The antipodal sets of compact symmetric spaces

Xingda Liu and Shaoqiang Deng

Abstract. We study the antipodal set of a point in a compact Riemannian symmetric space. It turns out that we can give an explicit description of the antipodal set of a point in any connected simply connected compact Riemannian symmetric space. In particular, we prove that if M is a connected simply connected Riemannian symmetric space such that the antipodal set of each point consists of a single point, then it must be the direct product of the manifolds of the following: $\mathrm{SU}(2n)$, $\mathrm{Spin}(5)$, $\mathrm{Spin}(7)$, $\mathrm{Sp}(n)$, E_7, $\mathrm{SU}(2n)/\mathrm{SO}(2n)$, $\mathrm{SU}(4n)/\mathrm{Sp}(2n)$, $G_{n,n}(\mathbb{C})$, $\mathrm{Sp}(n)/\mathrm{U}(n)$, $G_{n,n}(\mathbb{H})$, $\mathrm{G}_{p,q}$, $(p < q,\ p \leq 3)$, $\mathrm{SO}(4n)/\mathrm{U}(2n)$, $(\mathfrak{e}_7, \mathfrak{su}(8))$ and $(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R})$, endowed with a standard symmetric metric.

M.S.C. 2010: 22E46, 53C30.
Key words: Antipodal set, Riemannian symmetric space, cut lotus, conjugate lotus.

1 Introduction

Let (M, Q) be a connected compact Riemannian manifold with distance function d. Given $p \in M$, a point $x \in M$ is called an antipodal point of p if $d(p, x) = \max_{y \in M} d(p, y)$. The set of all antipodal point of p is called the antipodal set of p. It is an important problem in Riemannian geometry to determine the antipodal point set of a given point in a compact Riemannian manifold, and the case of rank one symmetric spaces has been settled in [7], see §10 of Chapter VII. In particular, it would be interesting to determine in which Riemannian manifold each point has exactly one single antipodal point.

In this paper we will give an answer to the above problem in the case of connected simply connected compact symmetric Riemannian manifold. The main result can be stated as the following

Theorem 1.1. *Let (M, Q) be a connected simply connected compact Riemannian symmetric space. Suppose each point of M has exactly one single antipodal point. Then (M, Q) must be the direct product of the manifolds of the following:* $\mathrm{SU}(2n)$, $\mathrm{Spin}(5)$, $\mathrm{Spin}(7)$, $\mathrm{Sp}(n)$, E_7, $\mathrm{SU}(2n)/\mathrm{SO}(2n)$, $\mathrm{SU}(4n)/\mathrm{Sp}(2n)$, $G_{n,n}(\mathbb{C})$, $\mathrm{Sp}(n)/\mathrm{U}(n)$, $G_{n,n}(\mathbb{H})$, $G_{p,q}$, $(p < q,\ p \leq 3)$, $\mathrm{SO}(4n)/\mathrm{U}(2n)$, $(\mathfrak{e}_7, \mathfrak{su}(8))$ *and* $(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R})$, *endowed with a standard symmetric metric.*

Balkan Journal of Geometry and Its Applications, Vol.19, No.1, 2014, pp. 73-79.

The proof of this theorem depends on an explicit description of the antipodal set of a given point for connected simply connected irreducible compact Riemannian symmetric spaces. In the study of the antipodal sets we use many known results on the cut lotus and conjugate lotus, see [3] [9], [10], [11] [12] [13] [14]; see also [1] for the classification of irreducible symmetric spaces. One can also consult [2] for some results on the antipodal points of Riemannian symmetric spaces.

It would be an interesting problem to consider the same problem for symmetric Finsler spaces. However, the computation would be much more complicated. We will take this problem up in a forthcoming paper; see [6, 5] for some information on symmetric Finsler spaces.

2 Preliminaries

In this section we recall some preliminaries and known results to establish our strategy to compute the antipodal sets.

Definition 2.1. Let M be a compact connected Riemannian manifold and $o \in M$. A point $p_0 \in M$ is called an antipodal point of o if

$$d(o, p_0) = \max_{p \in M} d(o, p),$$

where d is the distance function of M. The set consisting of all the antipodal points of o is called the antipodal set of o and is denoted by A_o.

The following lemma is obvious

Lemma 2.1. *Let p_0 be an antipodal point of o and γ a minimal geodesic connecting o and p_0. Then p_0 must be a cut point of o along γ.*

This lemma gives a method to find out the antipodal set of the point o, especially when M is a connected simply connected compact Riemannian symmetric space. Let us explain in some detail.

Let (M, Q) be a connected globally symmetric Riemannian manifold and G be its identity component of the full group of isometries of (M, Q). Let K be the isotropy subgroup of G at a fixed point in M. Let $\mathfrak{g}$, $\mathfrak{k}$ be respectively the Lie algebras of G and K. Then there is an involutive automorphism σ of G such that $(K_\sigma)_0 \subset K \subset K_\sigma$, where K_σ denote the set of the fixed point of σ on G and $(K_\sigma)_0$ the identity component of K_σ. Denote also by the differential of σ as σ. Then $(\mathfrak{g}, \sigma)$ is an orthogonal symmetric Lie algebra. Conversely, each effective orthogonal symmetric Lie algebra can determine in a unique way a connected simply connected Riemannian symmetric space (see [7]). If (M, Q) is a connected simply connected compact Riemannian symmetric space, then the corresponding orthogonal Lie algebra is of the compact type (see also [7]). Therefore, to study the problem we need only deal with orthogonal symmetric Lie algebras of the compact type.

Let $\mathfrak{u}$ be a compact semisimple Lie algebra and θ an involutive automorphism of $\mathfrak{u}$. Then θ extends uniquely to a complex involutive automorphism of $\mathfrak{u}^C$ (also denoted by θ), the complexification of $\mathfrak{u}$. We then have a decomposition:

$$\mathfrak{u} = \mathfrak{k}_0 + \mathfrak{p}_* \,;$$

where $\mathfrak{k}_0 = \{X \in \mathfrak{u} : \theta(X) = X\}$, and $\mathfrak{p}_* = \{X \in \mathfrak{u} : \theta(X) = -X\}$. Let $M = U/K$ be a compact symmetric space associated with $(\mathfrak{u}, \theta)$. Let $\langle , \rangle$ be an inner product on $\mathfrak{p}_*$ invariant under the action of $\mathrm{Ad}(K)$. Then we obtain a U-invariant metric g on M, and there is a natural correspondence between (T_oM, g) and $(\mathfrak{p}_*, \langle , \rangle)$, where $o = eK$ is the origin. Let exp be the exponential map of $\mathfrak{u}$, and Exp be the exponential map of $\mathfrak{p}_*$. Then we have $\mathrm{Exp}(X) = \exp(X)K$, for $X \in \mathfrak{p}^*$.

Let $\mathfrak{h}_{\mathfrak{p}_*}$ be a maximal abelian subalgebra of $\mathfrak{p}_*$ and denote the corresponding restricted root system by Σ (see [7]). Let C be the Weyl chamber with respect an ordering of Σ, i.e., $C = \{x \in \sqrt{-1}\mathfrak{h}_{\mathfrak{p}_*} : \gamma(x) > 0 \,\text{for every}\, \gamma \in \Sigma^+\}$. Denote by Π the set of simple roots. Let $\mathfrak{H}$ be the set of highest restricted roots of Σ. In [13] and [14], the author introduced the definition of *Cartan polyhedron*, which is defined by

$$\{x \in \sqrt{-1}\mathfrak{h}_{\mathfrak{p}_*} : \gamma(x) \geq 0,\ \beta(x) \leq 1,\ \gamma \in \Pi,\ \beta \in \mathfrak{H}\}.$$

For simplicity, we denote it as $\triangle$.

Let $(\mathfrak{u}, \theta)$ be an irreducible orthogonal symmetric Lie algebra. Then Σ is also irreducible and $\triangle$ is a simplex. Let ψ be the unique highest restricted root, $\Pi = \{\gamma_1, ..., \gamma_n\}$, $n = \mathrm{rank}\,(\Sigma) = \dim \mathfrak{h}_{\mathfrak{p}_*}$, and $d_1, ..., d_n \in \mathbb{Z}^+$ such that $\psi = \sum_{i=1}^n d_i \gamma_i$. Then the set of the vertices of $\triangle$, denoted by P, consists of:

$$0, e_1, ..., e_n;\ \gamma_i(e_j) = \frac{1}{d_j}\delta_{ij}$$

Let A_P be the subset of P defined by

$$A_P = \{X \in P |\, \mathrm{Exp}(\pi\sqrt{-1}X) \text{ is the antipodal point of } o\}.$$

From the Theorem 4.1 in [14], we have the following corollary:

Corollary 2.2. *Let $(\mathfrak{u}, \theta)$ be an irreducible orthogonal Lie algebra of compact type and $M = U/K$ be the simply connected Riemannian symmetric space associated with $(\mathfrak{u}, \theta)$. Then the antipodal set A_o is* $\mathrm{Exp}\,\mathrm{Ad}(K)(\pi\sqrt{-1}A_P)$.

The above corollary gives the strategy to determine the antipodal set of a connected simply connected compact Riermannian symmetric space. However, it is very difficult to obtain a complete description for an explicit symmetric space. In the following, we will give a partial describing of the antipodal sets for each irreducible compact symmetric space. We will also study some general properties of the antipodal sets. For example, it is an interesting problem to find out on which connected simply connected Riemannian manifold, each point has exactly one antipodal point. The symmetric case will be completely settled in this paper.

3 The set of vertices

In [13] and [14], the author has computed the set P and the diameter of the compact irreducible Riemannian symmetric space. It is easily seen that the diameter is the common length of all the elements in $\pi\sqrt{-1}A_P$. From this we can obtain directly the set A_P. The results is presented in Table 1. Here we adopt the Dynkin diagrams of the restricted root system in Section 5 of [14].

Table 1: **The vertices**

Σ	the highest restricted root ψ	the elements in A_P
$\mathfrak{a}_{n-1}$ $(n \geq 2)$	$\sum_{i=1}^{n-1} \gamma_i$	$e_{\frac{n}{2}}$, $2 \mid n$.
		$e_{\frac{n-1}{2}}$ and $e_{\frac{n+1}{2}}$, $2 \nmid n$.
$\mathfrak{b}_n$ $(n \geq 2)$	$\gamma_1 + 2\sum_{i=2}^{n} \gamma_i$	e_1, $n \leq 3$.
		e_1 and e_4, $n = 4$.
		e_n, $n \geq 5$.
$\mathfrak{c}_n, (n \geq 3)$	$2\sum_{i=1}^{n-1} \gamma_i + \gamma_n$	e_n.
$\mathfrak{d}_n$ $(n \geq 4)$	$\gamma_1 + 2\sum_{i=2}^{n} \gamma_i$ $+\gamma_{n-1} + \gamma_n$	e_1, e_3 and e_4, $n = 4$.
		e_{n-1} and e_n, $n \geq 5$.
$\mathfrak{e}_6$	$\gamma_1 + 2\gamma_2 + 3\gamma_3 + 2\gamma_4 + \gamma_5 + 2\gamma_6$	e_1 and e_5.
$\mathfrak{e}_7$	$\gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 3\gamma_5 + 2\gamma_6 + 2\gamma_7$	e_1.
$\mathfrak{e}_8$	$2\gamma_1 + 3\gamma_2 + 4\gamma_3 + 5\gamma_4 + 6\gamma_5 + 4\gamma_6 + 2\gamma_7 + 3\gamma_8$	e_7.
$\mathfrak{f}_4$	$2\gamma_1 + 3\gamma_2 + 4\gamma_3 + 2\gamma_4$	e_4.
$\mathfrak{g}_2$	$2\gamma_1 + 3\gamma_2$	e_2.
$(\mathfrak{bc})_n$	$2\sum_{i=1}^{n} \gamma_i$	e_n.

4 The antipodal sets for compact connected irreducible Riemannian symmetric spaces

From the previous sections, we can obtain the antipodal set A_o. However, for every compact connected irreducible Riemannian symmetric space U/K, the antipodal set constitutes of some K-orbits (actually, each K-orbit of dim ≥ 1 is also a Riemannian symmetric space). For the compact simply-connected irreducible Riemannian symmetric space, we need only know whether each K-orbit is a single point or not.

Now we introduce two conventions: given $p \in U/K$, if the K-orbit of p consists of a single point, we say that the orbit of p is of type $\mathcal{P}$. If the dimension of the orbit is ≥ 1, we say that the orbit is of type $\mathcal{O}$. From the definition of $\triangle$, we know that the number of the K-orbits of the antipodal set of p is the same as the number of elements in A_P. Then we define the type of the antipodal set of p to be $\mathcal{P}^m\mathcal{O}^n$, $m, n \in \mathbb{N}$, which means that there are m single points and n $K-$orbits of dim≥ 1 in the antipodal set.

Before stating the results, we give two lemmas.

Lemma 4.1. *Let $M = U/K$ be a compact connected simply-connected irreducible symmetric space, and $Z_M(K)$ be the set $\{p \in M : \tau(k)p = p, \forall k \in K\}$. Then for each $X \in \pi\sqrt{-1}A_P$, the following conditions are equivalence:*

(a) the K-action on $\exp(X)K$ is trivial;

(b) $\exp(X)K \in Z_M(K)$;

(c) $X = \pi\sqrt{-1}e_j$ and e_j satisties $(e_j, \gamma_i) = \delta_{ij}$, $1 \leq i \leq n$, as for the highest restricted root $\psi = \sum_{i=1}^{n} d_i\gamma_i$, there must be $d_j = 1$.

Proof. The lemma follows directly from Proposition 3.1 in [14]. □

Lemma 4.2. *Let U be a compact connected Lie group. If the type of the antipodal set of unit e is of type $\mathcal{P}^1$, then the single point must be the non-trivial center element of U.*

Proof. If we denote the single point as $\exp H$ (when we view U as a symmetric space), then $\exp H = \mathrm{Exp}(d\pi(\frac{H}{2}, -\frac{H}{2})) = (\exp\frac{H}{2}, \exp\frac{-H}{2})U^*$. Thus we have

$$\begin{aligned}&\mathrm{Exp}(d\pi(\mathrm{Ad}g\tfrac{H}{2}, \mathrm{Ad}g\tfrac{-H}{2}))) = \mathrm{Exp}(d\pi(\tfrac{H}{2}, \tfrac{-H}{2})) \Leftrightarrow (g,g)(\exp\tfrac{H}{2}, \exp\tfrac{-H}{2})U^* \\ &\quad = (\exp\tfrac{H}{2}, \exp\tfrac{-H}{2})U^* \Leftrightarrow g\exp H = \exp Hg, \quad \forall g \in U.\end{aligned}$$

This completes the proof of the lemma. □

Corollary 4.3. *The type of the antipodal sets for G_2, F_4, and E_8 is $\mathcal{O}^1$.*

From Table 1, Lemma 4.1 and Lemma 4.2, we can obtain the structure of the antipodal set for each compact connected simply connected irreducible symmetric space as follows:

4.1 Compact simple Lie Groups

M	the highest restricted root ψ	the antipodal sets A_o
SU(n), ($n \geq 2$)	$\sum_{i=1}^{n-1}\gamma_i$	$\mathcal{P}^1$, $2 \mid n$.
		$\mathcal{P}^2$, $2 \nmid n$.
Spin($2n+1$), ($n \geq 2$)	$\gamma_1 + 2\sum_{i=2}^{n}\gamma_i$	$\mathcal{P}^1$, $n \leq 3$.
		$\mathcal{P}^1\mathcal{O}^1$, $n = 4$.
		$\mathcal{O}^1$, $n \geq 5$.
Sp(n), ($n \geq 3$)	$2\sum_{i=1}^{n-1}\gamma_i + \gamma_n$	$\mathcal{P}^1$.
Spin($2n$), ($n \geq 4$)	$\gamma_1 + 2\sum_{i=2}^{n}\gamma_i$ $+\gamma_{n-1} + \gamma_n$	$\mathcal{P}^3$, $n = 4$.
		$\mathcal{P}^2$, $n \geq 5$.
G_2	$2\gamma_1 + 3\gamma_2$	$\mathcal{O}^1$.
F_4	$2\gamma_1 + 3\gamma_2 + 4\gamma_3 + 2\gamma_4$	$\mathcal{O}^1$.
E_6	$\gamma_1 + 2\gamma_2 + 3\gamma_3 + 2\gamma_4 + \gamma_5 + 2\gamma_6$	$\mathcal{P}^2$.
E_7	$\gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 3\gamma_5 + 2\gamma_6 + 2\gamma_7$	$\mathcal{P}^1$.
E_8	$2\gamma_1 + 3\gamma_2 + 4\gamma_3 + 5\gamma_4 + 6\gamma_5 + 4\gamma_6 + 2\gamma_7 + 3\gamma_8$	$\mathcal{O}^1$.

Remark 4.1. From the relationships $\mathrm{Spin}(4) \cong \mathrm{Sp}(1)\times\mathrm{Sp}(1)$ and $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ (see page 141 of [4]), one easily sees that the type of the antipodal set of the unit in Spin(4) is $\mathcal{P}^1$.

4.2 Simply-connected irreducible Riemannian symmetric spaces of type I

For the simply-connected irreducible Riemannian symmetric spaces of type I, we have:

the type	M	Σ	the antipodal sets A_o
AI	$\mathrm{SU}(n)/\mathrm{SO}(n)$, $(n \geq 2)$	$\mathfrak{a}_{n-1}$	$\mathcal{P}^1,\ 2 \mid n.$
			$\mathcal{P}^2,\ 2 \nmid n.$
AII	$\mathrm{SU}(2n)/\mathrm{Sp}(n)$, $(n \geq 2)$	$\mathfrak{a}_{n-1}$	$\mathcal{P}^1,\ 2 \mid n.$
			$\mathcal{P}^2,\ 2 \nmid n.$
$AIII$	$G_{p,q}(\mathbb{H})$	$(\mathfrak{bc})_p$	$\mathcal{O}^1,\ 2 \leq p < q$ or $p = 1.$
		$\mathfrak{c}_p$	$\mathcal{P}^1,\ 2 \leq p = q.$
CI	$\mathrm{Sp}(n)/\mathrm{U}(n)$	$\mathfrak{c}_n$	$\mathcal{P}^1$
CII	$G_{p,q}(\mathbb{H})$	$(\mathfrak{bc})_p$	$\mathcal{O}^1,\ 2 \leq p < q$ or $p = 1.$
		$\mathfrak{c}_p$	$\mathcal{P}^1.\ 2 \leq p = q.$
BDI	$\mathrm{G}_{p,q}$, $(p < q)$	$\mathfrak{b}_p$	$\mathcal{P}^1,\ p = 2, 3.$
		$\mathfrak{a}_1$	$\mathcal{P}^1,\ p = 1.$
		$\mathfrak{b}_4$	$\mathcal{P}^1\mathcal{O}^1,\ p = 4.$
		$\mathfrak{b}_p$	$\mathcal{O}^1,\ p \geq 5.$
	$\mathrm{G}_{p,p}$, $(p \geq 4)$	$\mathfrak{d}_4$	$\mathcal{P}^3$, $p = 4.$
		$\mathfrak{d}_p$	$\mathcal{P}^2$, $p \geq 5.$
$DIII$	$\mathrm{SO}(2n)/\mathrm{U}(n)$	$\mathfrak{c}_{\frac{n}{2}}$	$\mathcal{P}^1,\ 2 \mid n.$
		$(\mathfrak{bc})_{\frac{n-1}{2}}$	$\mathcal{O}^1,\ 2 \nmid n.$
EI	$(\mathfrak{e}_6, \mathfrak{sp}_4)$	$\mathfrak{e}_6$	$\mathcal{P}^2.$
EII	$(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2)$	$\mathfrak{f}_4$	$\mathcal{O}^1.$
$EIII$	$(\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{R})$	$(\mathfrak{bc})_2$	$\mathcal{O}^1.$
EIV	$(\mathfrak{e}_6, \mathfrak{f}_4)$	$\mathfrak{a}_2$	$\mathcal{P}^2.$
EV	$(\mathfrak{e}_7, \mathfrak{su}_8)$	$\mathfrak{e}_7$	$\mathcal{P}^1.$
EVI	$(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2)$	$\mathfrak{f}_4$	$\mathcal{O}^1.$
$EVII$	$(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R})$	$\mathfrak{c}_3$	$\mathcal{P}^1.$
$EVIII$	$(\mathfrak{e}_8, \mathfrak{so}_{16})$	$\mathfrak{e}_8$	$\mathcal{O}^1.$
EIX	$(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2)$	$\mathfrak{f}_4$	$\mathcal{O}^1.$
FI	$(\mathfrak{f}_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2)$	$\mathfrak{f}_4$	$\mathcal{O}^1.$
FII	$(\mathfrak{f}_4, \mathfrak{so}_9)$	$(\mathfrak{bc})_1$	$\mathcal{O}^1.$
G	$(\mathfrak{g}_2, \mathfrak{so}_4)$	$\mathfrak{g}_2$	$\mathcal{O}^1.$

5 Proof of Theorem 1.1

It is well known that any connected simply-connected compact Riemannian symmetric space M can be decomposed as:

$$M = M_1 \times \ldots \times M_r,$$

where the factors M_i are irreducible compact connected simply connected Riemannian symmetric spaces (see for example [7]). Combining this fact with the the above description of antipodal sets we get the proof of Theorem 1.1.

Acknowledgement. The present work was supported by NSFC (no. 11271198, 10971104) and SRFDP of China.

References

[1] S. I. Araki, *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ. 13 (1962), 1-34.

[2] B. Y. Chen and T. Nagano, *Totally geodesic submanifolds of symmetric spaces II*, Duke Mathematical Journal 45 (1978), 405-425.

[3] R. Crittenden, *Minimum and conjugate points in symmetric spaces*, Canadian J. Math. 14 (1962),320-328.

[4] M. L. Curtis, *Matrix Groups*, 2nd ed., Springer, 1984.

[5] S. Deng, *Homogeneous Finsler Spaces*, Springer Monographs in Mathematics, Springer, New York, 2012.

[6] S. Deng and Z. Hou, *On symmetric Finsler spaces*, Israel Journal of Mathematics, 162 (2007), 197-219.

[7] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Acaademic Press, New York, 1978.

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Volume 2, Intersecice Publishers, 1969.

[9] T. Sakai, *On cut loci of compact symmetric spaces*, Hokkaido Mathematical Journal, 6 (1977), 136-161.

[10] T. Sakai, *On the structure of cut loci in compact Riemannian symmetric spaces*, Math. Ann., 235 (1978), 129-148.

[11] M. Takeuchi, *On conjugate loci and cut loci of compact symmetric spaces, I*, Tsukuba J. Math., 2 (1978), 35-68. Correction in 7 (1983),203-204.

[12] A. D. Weinstein, *The cut locus and conjugate locus of a Riemannian manifold*, Ann. of Math. (2), 87 (1968), 29-41.

[13] L. Yang, *Injectivity radius and Cartan polyhedron for simply connected symmetric spaces*, Chin. Ann. Math., 28 B (6) (2007), 685-700.

[14] L. Yang, *Injectivity radius for non-simply connected symmetric spaces via Cartan polyhedron*, Osaka J. Math, 45 (2008), 511-540.

Authors' address:

Xingda Liu and Shaoqiang Deng (corresponding author)
School of Mathematical Sciences and LPMC, Nankai University
Tianjin 300071, People's Republic of China.
E-mail: dengsq@nankai.edu.cn