Codimension reduction on the contact CR-submanifolds of an odd-dimensional unit sphere

E. Abedi, M. Asadollahzadeh, G. Haghighatdoost

Abstract. Let M be an $(n+1)$-dimensional contact CR-submanifold of an odd-dimensional unit sphere S^{2m+1} of $(n-q)$ contact CR-dimension. We study the condition $h(FX,Y)+h(X,FY)=0$ on the structure tensor F which is naturally induced from the almost contact structure ϕ of the ambient manifold and the second fundamental form h of the submanifold M. We obtain two results on codimension reduction for such submanifolds.

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1 Brief overview

Let $\overline{M}$ be a $(2m+1)$-dimensional Sasakian manifold with the Sasakian structure tensors (ϕ, ξ, η, g) satisfying:

$$\phi^2 X = -X + \eta(X)\xi, \ \phi\xi = 0, \ \eta(\xi) = 1, \ \eta(\phi X) = 0, \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \tag{1.2}$$

for any vector fields X and Y on $\overline{M}$ [8]. Let M be a submanifold tangent to the structure vector field ξ isometrically immersed in the Sasakian manifold $\overline{M}$. Then M is called a contact CR-submanifold of $\overline{M}$ if there exists a differentiable distribution $D : x \longrightarrow D_x \subset T_xM$ on M such that: (i) D is invariant with respect to ϕ, i.e., $\phi D_x \subset D_x$; (ii) the complementary orthogonal distribution $D^\perp : x \longrightarrow D_x^\perp \subset T_xM$ is anti-invariant with respect to ϕ, i.e., $\phi D_x^\perp \subset T_x^\perp M$, for $x \in M$.

If $\dim D = 0$, then the contact CR-submanifold M is called an *anti-invariant submanifold* of $\overline{M}$ tangent to ξ. If $\dim D^\perp = 0$, then M is an *invariant submanifold* of $\overline{M}$ [8]. Contact CR-submanifold of maximal CR-dimension in an odd-dimensional unit sphere has been studied in [5], [6] and [7].

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In the present article we study connected $(n+1)$-dimensional real submanifolds of codimension $(2m-n)$ of the odd-dimensional unit sphere S^{2m+1} which are contact CR-submanifolds of contact CR-dimension $(n-q)$, that is, $\dim D^{\perp} = q+1$.

In Section 2 we collect some basic relations concerning submanifolds, in particular we discuss the notion of contact CR-submanifolds of the Sasakian manifold S^{2m+1}.

Section 3 is devoted to the study of contact CR-submanifolds which satisfy the condition $h(FX,Y)+h(X,FY)=0$ on the structure tensor F naturally induced from the almost contact structure ϕ of the ambient manifold and on the second fundamental form h of a submanifold M. M. Djoric studied these complex space forms in [2].

Finally, in Section 4, using the codimension reduction theorem in [4], we obtain codimension reduction results for contact CR-submanifolds of an odd-dimensional unit sphere similar to that in [2] and [5].

2 Preliminaries

Let S^{2m+1} be a $(2m+1)$-unit sphere and $z \in S^{2m+1}$. We put $\xi = Jz$ where J is the complex structure of the complex $(m+1)$-space $\mathbb{C}^{m+1}$. We consider the orthogonal projection $\pi : T_z\mathbb{C}^{m+1} \to T_zS^{2m+1}$, and put $\phi = \pi \circ J$. Then we see that (ϕ, ξ, η, g) is a Sasakian structure on S^{2m+1}, where η is an 1-form dual to ξ and g is the standard metric tensor field on S^{2m+1}. Hence, S^{2m+1} can be regarded as a Sasakian manifold of constant ϕ-sectional curvature 1 [1],[9].

Consider M, an $(n+1)$-dimensional contact CR-submanifold in S^{2m+1} which is tangent to the structure vector field ξ. The subspace D_x is the ϕ-invariant subspace $T_xM \cap \phi T_xM$ of the tangent space T_xM of M at $x \in M$. Then ξ is not in D_x at any x in M. Let $D_x^{\perp}$ denote the complementary orthogonal subspace to D_x in T_xM. For any nonzero vectors U_α orthogonal to ξ and contained in $D_x^{\perp}$, we have ϕU_α normal to M. In the following we assume that $\dim D_x = n-q$ and $\dim D_x^{\perp} = q+1$, at each point x in M. We observe that the definition for a contact CR-submanifold of S^{2m+1} given in [5], states that the maximal ϕ-invariant subspace D_x has constant dimension, for any $x \in M$. For the definition given above, the subspace D_x obviously has constant dimension for any $x \in M$, since D is a distribution. When the contact CR-submanifold is of maximal CR-dimension, the two definition are equivalent. In the general case this need not be so, see [3].

We denote by $\nu(M)$ the complementary orthogonal subbundle of $\phi D^{\perp}$ in the normal bundle $TM^{\perp}$. We have the following orthogonal direct sum decomposition $TM^{\perp} = \phi D^{\perp} \oplus \nu(M)$. It is easy to see that $\nu(M)$ is ϕ-invariant. For $Y \in \nu(M)$, $\phi Y \in TM^{\perp}$ and writing $\phi Y = Y_1 + Y_2$ with $Y_1 \in \phi D^{\perp}$ and $Y_2 \in \nu(M)$, we obtain that $Y_1 = 0$ by using (1.1) and hence $\phi Y \in \nu(M)$. We choose local orthonormal vector fields $N_1, \ldots, N_q, \lambda_1, \ldots, \lambda_{2m-n-q}$ normal to M, such that $N_1, \ldots, N_q$ span $\phi D^{\perp}$ while $\lambda_1, \ldots, \lambda_{2m-n-q}$ span $\nu(M)$ at each point.

For X tangent to M, we have the following decomposition into tangential and normal components:

$$\phi X = FX + \sum_{\alpha=1}^{q} u^{\alpha}(X) N_{\alpha}, \tag{2.1}$$

where FX is just the tangential component of ϕX, while for X tangent to M, the normal component is in $\phi D^{\perp}$ hence the second term in the expression on the right of (2.1). As $N_\alpha \in \phi D^{\perp}$, we have $N_\alpha = \phi U_\alpha$, for some $U_\alpha \in D^{\perp}$, hence

$$\phi N_\alpha = -U_\alpha, \qquad \alpha = 1, \dots, q \tag{2.2}$$

Since $\nu(M)$ is ϕ-invariant, then

$$\phi\lambda_c = \sum_{k=1}^{2m-n-q} \gamma_{ck}\lambda_k, \qquad c = 1, \dots, 2m-n-q \ ,$$

where F is a skew-symmetric linear endomorphism acting on T_xM,γ_{ck} are real valued functions and U_α and u^α, are tangent vector fields and 1-forms on M, respectively. Since ξ is tangent to M from (1.1), (1.2) and (2.1), we conclude that:

$$g(X, U_\alpha) = u^\alpha(X), \ \ F\xi = 0, \ \ u^\alpha(\xi) = 0, \ \ FU_\alpha = 0, \ \ u^\alpha(U_\alpha) = 1.$$

Using (2.1) again, we get:

$$F^2X = -X + \eta(X)\xi + \sum_{\alpha=1}^{q} u^\alpha(X)U_\alpha. \tag{2.3}$$

Let us denote by $\overline{\nabla}$ and ∇ the Riemannian connection of S^{2m+1} and M, respectively and by $\nabla^{\perp}$ the normal connection induced from $\overline{\nabla}$ in the normal bundle of M. Then the Gauss and Weingarten formulas for M are given by:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

for any vector fields X, Y tangent to M and any vector field N normal to M, where h denotes the second fundamental form and A_N denotes the shape operator (second fundamental tensor) corresponding to N.

Suppose that $\nu(M)$ is not necessarily invariant with respect to the normal connection, then the Weingarten formula becomes:

$$\overline{\nabla}_X\lambda_c = -A_cX + \sum_{\beta=1}^{q} S_{c\beta^*}(X)N_\beta + \sum_{d=1}^{2m-n-q} S_{cd}(X)\lambda_d \tag{2.4}$$

$$\overline{\nabla}_X N_\alpha = -A_{\alpha^*}X + \sum_{\beta=1}^{q} S_{\alpha^*\beta^*}(X)N_\beta + \sum_{c=1}^{2m-n-q} S_{\alpha^*c}(X)\lambda_c \tag{2.5}$$

where $c = 1, \dots, 2m-n-q$, $\alpha = 1, ..., q$ and the S's are the coefficients of the normal connection $\nabla^{\perp}$ and A_c, A_{α^*}, are the shape operators corresponding to the normals λ_c, N_α, respectively. Furthermore

$$\overline{\nabla}_X\xi = \phi X,$$

and hence, $\nabla_X\xi + h(X, \xi) = FX + \sum_{\alpha=1}^{q} u^\alpha(X)N_\alpha$, and so $\nabla_X\xi = FX$. Moreover,

$$A_{\alpha^*}\xi = U_\alpha, \quad \alpha = 1, \dots, q. \tag{2.6}$$

Also,

$$A_c\xi = 0, \quad c = 1, \dots, 2m-n-q. \tag{2.7}$$

In addition from the equation of Ricci:

$$g(\overline{R}(X,Y)\lambda_c, N_\alpha) = g(R^\perp(X,Y)\lambda_c, N_\alpha) + g([A_\alpha, A_c]X, Y),$$

where $\overline{R}$ and $R^\perp$ are the curvature tensors with respect to $\overline{\nabla}$ and $\nabla^\perp$ respectively. Because the ambient space is Sasakian, we have:

$$(\overline{\nabla}_X\phi)Y = -g(X,Y)\xi + \eta(Y)X. \tag{2.8}$$

From $\phi(\overline{\nabla}_X\lambda_c) = \overline{\nabla}_X(\phi\lambda_c) - (\overline{\nabla}_X\phi)\lambda_c$, using (2.1), (2.2),(2.4), (2.5) and (2.8), we obtain:

$$\phi(-A_cX + \sum_{\alpha=1}^{q} S_{c\alpha^*}(X)N_\alpha + \sum_{d=1}^{2m-n-q} S_{cd}(X)\lambda_d) = \overline{\nabla}_X\phi\lambda_c.$$

Thus,

$$\begin{aligned}-FA_cX - \textstyle\sum_{\alpha=1}^{q} u^\alpha(A_cX)N_\alpha - \sum_{\alpha=1}^{q} S_{c\alpha^*}(X)U_\alpha + \sum_{d=1}^{2m-n-q}\sum_{k=1}^{2m-n-q}\gamma_{dk}S_{cd}(X)\lambda_k \\ = \textstyle\sum_{k=1}^{2m-n-q}\{(X\gamma_{ck})\lambda_k + \gamma_{ck}(-A_kX + \sum_{\alpha=1}^{q} S_{k\alpha^*}(X)N_\alpha + \sum_{d=1}^{2m-n-q} S_{kd}(X)\lambda_d)\},\end{aligned}$$

for X tangent to M. Comparing the tangential part and the coefficients of N_α, we get:

$$FA_cX = \sum_{k=1}^{2m-n-q} \gamma_{ck}A_kX - \sum_{\alpha=1}^{q} S_{c\alpha^*}(X)U_\alpha, \tag{2.9}$$

$$u^\alpha(A_cX) = -\sum_{k=1}^{2m-n-q} \gamma_{ck}S_{k\alpha^*}(X).$$

Applying F to both sides of the relation (2.9) and using (2.3), we have:

$$A_cX = \sum_{\alpha=1}^{q} u^\alpha(A_cX)U_\alpha - \sum_{k=1}^{2m-n-q} \gamma_{ck}FA_kX,$$

for all X tangent to M and $c = 1, \dots, 2m-n-q$.

From now on we suppose that $\mu(M)$, $\dim\mu(M) = e$, is a subbundle of $\nu(M)$ which is not necessarily ϕ-invariant, but invariant with respect to the normal connection. We can select a local orthonormal frame $\lambda_1, \dots, \lambda_{2m-n-q}$ for $\nu(M)$ so that $\lambda_1, \dots, \lambda_e$ form a local orthonormal frame for $\mu(M)$. Then the Weingarten equation is:

$$\overline{\nabla}_X\lambda_i = -A_iX + \sum_{j=1}^{e} S_{ij}(X)\lambda_j, \quad i = 1, \dots, e. \tag{2.10}$$

Since (2.4) is true for $c = i$, we have:

$$\overline{\nabla}_X\lambda_i = -A_iX + \sum_{\alpha=1}^{q} S_{i\alpha^*}(X)N_\alpha + \sum_{d=1}^{2m-n-q} S_{id}(X)\lambda_d.$$

Comparing the last relation and (2.10) we conclude that:

$$S_{i\alpha^*}(X) = 0, \quad i = 1, \ldots, e \tag{2.11}$$

and $S_{id}(X) = 0$, $d = e+1, \ldots, 2m-n-q$. Since S^{2m+1} is of constant curvature 1, we have

$$\overline{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$

for all X, Y, Z tangent to $\overline{M}$. Since $\mu(M)$ is invariant with respect to the normal connection then from the equation of Ricci we get:

$$g((A_i A_{\alpha^*} - A_{\alpha^*} A_i)X, Y) = 0$$

and hence,

$$A_i A_{\alpha^*} X = A_{\alpha^*} A_i X, \tag{2.12}$$

for all X tangent to M, $\alpha = 1, \ldots, q$ and $i = 1, \ldots, e$.

3 Contact CR-submanifolds of odd-dimensional unit sphere satisfying $h(FX, Y) + h(X, FY) = 0$

Let M be a connected $(n+1)$-dimensional contact CR-submanifold of S^{2m+1} with $\dim D_x^{\perp} = q+1$. In this section we study submanifolds M which satisfy the condition

$$h(FX, Y) + h(X, FY) = 0, \ \textit{for all } X, Y \textit{ tangent to } M. \tag{3.1}$$

The second fundamental form h and the shape operators A_{α^*}, A_c corresponding to normals $N_\alpha \in \phi D^{\perp}$ and $\lambda_c \in \nu(M), c = 1, \ldots, 2m-n-q$, respectively, are related by:

$$h(X,Y) = \sum_{\alpha=1}^{q} g(A_{\alpha^*}X, Y)N_\alpha + \sum_{c=1}^{2m-n-q} g(A_c X, Y)\lambda_c,$$

for all X,Y in TM. Hence,

$$h(FX,Y) + h(X,FY) = 0 = \textstyle\sum_{\alpha=1}^{q}\{g(A_{\alpha^*}FX, Y) + g(A_{\alpha^*}X, FY)\}N_\alpha$$
$$+ \textstyle\sum_{c=1}^{2m-n-q}\{g(A_c FX, Y) + g(A_c X, FY)\}\lambda_c.$$

Since F is skew-symmetric, (3.1) is equivalent to $A_{\alpha^*}F = FA_{\alpha^*}$, i.e.,

$$A_c F = F A_c, \tag{3.2}$$

with $\alpha = 1, \ldots, q$, $c = 1, \ldots, 2m-n-q$.

Lemma 3.1. *Let M be a connected $(n+1)$-dimensional contact CR-submanifold of contact CR-dimension $(n-q)$ of S^{2m+1}. Suppose the subbundle $\mu(M)$ is invariant with respect to the normal connection. If the condition (3.1) is satisfied, then $FA_i = 0 = A_i F$, $i = 1, \ldots, e$, where A_i are the shape operators for the normals λ_i and $e = \dim \mu(M)$.*

Proof. Using (3.2) we have:

$$g(FA_cX,Y) - g(X,FA_cY) = g((FA_c + A_cF)X,Y) = 2g(FA_cX,Y)$$

and, using (2.9), we get

$$\begin{aligned} 2g(FA_cX,Y) &= \textstyle\sum_{k=1}^{2m-n-q} \gamma_{ck} g(A_kX,Y) - \sum_{\alpha=1}^{q} S_{c\alpha^*}(X)u^\alpha(Y) \\ &- \textstyle\sum_{k=1}^{2m-n-q} \gamma_{ck} g(A_kY,X) + \sum_{\alpha=1}^{q} S_{c\alpha^*}(Y)u^\alpha(X). \end{aligned}$$

Since the shape operators are self-adjoint, then the last relation reduces to:

$$2g(FA_cX,Y) = -\sum_{\alpha=1}^{q} S_{c\alpha^*}(X)u^\alpha(Y) + \sum_{\alpha=1}^{q} S_{c\alpha^*}(Y)u^\alpha(X).$$

Then, using (2.11) we get:

$$2g(FA_iX,Y) = -\sum_{\alpha=1}^{q} S_{i\alpha^*}(X)u^\alpha(Y) + \sum_{\alpha=1}^{q} S_{i\alpha^*}(Y)u^\alpha(X) = 0$$

and hence, $FA_iX = 0$, $i = 1,\dots,e$. □

Lemma 3.2. *Let M be a connected $(n+1)$-dimensional contact CR-submanifold of contact CR-dimension $(n-q)$ of S^{2m+1}. Suppose the subbundle $\mu(M)$ is invariant with respect to the normal connection. If the condition* (3.1) *is satisfied, then $A_i = 0$, $i = 1,\dots,e$, where A_i are the shape operators for the normals λ_i and $e = \dim\mu(M)$.*

Proof. Replacing X with ξ in equation (2.12) and using equations (2.6) and (2.7) we get $A_iA_{\alpha^*}\xi = A_{\alpha^*}A_i\xi = 0$, that is, $A_iU_\alpha = 0$, $i = 1,\dots,e$. From (2.3) and Lemma 3.1 we have $A_iX = \sum_{\alpha=1}^{q} u^\alpha(A_iX)U_\alpha$. Then, from the last two equations we conclude that $A_iX = 0$, for all X tangent to M and $i = 1,\dots,e$. □

4 Codimension reduction of contact CR-submanifolds in odd-dimensional unit sphere

In this section, we apply the Erbacher's reduction of codimension theorem to contact CR-submanifold in an odd-dimensional unit sphere.

Let M be a connected submanifold in a Riemannian manifold. The first normal space $N_1(x)$ is defined to be the orthogonal complement of the set $N_0(x) = \{\zeta \in T_x^\perp M | A_\zeta = 0\}$ in $T_x^\perp M$ [9]. Erbacher proved the following theorem [4]:

Theorem 4.1. *Let $\psi : M^n \longrightarrow \overline{M}^{n+p}(\widetilde{c})$ be an isometric immersion of a connected n-dimensional Riemannian manifold into an $n+p$-dimensional Riemannian manifold $\overline{M}^{n+p}(\widetilde{c})$ of constant sectional curvature $\widetilde{c}$. If $N \supset N_1$ and N is a subbundle of $TM^\perp$ invariant with respect to the normal connection and l is the dimension of N, then there exists a totally geodesic submanifold N^{n+l} of $\overline{M}^{n+p}(\widetilde{c})$ such that $\psi(M^n) \subset N^{n+l}$.*

Let M be a connected contact CR-submanifold of S^{2m+1} whose contact CR-dimension is $(n-q)$, i.e, $\dim D^{\perp} = q+1$. For any orthogonal direct sum decomposition $TM^{\perp} = V_1 \oplus V_2$, it is easy to see that V_1 is invariant with respect to the normal connection if and only if V_2 is invariant with respect to the normal connection.

Using the results of the previous section and Theorem 4.1, we have the following result without assuming that M is of maximal CR-dimension as was the case in [6, 7, 5].

Theorem 4.2. *Let M be an $(n+1)$-dimensional contact CR-submanifold of contact CR-dimension $(n-q)$ of S^{2m+1}. If $\phi D^{\perp}$ is invariant with respect to the normal connection and if the condition* (3.1) *is satisfied, then there exists a totally geodesic unit sphere of dimension $(n+q+1)$ of S^{2m+1} such that $M \subset S^{n+q+1}$.*

Proof. By Lemma 3.2, the first normal space $N_1(x) = \phi D_x^{\perp}$. Hence, by Theorem 4.1 we can conclude that there exists a $(n+q+1)$-dimensional totally geodesic unit sphere S^{n+q+1} such that $M \subset S^{n+q+1}$. □

Suppose $\mu(M)$ is a subbundle which is invariant with respect to the normal connection with $\lambda_1, \ldots, \lambda_e$ forming a local orthonormal frame for $\mu(M)$. At each point $x \in M$, consider the subspace $\widetilde{\mu}(M)_x$ of T_xM given by

$$\widetilde{\mu}(M)_x = span\{\lambda_1(x), \ldots, \lambda_e(x), \phi\lambda_1(x), \ldots, \phi\lambda_e(x)\}.$$

Then we have the following:

Lemma 4.3. *Let $\mu(M)$ be a subbundle of $\nu(M)$ invariant with respect to the normal connection. There is a ϕ-invariant subbundle $\widetilde{\mu}(M)$ invariant with respect to the normal connection with $\mu(M) \subset \widetilde{\mu}(M) \subset \nu(M)$, such that $A_\lambda = 0$, for any normal vector field λ in $\widetilde{\mu}(M)$.*

Proof. We first observe that

$$-A_{\phi\lambda_i}X + \nabla_X^{\perp}(\phi\lambda_i) = \overline{\nabla}_X(\phi\lambda_i) = \phi(\overline{\nabla}_X\lambda_i) = \phi(-A_iX + \nabla_X^{\perp}\lambda_i) = \phi(\nabla_X^{\perp}\lambda_i).$$

This shows that $\phi\mu(M)$ is invariant relative to the normal connection and $A_{\phi\lambda_i} = 0$.

Let $\gamma : [a,b] \to M$ be a smooth curve with $\gamma(a) = x$ and $\gamma(b) = y$. Consider orthonormal parallel vector fields $\lambda_1, \cdots, \lambda_e$ in $\mu(M)$ along γ. Then $\phi\lambda_1, \cdots, \phi\lambda_e$ are orthonormal parallel vector fields in $\phi\mu(M)$ along γ. Suppose $\dim\widetilde{\mu}(M)_x = r$, $\{v_1, \cdots, v_r\}$ an orthonormal basis for $\tilde{\mu}(M)_x$ and $V_1 \cdots, V_r$ parallel vector fields along γ with $V_1(a) = v_1, \cdots, V_r(a) = v_r$. Since each v_j is a linear combination of $\lambda_1(a), \cdots, \lambda_e(a)$, $\phi\lambda_1(a), \cdots, \phi\lambda_e(a)$, each V_j is a linear combination of $\lambda_1, \cdots, \lambda_e$, $\phi\lambda_1, \cdots, \phi\lambda_e$, this shows that $\widetilde{\mu}(M)$ is invariant under parallel transport with respect to the normal connection and so $\{V_1(b), \cdots, V_r(b)\}$ is orthonormal in $\widetilde{\mu}(M)_y$. Hence, $\dim\tilde{\mu}(M)_y \geq r = \dim\tilde{\mu}(M)_x$. By switching the role of x and y, we see that $\dim\widetilde{\mu}(M)_x \geq \dim\tilde{\mu}(M)_y$ and so $\dim\tilde{\mu}(M)_x = \dim\tilde{\mu}(M)_y$.

In general, any two points x, $y \in M$ can be joined by a piecewise smooth curve, since M is connected. We can deduce that $\widetilde{\mu}(M)$ has constant dimension at each point in M and conclude that $\tilde{\mu}(M)$ defines a vector subbundle of $\nu(M)$. Moreover, it is clear that $\widetilde{\mu}(M)$ is ϕ-invariant with $\mu(M) \subset \widetilde{\mu}(M) \subset \nu(M)$. Then by Lemma 3.2 we obtain $A_\lambda = 0$, for any normal vector field λ in $\widetilde{\mu}(M)$. Also, $\widetilde{\mu}(M)$ is a maximal subbundle of $\nu(M)$ which is invariant with respect to the normal connection. If $\nabla_X^{\perp}N = 0$, then $N \in \widetilde{\mu}(M)$. Let $\{\lambda_1(p), \ldots, \lambda_e(p)\}$. □

We now have a result similar to that in [2]. We do not assume that $\mu(M)$ is ϕ-invariant and M is of maximal CR-dimension.

Theorem 4.4. *Let M be an $(n+1)$-dimensional contact CR-submanifold of contact CR-dimension $(n-q)$ of S^{2m+1}. Let $\mu(M)$ be a subbundle of $\nu(M)$ which is also invariant with respect to the normal connection with $\dim \mu(M) = e$. If the condition* (3.1) *is satisfied, then there exists a totally geodesic odd-dimensional unit sphere of dimension $(2m+1-l)$ in S^{2m+1} such that $M \subset S^{2m+1-l}$ with $l \geq e$.*

Proof. From Lemma 4.3 we have a ϕ-invariant subbundle $\widetilde{\mu}(M)$ which is invariant with respect to the normal connection with $\mu(M) \subset \widetilde{\mu}(M) \subset \nu(M)$. Since $\widetilde{\mu}(M)$ is ϕ-invariant, it is of even dimension and $\dim \nu(M) \geq \dim \widetilde{\mu}(M) = l \geq e$. Also since $\widetilde{\mu}(M)$ is invariant with respect to the normal connection, we have $\widetilde{\mu}(M)_x \subset N_0(x)$. Hence the first normal space $N_1(x) \subset N_x = \phi D_x^{\perp} \oplus \sigma(M)_x$ where $\nu(M) = \widetilde{\mu}(M) \oplus \sigma(M)$. Since $\widetilde{\mu}(M)$ is invariant with respect to the normal connection, so is N. Applying Theorem 4.1, there exists a totally geodesic odd-dimensional unit sphere S^{2m+1-l} such that $M \subset S^{2m+1-l}$. □

References

[1] A. Bejancu, *Geometry of CR-submanifolds*, D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, Tokyo, 1986.

[2] M. Djoric, *Codimension reduction and second fundamental form of CR-submanifolds in complex space forms*, J. Math. Anal. Appl. 356 (2009), 237–241.

[3] M. Djoric, M. Okumura, *CR-Submanifolds of Complex Projective Spaces*, Springer, New York, Dordrecht, Heidelberg, London, 2010.

[4] J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differ. Geom. 5 (1971), 333–340.

[5] H. S. Kim, J. S. Pak, *Certain contact CR-submanifolds of an odd-dimensional uint sphere*, Bull. Korean Math. Soc. 44 (2007), 109–116.

[6] J. H. Kwon, J. S. Pak, *On some contact CR-submanifolds of an odd dimensional unit sphere*, Soochow J. Math. 26 (2000), 427–439.

[7] J. S. Pak, J. H. Kwon, H. S. Kim, Y. M. Kim, *Contact CR-submanifolds of an odd-dimensional unit sphere*, Geom. Dedic. 114 (2005), 1–11.

[8] K. Yano, M. Kon, *Contact CR-submanifolds*, Kodai Math. J. 5 (1982), 238–252.

[9] K. Yano, M. Kon, *CR-submanifolds of Keahlerian and Sasakian manifolds*, Brikhauser, Boston, Basel, Stuttgart, 1983.

Author's address:

Esmaiel Abedi, Ghorbanali Haghightdoost
Department of Mathematics,
Azarbaijan Shahid Madani University, Tabriz, Iran.
E-mail: esabedi@azaruniv.edu, gorbanali@azaruniv.edu

Mehri Asadollahzadeh
Institute of Mathematical Sciences,
University of Bonab, Iran.
E-mail: m.a.zadeh@bonabu.ac.ir