Characterization of locally dually flat first approximate Matsumoto metric

S. K. Narasimhamurthy, A. R. Kavyashree and Y.K. Mallikarjun

Abstract. The concept of locally dually flat Finsler metrics originate from information geometry. As we know, (α, β)-metrics defined by a Riemannian metric α and an 1-form β, represent an important class of Finsler metrics, which contains the Matsumoto metric. In this paper, we study and characterize locally dually flat first approximation of the Matsumoto metric with isotropic S-curvature, which is not Riemannian.

M.S.C. 2010: 53C60, 53B40.
Key words: Matsumoto metric; S-curvature; locally dually flat metric; locally Minkowski metric.

1 Introduction

The notion of dually flat metric was first introduced by S. I. Amari and H. Nagaoka, while studying the information geometry on Riemannian spaces [1]. Later, Z. Shen extended the notion of dually flatness to Finsler metrics [6]. Dually flat Finsler metrics form a special important class of Finsler metrics in Finsler information geometry, which play a very important role in studying flat Finsler information structures ([3],[4],[8],[9],[10]).

In 2009, the authors of [3] classified the locally dual flat Randers metrics with almost isotropic flag curvature. Recently, Q. Xia worked on the dual flatness of Finsler metrics of isotropic flag curvature as well as scalar flag curvature ([9],[10]). Also, Q. Xia studied and gave a characterization of locally dually flat (α, β)-metrics on an n-dimensional manifold M $(n \geq 3)$ [8].

The first example of non-Riemannian dually flat metrics is *the Funk metric* given by ([3],[6]):

$$F = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2} \pm \frac{\langle x, y\rangle}{1-|x|^2}.$$

This metric is defined on the unit ball $\mathbf{B}^n \subset \mathbb{R}^n$ and is a Randers metric with constant flag curvature $K = -\frac{1}{4}$. This is the only known example of locally dually flat metric with non-zero constant flag curvature up to now.

Balkan Journal of Geometry and Its Applications, Vol.19, No.1, 2014, pp. 80-87.

In this paper, we study and characterize locally dually flat first approximate of the Matsumoto metric with isotropic S-curvature, which is not Riemannian.

2 Preliminaries

Let M be an n-dimensional smooth manifold. We denote by TM the tangent bundle of M and by $(x, y) = (x^i, y^i)$ the local coordinates on the tangent bundle *TM.* A Finlser manifold (M, F) is a smooth manifold equipped with a function $F : TM \to [0, \infty)$, which has the following properties:

- Regularity: F is smooth in $TM \setminus \{0\}$;
- Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$;
- Strong convexity: the Hessian matrix of F^2, $g_{ij}(x, y) = \frac{1}{2}(\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j})$, is positive definite on $TM \setminus \{0\}$. We call F and the tensor g_{ij} *the Finsler metric* and *the fundamental tensor of* M, respectively.

For a Finser metric $F = F(x, y)$, its geodesic curves are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called *the spray coefficients* and given by

$$G^i = \frac{1}{4} g^{il} \left\{ \frac{\partial^2 (F^2)}{\partial x^k \partial y^l} y^k - \frac{\partial (F^2)}{\partial x^l} \right\}, \quad \forall y \in T_x M.$$

Definition 2.1 A Finsler metric $F = F(x, y)$ on a manifold M is said to be *locally dually flat* if at any point there is a standard coordinate system (x^i, y^i) in TM which satisfies

$$(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}.$$

In this case, the system of coordinates (x^i) is called an *adapted local coordinate system.* It is easy to see that every locally Minkowskian metric is locally dually flat. But the converse is not generally true [3].

Definition 2.2: A Finsler metric is said to be *locally projectively flat* if at any point there is a local coordinate system in which the geodesics are straight lines as point sets.

It is known that a Finsler metric $F(x, y)$ on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if its geodesic coefficients G^i are of the form

$$G^i = P y^i,$$

where $P : TU = U \times \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous of degree one, $P(x, \lambda y) = \lambda P(x, y)$, $\forall \lambda > 0$. We call $P(x, y)$ *the projective factor* of $F(x, y)$.

Lemma 2.1 ([3]). *Let $F = F(x, y)$ be a Finsler metric on an open subset $U \subset \mathbb{R}^n$. Then F is locally flat and projectively flat on U if and only if $F_{x^k} = CFF_{y^k}$, where C is a constant.*

The S-curvature is a scalar function on TM, which was introduced by Z. Shen to study volume comparison in Riemann-Finsler geometry [2]. The S-curvature measures the average rate of change of $(T_xM, F_x = F|_{T_xM})$ in the direction $y \in T_xM$. It is known that $S = 0$ for Berwald metrics.

Definition 2.3. A Finsler metric F on an n-dimensional manifold M is said to have isotropic S-curvature if $S = (n+1)c(x)F$, for some scalar function c on M.

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \dots dx^n$ is defined by

$$\sigma(F) = \frac{Vol(B^n(1))}{Vol\{(y^i) \in \mathbb{R}^n | F(y^i \frac{\partial}{\partial x^i}|_x)\}}.$$

Here Vol denotes the Euclidean volumes and $B^n(1)$ denotes the unit ball in $\mathbb{R}^n$. Then the S-curvature is defined by

$$S(y) = \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}[ln\sigma_F(x)],$$

where $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$ [7].

For an (α,β)-metric, one can write $F = \alpha\phi(s)$, where $s = \beta/\alpha$ and $\phi = \phi(s)$ is a C^∞ function on the interval $(-b_0, b_0)$ with certain regularity properties, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is an 1-form on M.

We further denote

$$b_{i|j}\theta^j = db_i - b_j\theta_i^j,$$

where $\theta^i = dx^i$ and $\theta_i^j = \Gamma_{ik}^j dx^k$ denotes the coefficients of the Levi-Civita connection form of α. Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Clearly, β is closed if and only if $s_{ij} = 0$. An (α,β)-metric is said to be *trivial* if $r_{ij} = s_{ij} = 0$. We put

$$\begin{cases} r_{i0} = r_{ij}y^j, \quad r_{00} = r_{ij}y^iy^j, \quad r_j = b^ir_{ij}, \\ s_{i0} = s_{ij}y^j, \quad s_j = b^is_{ij}, \quad r_0 = r_jy^j, \quad s_0 = s_jy^j. \end{cases}$$

By direct computation, we can obtain a formula for the mean Cartan torsion of an (α,β)-metric as follows:

$$I_i = -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i).$$

Clearly, an (α,β)-metric $F = \alpha\phi(s), s = \beta/\alpha$ is Riemannian if and only if $\Phi = 0$. Hence, we further we assume that $\Phi \neq 0$.

Theorem 2.2. *[8] Let $F = \alpha\phi(s), s = \beta/\alpha$ be an (α,β)-metric on an n-dimensional manifold $M^n (n \geq 3)$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i \neq 0$ is an 1-form on M. Suppose that F is not Riemannian and $\phi'(s) \neq 0$. Then F is locally dually flat on M if and only if α, β and $\phi = \phi(s)$ satisfy*

1. $s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l)$,
2. $r_{00} = \frac{2}{3}\theta\beta + [\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3 b^2)\tau\beta^2$,
3. $G^l_\alpha = \frac{1}{3}[2\theta + (3k_1 - 2)\tau\beta]y^l + \frac{1}{3}(\theta^l\tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2 b^l$,
4. $\tau[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')] = 0$,

where $\tau = \tau(x)$ is a scalar function, $\theta = \theta_i(x)y^i$ is an 1-form on M, $\theta^l = a^{lm}\theta_m$,

$$k_1 = \Pi(0), \quad k_2 = \frac{\Pi'(0)}{Q(0)}, \quad k_3 = \frac{1}{6Q(0)^2}[3Q''(0)\Pi'(0) - 6\Pi(0)^2 - Q(0)\Pi'''(0)],$$

and $Q = \frac{\phi'}{\phi - s\phi'}$, $\Pi = \frac{\phi'^2 + \phi\phi''}{\phi(\phi - s\phi')}$.

In [3], Cheng-Shen studied the class of (α,β)-metrics of non-Randers type $\phi \neq t_1\sqrt{1 + t_2 s^2} + t_3 s$ with isotropic S-curvature and obtained the following

Theorem 2.3 ([2])**.** *Let $F = \alpha\phi(s), s = \beta/\alpha$ be an non-Riemannian (α,β)-metric on a manifold and $b = \|\beta_x\|_\alpha$. Suppose that $\phi \neq t_1\sqrt{1 + t_2 s^2} + t_3 s$ for any constants $t_1 > 0$, t_2 and t_3. Then F is of isotropic S-curvature $S = (n+1)cF$ if and only if one of the following assertions holds*

i) β satisfies

$$r_{ij} = \varepsilon\{b^2 a_{ij} - b_i b_j\}, \quad s_j = 0, \tag{2.1}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $c = c(x)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{2.2}$$

where k is a real constant. In this case, $S = (n+1)cF$ with $c = k\varepsilon$.
ii) β satisfies

$$r_{ij} = 0, \quad s_{ij} = 0. \tag{2.3}$$

In this case, $S = 0$, regardless of the choice of a particular ϕ.

3 Characterization of locally dually flat first approximate Matsumoto metric

Theorem 3.1. *Let $F = \alpha + \beta + \frac{\beta^2}{\alpha}$ be a first approximate Matsumoto metric on a manifold M of dimension $n \geq 3$. Then the necessary and sufficiency conditions for F to be locally dually flat on M are the following:*

1. $s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l)$;
2. $r_{00} = \frac{2}{3}\theta\beta + [\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)]\alpha^2 + \frac{1}{3}(7 + 18b^2)\tau\beta^2$;
3. $G^l_\alpha = \frac{1}{3}[2\theta + 7\tau\beta]y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 - 3\tau\beta^2 b^l$,

where $\tau = \tau(x)$ is a scalar function and $\theta = \theta_k y^k$ is an 1-form on M.

Proof. For a Finsler metric $F = \alpha + \beta + \frac{\beta^2}{\alpha}$, we obtain $k_1 = 3,\ k_2 = 3,\ k_3 = -6$, and

$$\phi = 1 + s + s^2, \quad \phi' = 1 + 2s, \quad \Pi = \tfrac{3(1+2s+2s^2)}{(1+s-s^3-s^4)}$$

$$Q = \tfrac{1+2s}{1-s^2}, \quad Q' = \tfrac{2\phi}{(1-s^2)^2}, \quad Q'' = \tfrac{2(1+4s+s^2)}{(1-s^2)^3}.$$

By using the above values in Lemma 2.1, we get

$$[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')] = 0, \text{ and } \tau = 0.$$

Then, finally, by substituting k_1,k_2 and k_3 in Lemma 2.1, we infer the claim □

Now, let $\phi = \phi(s)$ be a positive C^∞ function on $(-b_0, b_0)$. For a number $b \in [0, b_0]$, let

$$\Phi = -(Q - sQ') \cdot (n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'', \tag{3.1}$$

where $\Delta = 1 + sQ + (b^2 - s^2)Q'$. This implies that

$$\Delta = \frac{\phi(1 + 2b^2 - 3s^2)}{(1 - s^2)^2}.$$

Then the equation (3.1) can be written as follows:

$$\Phi = -(Q - sQ')(n + 1)\Delta + (b^2 - s^2)\{(Q - sQ')Q' - (1 + sQ)Q''\}.$$

By using Theorem 2.3, now we will consider a locally dually flat (α, β)-metric with isotropic S-curvature.

Theorem 3.2. *Let $F = \alpha + \beta + \frac{\beta^2}{\alpha}$ be a locally dually flat non-Randers type (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F is of isotropic S-curvature $S = (n + 1)cF$, where $c = c(x)$ is a scalar function on M. Then F is a locally projectively flat in adapted coordinate system and $G^i = 0$.*

Proof. Let $G^i = G^i(x, y)$ and $\overline{G}^i_\alpha = \overline{G}^i_\alpha(x, y)$ denote the coefficients of F and α respectively, in the same coordinate system. By definition, we have

$$G^i = \overline{G}^i_\alpha + Py^i + Q^i, \tag{3.2}$$

where

$$P = \alpha^{-1}\Theta{-}2Q\alpha s_0 + r_{00}, \tag{3.3}$$

$$Q^i = \alpha Q s^i_0 + \Psi{-}2Q\alpha s_0 + r_{00}b^i, \tag{3.4}$$

$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \quad \Psi = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

First, we suppose that case (i) of Theorem 2.3 holds. It is remarkable that, for a 1st approximation Matsumoto metric, we have

$$\Delta = \frac{(1 + 2b^2 - 3s)\phi}{(1 - s^2)^2}.$$

It follows that $(1-s^2)^2\Delta$ is a polynomial in s of degree 3. On the other hand we have

$$\phi\Delta^2 = \frac{\phi^2(1+2b^2-3s^2)^2}{(1-s^2)^4}. \tag{3.5}$$

Hence, if case(ii) of Theorem (2.3) holds, then substituting (3.5) we obtaine that

$$(b^2-s^2)(1-s^2)^4\Phi = -2(n+1)k\phi^2(1+2b^2-3s^2)^2. \tag{3.6}$$

It follows that $(b^2-s^2)(1-s^2)^4\Phi$ is not a polynomial in s (if $k=0$, then by considering the Cartan torsion equation, we get a contradiction). Then, we put

$$\phi\Delta^2 = \frac{\bar{\Delta}}{(1-s^2)^4},$$

where

$$\bar{\Delta} = \phi^2(1+2b^2-3s^2)^2.$$

By assumption, F is a non-Randers type metric. Thus $\bar{\Delta}$ is not a polynomial in s, and then $(b^2-s^2)(1-s^2)^4\Phi$ is not a polynomial in s. Now, let us consider another form of Φ:

$$\Phi = -(Q-sQ')(n+1)\Delta + (b^2-s^2)\{(Q-sQ')Q' - (1+sQ)Q''\},$$

where

$$Q - sQ' = \frac{1-3s^2-4s^3}{(1-s^2)^2}.$$

Then

$$\Phi = \frac{(n+1)\phi(1-3s^2+4s^3)(1+2b^2-3s^2) - 12\phi^2(b^2-s^2)s(1-s^2)^2}{(1-s^2)^6}. \tag{3.7}$$

From equations (3.6) and (3.7), the relation $(b^2-s^2)(1-s^2)^4\Phi$ is a polynomial in s and b of degree 8 and 4 respectively. The coefficient of s^8 is not equal to zero. Hence its impossible that $\Phi = 0$. Therefore, we can conclude that equation (2.2) does not hold. So, the case (ii) of Theorem 2.3 holds. In this case, we have

$$r_{00} = 0, \quad s_j = 0.$$

In Theorem 3.1(2), by taking $r_{00} = 0$, we obtain

$$[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)]\alpha^2 = \frac{1}{3}\beta[-2\theta - (7+18b^2)]\beta\tau. \tag{3.8}$$

Since α^2 is an irreducible polynomial of y^i , equation (3.8) reduces to the following

$$\tau + \frac{2}{3}(b^2\tau - \theta^m b_m) = 0, \tag{3.9}$$

$$\frac{2}{3}\theta + \frac{1}{3}(7+18b^2)\beta\tau = 0, \tag{3.10}$$

whence

$$\theta = -\frac{1}{2}(7 + 18b^2)\beta\tau. \tag{3.11}$$

Then Theorem 3.1(1) yields

$$s_0 = -\frac{1}{3(\theta b^2 - \beta b_m \theta^m)}.$$

This implies

$$\theta b^2 - \beta b_m \theta^m = 0.$$

From (3.8), (3.9) and (3.11), we obtain

$$\theta = -\frac{1}{2}(7 + 18b^2)\beta\tau. \tag{3.12}$$

From equations (3.9) and (3.12), it follows that $\tau = 0$ and substituting $\tau = 0$ in equation (3.12), we get $\theta = 0$. Thus finally (1),(2) and (3) reduce to the following

$$s_{ij} = 0, \quad G^l_\alpha = 0, \quad r_{00} = 0.$$

Since $s_0 = r_{00} = 0$, then equations (3.3) and (3.4) reduce to

$$P = 0 \text{ and } Q^i = 0.$$

Then the relation (3.2) becomes $G^i_\alpha = 0$, which completes the proof. □

Theorem 3.3. *Let $F = \alpha + \beta + \frac{\beta^2}{\alpha}$ be a non-Riemannian metric on n-dimensional ($n \geq 3$) manifold M. Then F is locally dually flat with isotropic S-curvature. Moreover, $S = (n+1)cF$ if and only if the structure is locally Minkowskian.*

Proof. From Theorem 3.2 we have that $F = \alpha + \beta + \frac{\beta^2}{\alpha}$ is dually flat and projectively flat in any adapted coordinate system. By Lemma 2.1, we infer

$$F_{x^k} = CFF_{y^k}.$$

Hence the spray coefficients $G^i = Py^i$ are given by $P = \frac{1}{2}CF$. Since $G^i = 0$, then $P = 0$, and hence $C = 0$. This implies that $F_{x^k} = 0$, and then F is a locally Minkowskian metric in the adapted coordinate system. □

4 Conclusions

The authors S. I. Amari and H. Nagaoka ([1]) introduced the notion of dually flat Riemannian metrics, while studying information geometry on Riemannian manifolds. Information geometry emerged from investigating the geometrical structure of a family of probability distributions and was successfully applied to various areas, including statistical inference, control system theorem and multi-terminal information theorem.

As known, Finsler geometry is just Riemannian geometry without the quadratic restriction. Therefore, it is natural to extend the construction of locally dually flat metrics to Finsler geometry. In Finsler geometry, Z.Shen[6] extended the notion of locally dually flat metric in Finsler information geometry, which plays a very important role in studying many applications in Finsler information structures.

In this article, we study and characterize the locally dually flat first approximate Matsumoto metric with isotropic S-curvature which is not Riemannian.

References

[1] S. I. Amari and H. Nagaoka, *Methods of Information Geometry*, Oxford University Press, AMS Translations of Math. Monographs 191, 2000.

[2] S. Bacso, X. Cheng and Z. Shen, *Curvature properties of* (α, β)*-metrics*, In: "Finsler Geometry", Sapporo 2005 - In Memory of Makoto Matsumoto, Ed. S. Sabău and H. Shimada, Adv. Stud. Pure Math. 48, Math. Soc. Japan, 2007, 73-110.

[3] X. Cheng, Z. Shen and Y. Zhou, *On a class of locally dually flat Finsler metrics*, J. Math. 21, 11 (2005), 1-13.

[4] X. Cheng and Y. Tian, *Locally dually flat Finsler metrics with special curvature properties*, J. Diff. Geom. Appl. 29, Supplement 1, August 2011, S98-S106.

[5] S. K. Narasimhamurthy, A. R. Kavyashree and M. Y. Kumbar, *On a class of weakly Berwald* (α, β)*-metrics of scalar flag curvature*, Int. J. Math. Res. 5 (2013), 97-108.

[6] Z. Shen, *Riemann-Finsler geometry with applications to information geometry*, J. Chin. Ann. Math. 27B (1) (2006), 73-94.

[7] Z. Shen, *Volume comparison and its applications in Riemann-Finsler Geometry*, J. Adv. in Math. 128 (1997), 306-328.

[8] Q. Xia, *On locally dually flat* (α, β)*-metrics*, J. Diff. Geom. Appl. 29 (2) (2011), 233-243.

[9] Q. Xia, *On a class of locally dually flat Finsler metrics of isotropic flag curvature*, J. Publ. Math. Debrecen. 78 (1) (2011), 169-190.

[10] Q. Xia, *On a class of locally dually flat Finsler metrics of scalar flag curvature*, J. Publ. Math. Debrecen, in press.

Authors' address:

Senajji Kampalappa Narasimhamurthy
Ambale Rajegowda Kavyashree
Yallappa Kumbar Mallikarjun

Department of P.G. Studies and Research in Mathematics,
Kuvempu University, Shankaraghatta - 577451, Shimoga, Karnataka, India.

E-mail: nmurthysk@gmail.com
kavyashree_86@yahoo.com
mallikarjunykumbar@gmail.com