_{ij} \mathfrak{N}_a^i \mathfrak{N}_b^j = \delta_{ab}; \quad \forall a, b \in \{m+1, \dots, m+n\}.$$

Denote by $[B_\alpha^i \mathfrak{N}_a^i]$ the transition matrix from $\{\frac{\partial}{\partial x^i}\}$ to $\{\frac{\partial}{\partial u^\alpha}, \mathfrak{N}_a\}$ and, $[\widetilde{B}_i^\alpha \widetilde{\mathfrak{N}}_i^a]$ his inverse. We have

$$(3.10) \quad \widetilde{B}_i^\alpha B_\beta^i = \delta_\beta^\alpha; \quad \widetilde{B}_i^\alpha \mathfrak{N}_a^i = 0, \quad \widetilde{\mathfrak{N}}_i^a B_\alpha^i = 0, \quad \widetilde{\mathfrak{N}}_i^a \mathfrak{N}_b^i = \delta_b^a,$$

$$(3.11) \quad \text{and } B_\alpha^i \widetilde{B}_j^\alpha + \widetilde{\mathfrak{N}}_j^a \mathfrak{N}_a^i = \delta_j^i.$$

In the sequel we use the notations: $B_{\alpha\beta\dots}^{ij\dots} = B_\alpha^i B_\beta^j \dots$ and $B_{\alpha 0}^i = B_{\alpha\beta}^i v^\beta$.

3.2 Induced and Intrinsic Finsler-Ehresmann connection

Recall that the kernel of $\widetilde{\theta}$ represents the Finsler-Ehresmann connection, and $V\widetilde{TM}_0$ its complementary distribution in $T\widetilde{TM}_0$, orthogonal with respect to the Sasaki-type metric \widetilde{G}_s on $T\widetilde{M}_0$ given by

$$(3.12) \quad \widetilde{G}_s = \widetilde{g}_{ij} dx^i \otimes dx^j + \widetilde{g}_{ij} \frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}.$$

By \widetilde{G}_s , we obtain the induced Finsler-Ehresmann form θ defined by

$$\theta := \frac{\delta v^\alpha}{F} \otimes \frac{\partial}{\partial u^\alpha},$$

where $\frac{\delta v^\alpha}{F} = \frac{1}{F}(dv^\alpha + N_\beta^\alpha du^\beta)$ and N_β^α is related to \widetilde{N}_j^i by (see [8], [3]):

$$(3.13) \quad N_\beta^\alpha = \widetilde{B}_i^\alpha (B_{\beta 0}^i + B_\beta^j \widetilde{N}_j^i).$$

On the other hand, the coefficients N_β^α has an intrinsic analogous \hat{N}_β^α , obtained by the spray coefficients of F on M , and related to N_β^α , by

$$(3.14) \quad \hat{N}_\beta^\alpha = N_\beta^\alpha + \frac{A_{\beta a}^\alpha}{F} H_\lambda^a v^\lambda,$$

where the functions H_λ^a and $A_{\beta a}^\alpha$ are given, respectively, by

$$(3.15) \quad H_\lambda^a = \widetilde{\mathfrak{N}}_k^a (B_{\lambda 0}^k + B_\lambda^i \widetilde{N}_i^k) \quad \text{and} \quad A_{\beta a}^\alpha = g^{\alpha\lambda} A_{\lambda\beta a}.$$

By (3.14), we obtain the intrinsic Finsler-Ehresmann π^*TM -valued form $\hat{\theta}$ given by

$$(3.16) \quad \hat{\theta} = \frac{1}{F}(dv^\alpha + \hat{N}_\beta^\alpha du^\beta) \otimes \frac{\partial}{\partial u^\alpha}.$$

Proposition 3.1. *Let $\tilde{\theta}$ be the Finsler-Ehresmann $\pi^*T\tilde{M}$ -valued form on $T\tilde{M}_0$. Then the induced Finsler-Ehresmann π^*TM -valued form θ coincides with θ on TM_0 . Moreover the intrinsic Finsler-Ehresmann form $\hat{\theta}$, and the induced one θ are related by*

$$(3.17) \quad \hat{\theta} = \theta + \frac{D}{F^2},$$

where D is the $(0, 1; 1)$ -tensor called deformation tensor and given by

$$(3.18) \quad D = D_{\beta}^{\alpha} du^{\beta} \otimes \frac{\partial}{\partial u^{\alpha}} = H_{\lambda}^{\alpha} v^{\lambda} A_{\beta\alpha}^{\alpha} du^{\beta} \otimes \frac{\partial}{\partial u^{\alpha}}.$$

Proof. We have

$$(3.19) \quad \begin{aligned} \frac{\delta y^i}{\tilde{F}} &= \frac{1}{\tilde{F}} (dy^i - \tilde{N}_j^i dx^j) = \frac{1}{\tilde{F}} \left(B_{\alpha 0}^i du^{\alpha} + B_{\alpha}^i dv^{\alpha} + \tilde{N}_j^i B_{\alpha}^j du^{\alpha} \right) \\ &= \frac{1}{\tilde{F}} (B_{\beta}^i N_{\alpha}^{\beta} du^{\alpha} + B_{\beta}^i dv^{\beta}) = B_{\beta}^i \frac{\delta v^{\beta}}{F} \end{aligned}$$

Since $\frac{\partial}{\partial x^i} = \tilde{B}_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \tilde{\mathfrak{N}}_i^{\alpha} \mathfrak{N}_{\alpha}$ are sections of $\pi^*T\tilde{M}|_{TM_0}$ and $B_{\beta}^i \tilde{\mathfrak{N}}_i^{\alpha} = 0$, it follows that $\tilde{\theta} = \theta$ on TM_0 . Also, we have

$$(3.20) \quad \begin{aligned} \hat{\theta} &= \frac{1}{F} (dv^{\alpha} + \hat{N}_{\beta}^{\alpha} du^{\beta}) \otimes \frac{\partial}{\partial u^{\alpha}} \\ &= \frac{1}{F} \left(dv^{\alpha} + (N_{\beta}^{\alpha} + \frac{A_{\beta\alpha}^{\alpha}}{F} H_{\lambda}^{\alpha} v^{\lambda}) du^{\beta} \right) \otimes \frac{\partial}{\partial u^{\alpha}} = \theta + \frac{D}{F^2}, \end{aligned}$$

which completes the proof. \square

Note that the corresponding horizontal section l^H of the distinguished section l defined in (2.4) is given by $l^H = l^{\alpha} \frac{\delta}{\delta u^{\alpha}}$. Therefore, we have the following.

Lemma 3.2. *The action of the deformation $(0, 1; 1)$ -tensor D defined in (3.18) on l^H vanishes, that is $D(l^H) = 0$.*

Proof. The proof follows from a direct calculations using the fact that the Cartan tensor A vanishes along the distinguished section l . \square

Let us denote by $\tilde{\theta}$ the Finsler-Ehresmann form on $V\tilde{M}_0$. It is easy to see that $\tilde{\theta}$ is a bundle isomorphism of $V\tilde{M}_0$ onto $\pi^*T\tilde{M}$ (see [1] for more details). Therefore, we have the following result.

Lemma 3.3. *Let (M, F) be a Finsler submanifold of (\tilde{M}, \tilde{F}) , $HT\tilde{M}_0|_{TM_0}$ be a Finsler-Ehresmann connection, and π^*TM^{\perp} be the normal pull-back bundle on TM_0 . Then the induced Finsler-Ehresmann connection HTM_0 is a vector subbundle of $HT\tilde{M}_0|_{TM_0} \oplus \tilde{\theta}^{-1}(\pi^*TM^{\perp})$.*

Proof. Let $\tilde{\theta}^{-1}$ the inverse of $\tilde{\theta}$, we have

$$(3.21) \quad \begin{aligned} T\tilde{M}_0|_{TM_0} &= HT\tilde{M}_0|_{TM_0} \oplus VT\tilde{M}_0|_{TM_0} \\ &= HT\tilde{M}_0|_{TM_0} \oplus \tilde{\theta}^{-1}(\pi^*T\tilde{M}|_{TM_0}) \\ &= HT\tilde{M}_0|_{TM_0} \oplus \tilde{\theta}^{-1}(\pi^*TM) \oplus \tilde{\theta}^{-1}(\pi^*TM^{\perp}). \end{aligned}$$

Now, using the Proposition 3.1 we have $\tilde{\theta}^{-1}(\pi^*TM) = \theta^{-1}(\pi^*TM) = VTM_0$. It follows that $HT\tilde{M}_0|_{TM_0} \oplus \tilde{\theta}^{-1}(\pi^*TM^\perp)$ is the orthogonal complementary vector bundle to VTM_0 , in $TT\tilde{M}_0|_{TM_0}$, and HTM_0 is orthogonal to VTM_0 . We deduce that HTM_0 is a vector subbundle of $HT\tilde{M}_0|_{TM_0} \oplus \tilde{\theta}^{-1}(\pi^*TM^\perp)$, hence the result. \square

Definition 3.2. Let HTM_0 be a Finsler-Ehresmann connection on the Finsler manifold (M, F) , and π^*TM the pull-back bundle over the slit tangent bundle TM_0 . The pull-back Finsler connection is the pair (HTM_0, ∇) , where ∇ is a linear connection on π^*TM .

Considering the induced Finsler-Ehresmann connection on \tilde{M} , we proceed with the study of the geometric objects induced by $\tilde{\nabla}$ on π^*TM . Then according to the orthogonal decomposition (3.7), the Gauss and Weingarten formulas are given by

$$(3.22) \quad \tilde{\nabla}_X \xi = \nabla_X \xi + \mathcal{S}(\xi, X),$$

$$(3.23) \quad \tilde{\nabla}_X \eta = -\mathcal{A}_\eta X + \nabla_X^\perp \eta,$$

where, $X \in \Gamma(TTM_0)$, $\xi \in \Gamma(\pi^*TM)$, $\eta \in \Gamma(\pi^*TM^\perp)$, $\mathcal{S}(\xi, X) \in \Gamma(\pi^*TM^\perp)$ and $\mathcal{A}_\eta X \in \Gamma(\pi^*TM)$. Here \mathcal{A}_η represents the shape operator and \mathcal{S} the second fundamental form of π^*TM , and these are the Finslerian tensors of type $(0, 1; 1)$ and $(1, 1; 1)$, respectively. It is easy to check that, ∇ and ∇^\perp are respectively the linear connection on π^*TM and π^*TM^\perp . Thus, with the induced Finsler-Ehresmann connection HTM_0 and the linear connection $\tilde{\nabla}$, we can define pull-back Finsler connections on π^*TM and π^*TM^\perp . The one on π^*TM will be called the induced pull-back Finsler connection and the another one the induced normal pull-back Finsler connection.

Now, for any $X \in \Gamma(TTM_0)$, we define the differential operator,

$$(3.24) \quad \begin{aligned} \bar{\nabla}_X : \Gamma(\pi^*T\tilde{M}|_{TM_0}) &\longrightarrow \Gamma(\pi^*T\tilde{M}|_{TM_0}) \\ \xi &\longmapsto \bar{\nabla}_X \xi := \tilde{\nabla}_X \xi. \end{aligned}$$

Clearly, $\bar{\nabla}$ is a linear connection on $\pi^*T\tilde{M}|_{TM_0}$. So $(HTM_0, \bar{\nabla})$ defines the restriction of the pull-back Finsler connection $\tilde{\nabla}$ on $\pi^*T\tilde{M}|_{TM_0}$.

We now consider the local coefficients of $\tilde{\nabla}$, ∇ and ∇^\perp given respectively by

$$(3.25) \quad \tilde{\nabla}_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial x^i} = \tilde{\Gamma}_{ij}^k \frac{\partial}{\partial x^k}, \quad \tilde{\nabla}_{F \frac{\partial}{\partial y^j}} \frac{\partial}{\partial x^i} = \tilde{\gamma}_{ij}^k \frac{\partial}{\partial x^k},$$

$$(3.26) \quad \nabla_{\frac{\delta}{\delta u^\beta}} \frac{\partial}{\partial u^\alpha} = \Gamma_{\alpha\beta}^\lambda \frac{\partial}{\partial u^\lambda}, \quad \nabla_{F \frac{\partial}{\partial v^\beta}} \frac{\partial}{\partial u^\alpha} = \gamma_{\alpha\beta}^\lambda \frac{\partial}{\partial u^\lambda},$$

and

$$(3.27) \quad \nabla_{\frac{\delta}{\delta u^\alpha}}^\perp \mathfrak{N}_a = \Gamma_{a\alpha}^b \mathfrak{N}_b, \quad \nabla_{F \frac{\partial}{\partial v^\alpha}}^\perp \mathfrak{N}_a = \gamma_{a\alpha}^b \mathfrak{N}_b,$$

where $\tilde{\Gamma}_{ij}^k$ and $\tilde{\gamma}_{ij}^k$ are the ‘‘Christoffel symbols’’ with respect to $\frac{\delta}{\delta x^i}$ and $F \frac{\partial}{\partial y^i}$, respectively and given by

$$(3.28) \quad \tilde{\Gamma}_{ij}^k = \frac{\tilde{g}^{is}}{2} \left(\frac{\delta \tilde{g}_{sj}}{\delta x^k} - \frac{\delta \tilde{g}_{jk}}{\delta x^s} + \frac{\delta \tilde{g}_{ks}}{\delta x^j} \right),$$

$$(3.29) \quad \text{and } \tilde{\gamma}_{ij}^k = \frac{\tilde{F}}{2} \tilde{g}^{ks} \frac{\partial \tilde{g}_{ij}}{\partial y^j}.$$

We also define locally, the horizontal and vertical part of second fundamental form respectively by

$$(3.30) \quad \mathcal{S}\left(\frac{\delta}{\delta u^\alpha}, \frac{\partial}{\partial u^\beta}\right) = \mathcal{S}_{\alpha\beta}^h \mathfrak{N}_a \quad \text{and} \quad \mathcal{S}\left(F \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}\right) = \mathcal{S}_{\alpha\beta}^v \mathfrak{N}_a.$$

Likewise, the horizontal and vertical part of shape operator with respect to the normal section \mathfrak{N}_a are given, respectively, by

$$(3.31) \quad \mathcal{A}(\mathfrak{N}_a, \frac{\delta}{\delta u^\alpha}) = \mathcal{A}_{\alpha\beta}^h \frac{\partial}{\partial u^\alpha} \quad \text{and} \quad \mathcal{A}(\mathfrak{N}_a, F \frac{\partial}{\partial v^\alpha}) = \mathcal{A}_{\alpha\beta}^v \frac{\partial}{\partial u^\alpha}.$$

By Lemma 3.3, it is easy to check that the local coefficients of $\bar{\nabla}$ and $\tilde{\nabla}$ are related by

$$(3.32) \quad \bar{\Gamma}_{j\alpha}^i = B_\alpha^k \tilde{\Gamma}_{jk}^i + H_\alpha^a \mathfrak{N}_a^k \tilde{\gamma}_{jk}^i,$$

$$(3.33) \quad \bar{\gamma}_{j\alpha}^i = \tilde{\gamma}_{jk}^i B_\alpha^k,$$

where H_α^a is given by (3.15).

The local coefficients of the induced pull-back Finsler connection ∇ are given in terms of the local coefficients of $\tilde{\nabla}$ by [3]

$$(3.34) \quad \Gamma_{\alpha\beta}^\lambda = \tilde{B}_i^\lambda \left(B_{\alpha\beta}^i + \tilde{\Gamma}_{jk}^i B_{\alpha\beta}^{jk} + \tilde{\gamma}_{jk}^i B_\alpha^j H_\beta^a \mathfrak{N}_a^k \right),$$

$$(3.35) \quad \gamma_{\alpha\beta}^\lambda = \tilde{B}_i^\lambda \tilde{\gamma}_{jk}^i B_{\alpha\beta}^{jk}.$$

Likewise, the local coefficients of ∇^\perp are given in function of the local coefficients of $\tilde{\nabla}$ by:

$$(3.36) \quad \Gamma_{a\alpha}^b = \left(\frac{\delta B_a^k}{\delta u^\alpha} + B_a^j B_\alpha^k \tilde{\Gamma}_{jk}^i + B_a^j H_\alpha^a \mathfrak{N}_a^k \tilde{\gamma}_{jk}^i \right) \tilde{\mathfrak{N}}_i^b,$$

$$(3.37) \quad \gamma_{a\alpha}^b = \left(F \frac{\partial B_a^k}{\partial v^\alpha} + B_a^j \tilde{\gamma}_{jk}^i B_\alpha^k \right) \tilde{\mathfrak{N}}_i^b.$$

Furthermore, the local components of the horizontal and vertical part of second fundamental form are given respectively by

$$(3.38) \quad \mathcal{S}_{\alpha\beta}^h = \tilde{\mathfrak{N}}_i^a \left(B_{\alpha\beta}^i + \tilde{\Gamma}_{jk}^i B_{\alpha\beta}^{jk} + \tilde{\gamma}_{jk}^i B_\alpha^j H_\beta^a \mathfrak{N}_a^k \right),$$

$$(3.39) \quad \text{and} \quad \mathcal{S}_{\alpha\beta}^v = \tilde{\mathfrak{N}}_i^a \tilde{\gamma}_{jk}^i B_{\alpha\beta}^{jk}.$$

Finally the local components of the horizontal and vertical part of shape operator are given respectively by:

$$(3.40) \quad \mathcal{A}_{\alpha\alpha}^h = - \left(\frac{\delta B_a^k}{\delta u^\alpha} + B_a^j B_\alpha^k \tilde{\Gamma}_{jk}^i + B_a^j H_\alpha^a \mathfrak{N}_a^k \tilde{\gamma}_{jk}^i \right) \tilde{B}_i^\lambda,$$

$$(3.41) \quad \text{and} \quad \mathcal{A}_{\alpha\alpha}^v = - \left(F \frac{\partial B_a^k}{\partial v^\alpha} + B_a^j \tilde{\gamma}_{jk}^i B_\alpha^k \right) \tilde{B}_i^\lambda.$$

4 Induced and intrinsic Hashiguchi connection

As an application to the general theory of pull-back Finsler connection developed in the previous section, we consider in the following the ambient manifold $(\widetilde{M}, \widetilde{F})$ endowed with a Hashiguchi connection, and study the induced and intrinsic ones in (M, F) .

Denote by $\overset{H}{\nabla}$ the Hashiguchi connection on the pull-back bundle $\pi^*T\widetilde{M}$, given locally by:

$$(4.1) \quad \overset{H}{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial x^j} = \widetilde{\mathfrak{H}}_{ij}^k \frac{\partial}{\partial x^k} \quad \text{and} \quad \overset{H}{\nabla}_{F \frac{\partial}{\partial y^i}} \frac{\partial}{\partial x^j} = \widetilde{\mathfrak{h}}_{ij}^k \frac{\partial}{\partial x^k}.$$

Recall that $\widetilde{\mathfrak{H}}_{ij}^k$ and $\widetilde{\mathfrak{h}}_{ij}^k$ are given, respectively, by [6]:

$$(4.2) \quad \widetilde{\mathfrak{H}}_{ij}^k = \widetilde{\Gamma}_{ij}^k + \widetilde{L}_{ij}^k \quad \text{and} \quad \widetilde{\mathfrak{h}}_{ij}^k = \widetilde{\gamma}_{ij}^k,$$

where \widetilde{L}_{ij}^k are the coefficients of Landsberg tensor with respect to \widetilde{g} (see [9] for more details).

Lemma 4.1. *Let $(\widetilde{M}, \widetilde{F})$ be a Finsler manifold and \widetilde{L} be the Landsberg tensor with respect to the pull-back bundle $\pi^*T\widetilde{M}$ on $(\widetilde{M}, \widetilde{F})$. Then the components of the restriction \bar{L} of \widetilde{L} to $\pi^*T\widetilde{M}|_{TM_0}$ are given locally by*

$$\bar{L}_{ij\alpha} = B_\alpha^k \widetilde{L}_{ijk} - H_\alpha^a \mathfrak{N}_a^k \widetilde{\gamma}_{ijk}.$$

Proof. Recall that the coefficients of Landsberg tensor \widetilde{L}_{ijk} are given by the horizontal covariant derivatives of \widetilde{g}_{ij} denoted $\widetilde{g}_{ij;k}$, more precisely $\widetilde{L}_{ijk} := -\frac{1}{2}\widetilde{g}_{ij;k}$ (see [9]). By lemma 3.3 we have $\frac{\delta}{\delta u^\alpha} \in HT\widetilde{M}_0|_{TM_0} \oplus \widetilde{\theta}^{-1}(\pi^*TM^\perp)$, and denoting the vertical correspondent of \mathfrak{N}_a by $\overset{V}{\mathfrak{N}}_a := \widetilde{\theta}^{-1}(\mathfrak{N}_a)$, and using (3.13) and (3.11), we obtain

$$\begin{aligned} \frac{\delta}{\delta u^\alpha} &= B_\alpha^i \frac{\delta}{\delta u^i} + \overset{V}{\mathfrak{N}}_a^i (B_{\alpha 0}^i + B_\alpha^j \widetilde{N}_j^i) \overset{V}{\mathfrak{N}}_a \\ &= B_\alpha^i \frac{\delta}{\delta u^i} + H_\alpha^a \overset{V}{\mathfrak{N}}_a, \end{aligned}$$

where $H_\alpha^a = \overset{V}{\mathfrak{N}}_a^i (B_{\alpha 0}^i + B_\alpha^j \widetilde{N}_j^i)$. Hence, one has

$$\begin{aligned} \bar{L}_{ij\alpha} &= -\frac{1}{2}\widetilde{g}_{ij;\alpha} = -\frac{1}{2}B_\alpha^k \widetilde{g}_{ij;k} - \frac{1}{2}\overset{V}{\mathfrak{N}}_a^k H_\alpha^a \widetilde{\gamma}_{ijk} \\ &= B_\alpha^k \widetilde{L}_{ijk} - \overset{V}{\mathfrak{N}}_a^k H_\alpha^a \widetilde{\gamma}_{ijk}, \end{aligned}$$

as required. \square

Theorem 4.2. *Let $(\widetilde{M}, \widetilde{F})$ be a $(m+n)$ -dimensional Finsler manifold endowed with Hashiguchi connection $\overset{H}{\nabla}$ and (M, F) be a m -dimensional Finsler submanifold of*

$(\widetilde{M}, \widetilde{F})$. Then the local coefficients of the induced Hashiguchi connection are given by the following formulas:

$$(4.3) \quad \mathfrak{H}_{\alpha\beta}^\lambda = \left(B_{\alpha\beta}^k + B_{\alpha\beta}^{ij} \widetilde{\mathfrak{H}}_{ij}^k \right) \widetilde{B}_k^\lambda,$$

$$(4.4) \quad \text{and } \mathfrak{h}_{\alpha\beta}^\lambda = B_{\alpha\beta}^{ij} \widetilde{\mathfrak{h}}_{ij}^k \widetilde{B}_k^\lambda.$$

Proof. By lemma 4.1 we obtain $\overline{\mathfrak{H}}_{i\beta}^k = B_{\beta}^j \widetilde{\mathfrak{H}}_{ij}^k$. Writing (3.34) for the Hashiguchi connection and replacing $\overline{\mathfrak{H}}_{i\beta}^k$ by its value we obtain the relation (4.3). One has (4.4) in the similar way the relation (3.35). \square

Theorem 4.3. Let (M, F) be a m -dimensional Finsler submanifold of $(\widetilde{M}, \widetilde{F})$ endowed with Hashiguchi connection $\overset{H}{\nabla}$. Locally, the horizontal part $\mathfrak{S}_{\alpha\beta}^h$ and vertical part $\mathfrak{S}_{\alpha\beta}^v$ of second fundamental form are given by:

$$(4.5) \quad \mathfrak{S}_{\alpha\beta}^h = \left(B_{\alpha\beta}^k + B_{\alpha\beta}^{ij} \widetilde{\mathfrak{H}}_{ij}^k \right) \widetilde{\mathfrak{N}}_k^a \quad \text{and} \quad \mathfrak{S}_{\alpha\beta}^v = B_{\alpha\beta}^{ij} \widetilde{\mathfrak{h}}_{ij}^k \widetilde{\mathfrak{N}}_k^a,$$

respectively.

Proof. The proof follows from the relations (3.38) and (3.39). \square

Now we consider the normal Hashiguchi connection $\overset{H}{\nabla}^\perp$ and set

$$(4.6) \quad \overset{H}{\nabla}^\perp_{\frac{\delta}{\delta u^\alpha}} \mathfrak{N}_a = \mathfrak{H}_{a\alpha}^b \mathfrak{N}_b \quad \text{and} \quad \overset{H}{\nabla}^\perp_{F \frac{\partial}{\partial v^\alpha}} \mathfrak{N}_a = \mathfrak{h}_{a\alpha}^b \mathfrak{N}_b.$$

Theorem 4.4. Let (M, F) be a m -dimensional Finsler submanifold of Finsler manifold $(\widetilde{M}, \widetilde{F})$, endowed with Hashiguchi connection. Then the local coefficients of the normal Hashiguchi connection on $(\pi^*TM)^\perp$, are given by the following formulas:

$$(4.7) \quad \mathfrak{H}_{a\alpha}^b = \left(\frac{\delta B_a^k}{\delta u^\alpha} + B_a^j B_\alpha^k \widetilde{\mathfrak{H}}_{jk}^i \right) \widetilde{\mathfrak{N}}_i^b,$$

$$(4.8) \quad \text{and } \mathfrak{h}_{a\alpha}^b = \left(F \frac{\partial B_a^k}{\partial v^\alpha} + B_a^j B_\alpha^k \widetilde{\mathfrak{h}}_{jk}^i \right) \widetilde{\mathfrak{N}}_i^b.$$

Moreover the local coefficients of the horizontal and vertical part of shape operator are given respectively by:

$$(4.9) \quad \mathfrak{A}_{a\alpha}^h = - \left(\frac{\delta B_a^k}{\delta u^\alpha} + B_a^j B_\alpha^k \widetilde{\mathfrak{H}}_{jk}^i \right) \widetilde{B}_i^\beta,$$

$$(4.10) \quad \text{and } \mathfrak{A}_{a\alpha}^v = - \left(F \frac{\partial B_a^k}{\partial v^\alpha} + B_a^j B_\alpha^k \widetilde{\mathfrak{h}}_{jk}^i \right) \widetilde{B}_i^\beta.$$

Proof. Writing the relations (3.36), (3.37), (3.40) and (3.41) for the Hashiguchi connection and using the relation $\overline{\mathfrak{H}}_{i\beta}^k = B_{\beta}^j \widetilde{\mathfrak{H}}_{ij}^k$, the result follows. \square

In the previous paragraph, we constructed the induced Hashiguchi connection, whose local coefficients are given by (4.3) and (4.4). On the other hand, on Finsler submanifold lives the intrinsic Hashiguchi connection $\overset{H}{\nabla}^*$. More precisely, the canonical Finsler-Ehresmann connection is locally spanned by the vector fields

$$(4.11) \quad \frac{\delta^*}{\delta^* u^\alpha} = \frac{\partial}{\partial u^\alpha} - \hat{N}_\alpha^\lambda \frac{\partial}{\partial v^\lambda}.$$

Then, using (3.14) and taking into account (4.11), we derive that

$$(4.12) \quad \frac{\delta^*}{\delta^* u^\alpha} = \frac{\delta}{\delta u^\alpha} - D_\alpha^\lambda \frac{\partial}{\partial v^\lambda},$$

where D_α^λ are the coefficients of deformation tensor D . Moreover, the Hashiguchi connection is given by [6]

$$(4.13) \quad \begin{aligned} 2g(\overset{H}{\nabla}_X \pi_* Y, \pi_* Z) &= X.g(\pi_* Y, \pi_* Z) + Y.g(\pi_* Z, \pi_* X) - Z.g(\pi_* X, \pi_* Y) \\ &+ g(\pi_*[X, Y], \pi_* Z) - g(\pi_*[Y, Z], \pi_* X) + g(\pi_*[Z, X], \pi_* Y) \\ &+ 2A(\theta(Z), \pi_* X, \pi_* Y) - 2A(\theta(Y), \pi_* X, \pi_* Z) + 2L(\pi_* X, \pi_* Y, \pi_* Z). \end{aligned}$$

Then, replacing X, Y and Z from (4.13) by $\frac{\delta}{\delta u^\alpha}$, $\frac{\delta}{\delta u^\beta}$ and $\frac{\delta}{\delta u^\lambda}$ respectively, we obtain

$$(4.14) \quad 2g_{\mu\lambda} \mathfrak{H}_{\alpha\beta}^\mu = \frac{\delta}{\delta u^\alpha} g_{\beta\lambda} + \frac{\delta}{\delta u^\beta} g_{\alpha\lambda} - \frac{\delta}{\delta u^\lambda} g_{\alpha\beta} + 2L_{\alpha\beta\lambda}.$$

Theorem 4.5. *Let $(\widetilde{M}, \widetilde{F})$ be a $(m+n)$ -dimensional Finsler manifold endowed with the Hashiguchi connection $\overset{H}{\nabla}$ and (M, F) be m -dimensional Finsler submanifold of $(\widetilde{M}, \widetilde{F})$. Then the local coefficients of the induced Hashiguchi connection $(\mathfrak{H}_{\alpha\beta}^\mu, \mathfrak{h}_{\alpha\beta}^\mu)$ are related to the local coefficients of intrinsic Hashiguchi connection $(\mathfrak{H}_{\alpha\beta}^{*\mu}, \mathfrak{h}_{\alpha\beta}^{*\mu})$, by the following relations:*

$$(4.15) \quad \mathfrak{H}_{\alpha\beta}^\mu = \mathfrak{H}_{\alpha\beta}^{*\mu} + D_\alpha^\tau \widetilde{B}_k^\mu \widetilde{\mathfrak{h}}_{ij}^k B_{\tau\beta}^{ij} + D_\beta^\tau \widetilde{B}_k^\mu \widetilde{\mathfrak{h}}_{ij}^k B_{\tau\alpha}^{ij} - D_\epsilon^\mu B_{\alpha\beta}^{ij} \widetilde{\mathfrak{h}}_{ij}^k \widetilde{B}_k^\epsilon,$$

$$(4.16) \quad \mathfrak{h}_{\alpha\beta}^\mu = \mathfrak{h}_{\alpha\beta}^{*\mu} = B_{\alpha\beta}^{ij} \widetilde{\mathfrak{h}}_{ij}^k \widetilde{B}_k^\mu.$$

Proof. Contracting (4.14) by $g^{\mu\lambda}$, and using (4.12) and (4.4), the result follows. \square

Theorem 4.6. *Let $(\widetilde{M}, \widetilde{F})$ be a $(m+n)$ -dimensional Finsler manifold endowed with the Hashiguchi connection $\overset{H}{\nabla}$ and (M, F) be m -dimensional Finsler submanifold of $(\widetilde{M}, \widetilde{F})$, Then the induced Hashiguchi connection coincides with the intrinsic Hashiguchi connection on submanifold M if and only if the deformation $(0, 1; 1)$ -tensor D vanishes on TM_0 . Moreover, the covariant differentiation in the direction of the horizontal correspondent of distinguished section l^H , is the same for both Hashiguchi connections on the submanifold.*

Proof. If the induced Hashiguchi connection coincides with the intrinsic one, then the Finsler-Ehresmann form $\hat{\theta} = \theta$ and by the Proposition 3.1, the deformation tensor $D = 0$. Reciprocally, if the tensor $D = 0$ then by the relation (4.15), the induced and intrinsic Hashiguchi connection coincides. The last assertion is obtained by the Lemma 3.2. \square

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