Indefinite Kähler manifolds with the Krupka-type curvature tensor

Hyang Sook Kim, Jung-Hwan Kwon and Jin Suk Pak

Abstract. In this paper we investigate several properties of indefinite Kähler manifold of complex dimension n $(n > 2)$ with the Krupka-type curvature tensor, and present several classes of indefinite complex submanifolds of an indefinite complex space form.

M.S.C. 2010: 53C40, 53C25.
Key words: indefinite Kähler manifolds; conformal curvature tensors; Krupka-type curvature tensor; indefinite complex submanifolds.

1 Introduction

In 1990, H. Kitahara, K. Matsuo and J. S. Pak ([6, 7]) defined a new tensor field on a Hermitian manifold which is conformally invariant and studied several properties of the new tensor field. This new tensor field is said to be the *conformal curvature tensor* for briefness.

In 2006, S. Funabashi, Y.-M. Kim, the first and third authors ([5]) defined *traceless component of the conformal curvature tensor field* $\hat{C}$ on a Kähler manifold analogous to the trace decomposition problems of D. Krupka ([8]). Hereafter, this tensor $\hat{C}$ is called the *Krupka-type curvature tensor*.

In this point of view, we investigate several properties of an indefinite Kähler manifold of the complex dimension n $(n > 2)$ with the Krupka-type curvature tensor, and study the relations between the Krupka-type curvature tensor $\hat{C}$, the Bochner curvature tensor B, the conformal curvature tensor C, the Weyl curvature tensor W and the concircular curvature tensor Z, and determine several classes of indefinite complex submanifolds of an indefinite complex space form. Specifically, in Section 2 of this paper we recall a brief summary of the complex version of indefinite Kähler manifolds and some fundamental formulas of indefinite complex submanifolds of an indefinite Kähler manifold. Section 3 is devoted to investigate some properties of an indefinite Kähler manifold with parallel or vanishing Krupka-type curvature tensor, and study the relations between $\hat{C}$, B, C, W and Z. In section 4 we present several classes of indefinite complex submanifolds of an indefinite complex space form.

Balkan Journal of Geometry and Its Applications, Vol.19, No.1, 2014, pp. 62-72.

All manifolds are assumed connected and all manifolds and maps are assumed smooth(class C^∞) unless otherwise stated. Notation and definitions not explicitly introduced may be found in [11] or [13].

2 Indefinite Kähler manifolds

We adopt the notation and terminology from [11]. We start this section by introducing some basic formulas concerning indefinite Kähler manifolds. Let M be a complex $n(\geq 2)$-dimensional connected indefinite Kähler manifold equipped with Kähler metric tensor g and almost complex structure J. Then for the indefinite Kähler structure (g, J), it is known that J is integrable and the index of g is even, say $2s(0 \leq s \leq n)$.

A local unitary frame field $\{E_1, \dots, E_n\}$ on a neighborhood of M can be chosen. This is a complex linear frame which is orthonormal with respect to the Kähler metric, that is, $(E_i, \overline{E}_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = \pm 1$ and $i, j = 1, 2, \cdots, n$. The dual frame field $\{\omega_1, \dots, \omega_n\}$ $(i, j = 1, 2, \cdots, n)$ of the frame field $\{E_j\}$ consists of complex-valued 1-forms ω_i of type $(1, 0)$ on M such that $\omega_i(E_j) = \varepsilon_i \delta_{ij}$ and $\{\omega_1, \dots, \omega_n, \overline{\omega}_1, \dots, \overline{\omega}_n\}$ is linearly independent. Then we see that the Kähler metric g of M can be expressed as $g = 2\sum_j \varepsilon_j \omega_j \otimes \overline{\omega}_j$. Associated with the frame field $\{E_j\}$, there exist complex-valued 1-forms ω_{ij}, which are usually *connection forms* on M such that they satisfy the structure equations of M:

$$
\begin{aligned}
&d\omega_i + \sum_j \varepsilon_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \overline{\omega}_{ji} = 0, \\
&d\omega_{ij} + \sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\
&\Omega_{ij} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{i}jk\bar{l}} \omega_k \wedge \overline{\omega}_l,
\end{aligned} \tag{2.1}
$$

where Ω_{ij}(resp. $R_{\bar{i}jk\bar{l}}$) denotes the curvature form (resp. the components of the Riemannian curvature tensor R) on M. The second equation of (2.1) means the skew-hermitian symmetry of Ω_{ij}, which is equivalent to the symmetric condition

$$R_{\bar{i}jk\bar{l}} = \overline{R}_{\bar{j}il\bar{k}}\ , \tag{2.2}$$

for $i, j, k, l = 1, 2, \cdots, n$. The Bianchi identity obtained by the exterior derivatives of (2.1) gives $\sum_j \varepsilon_j \Omega_{ij} \wedge \omega_j = 0$, which yields the following further symmetric relations

$$R_{\bar{i}jk\bar{l}} = R_{\bar{i}kj\bar{l}} = R_{\bar{l}jk\bar{i}} = R_{\bar{l}kj\bar{i}}. \tag{2.3}$$

Now, with respect to the frame field chosen above, the Ricci tensor S of M is given by

$$S = 2\sum_{i,j} \varepsilon_i \varepsilon_j S_{i\bar{j}} \omega_i \otimes \overline{\omega}_j,$$

where $S_{i\bar{j}} = \sum_k \varepsilon_k R_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \overline{S}_{\bar{i}j}$. Moreover we can express the scalar curvature r as the identity $r = 2\sum_j \varepsilon_j S_{j\bar{j}}$.

The indefinite Kähler manifold M is said to be *Einstein* if the Ricci tensor S is given by

$$S_{\bar{j}i} = \alpha\varepsilon_i\delta_{ij}, \tag{2.4}$$

where $\alpha = \frac{r}{2n}$.

The components $R_{\bar{i}jk\bar{l}m}$ and $R_{\bar{i}jk\bar{l}\overline{m}}$ (resp. $S_{\bar{j}ik}$ and $S_{\bar{j}i\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor R (resp. the Ricci tensor S) are defined by the following equation (2.5) (resp. (2.6))

$$\begin{aligned}&\sum_m \varepsilon_m(R_{\bar{i}jk\bar{l}m}\omega_m + R_{\bar{i}jk\bar{l}\overline{m}}\overline{\omega}_m) = dR_{\bar{i}jk\bar{l}}\\ &-\sum_m \varepsilon_m(R_{\overline{m}jk\bar{l}}\overline{\omega}_{mi} + R_{\bar{i}mk\bar{l}}\omega_{mj} + R_{\bar{i}jm\bar{l}}\omega_{mk} + R_{\bar{i}jk\overline{m}}\overline{\omega}_{ml}),\end{aligned} \tag{2.5}$$

$$\sum_k \varepsilon_k(S_{\bar{j}ik}\omega_k + S_{\bar{j}i\bar{k}}\overline{\omega}_k) = dS_{\bar{j}i} - \sum_k \varepsilon_k(S_{\bar{j}k}\omega_{ki} + S_{\bar{k}i}\overline{\omega}_{kj}). \tag{2.6}$$

The second Bianchi formula is given by $R_{\bar{i}jk\bar{l}m} = R_{\bar{i}jm\bar{l}k}$ and hence we have

$$S_{\bar{j}ik} = S_{\bar{j}ki} = \sum_l \varepsilon_l R_{\bar{j}ik\bar{l}l}, \quad r_j = 2\sum_k \varepsilon_k S_{\bar{k}jk}, \tag{2.7}$$

where $dr = \sum_j \varepsilon_j(r_j\omega_j + r_{\bar{j}}\overline{\omega}_j)$. A plane section P of the tangent space T_xM of M at any point x is said to be *non-degenerate*, provided that $g_x \mid_{T_xM}$ is non-degenerate. It is easily seen that P is non-degenerate if and only if it has a basis $\{u, v\}$ such that $g(u,u)g(v,v) - g(u,v)^2 \neq 0$, and a holomorphic plane spanned by u and Ju is non-degenerate if and only if it contains some v with $g(v,v) \neq 0$. The sectional curvature of the non-degenerate holomorphic plane P spanned by u and Ju is called the *holomorphic sectional curvature* which is denoted by $H(P) = H(u)$. The indefinite Kähler manifold M is said to be *of constant holomorphic sectional curvature* if its holomorphic sectional curvature $H(P)$ is constant for all P and for all points of M. An indefinite Kähler manifold M of constant holomorphic sectional curvature, say c, is called an *indefinite complex space form* and is denoted by $M_s^n(c)$ if M is of complex dimension n and of index $2s$.

It is known that the standard models of indefinite complex space forms are the following ([2, 13]):

(1) indefinite complex Euclidean space C_s^n,

(2) indefinite complex projective space P_s^nC,

(3) indefinite complex hyperbolic space H_s^nC.

It is shown in [2] and [13] that for any integer $s(0 \leq s \leq n)$ the above three models are the only complete, simply connected and connected indefinite complex space forms of dimension n and of index $2s$, according as $c = 0$, $c > 0$ and $c < 0$ respectively. We also we recall that the Riemannian curvature tensor $R_{\bar{i}jk\bar{l}}$ of $M_s^n(c)$ is given by

$$R_{\bar{i}jk\bar{l}} = \frac{c}{2}\varepsilon_j\varepsilon_k(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}). \tag{2.8}$$

From now on let M' be an $(n+p)$-dimensional connected indefinite Kähler manifold of index $2(s+t)(0 \le s \le n,\ \ 0 \le t \le p)$ and let M be an n-dimensional connected indefinite complex submanifold of M' of index $2s$.

Then M is the indefinite Kähler manifold endowed with the induced metric tensor g. We choose a local unitary frame field $\{E_A\} = \{E_1, \ldots, E_{n+p}\}$ on a neighborhood of M' in such a way that restricted to M, $E_1, \ldots, E_n$ are tangent to M and the others are normal to M. Here and in the sequel the following convention on the range of indices is used unless otherwise stated:

$$\begin{aligned} A,\ B,\ C, \ldots\ &= 1, \ldots,\ n,\ n+1, \ldots,\ n+p, \\ i,\ j,\ k, \ldots\ &= 1, \ldots,\ n, \\ x,\ y,\ z, \ldots\ &= n+1, \ldots,\ n+p. \end{aligned}$$

With respect to the above frame field $\{E_A\}$, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame field. Then the Kähler metric tensor g' of M' is given by

$$g' = 2\sum_A \varepsilon_A \omega_A \otimes \overline{\omega}_A.$$

The connection forms on M' are denoted by ω_{AB}. The canonical forms ω_A and the connection forms ω_{AB} of the ambient space satisfy the structure equations (2.1).

Restricting these forms to the submanifold M, we have $\omega_x = 0$ and the induced indefinite Kähler metric tensor g of index $2s$ of M is given by $g = 2\sum_j \varepsilon_j \omega_j \otimes \overline{\omega}_j$. Then $\{E_j\}$ is a local unitary frame field with respect to this metric and $\{\omega_j\}$ is a local dual frame field due to $\{E_j\}$ which consists of complex-valued 1-forms of type $(1,0)$ on M. Moreover $\{\omega_1, \cdots, \omega_n, \overline{\omega}_1, \cdots, \overline{\omega}_n\}$ is linearly independent and they are canonical forms on M. It follows from $\omega_x = 0$ and the Cartan lemma that the exterior derivatives of $\omega_x = 0$ give rise to

$$\omega_{xi} = \sum_j \varepsilon_j h^x_{ij} \omega_j, \quad h^x_{ij} = h^x_{ji}. \tag{2.9}$$

The quadratic form $\sum_{i,j,x} \varepsilon_i \varepsilon_j \varepsilon_x h^x_{ij} \omega_i \otimes \omega_j \otimes E_x$ with values in the normal bundle is called the *second fundamental form* of the submanifold M ([1]). From the structure equations of M' it follows that the structure equations of M are similarly given by (2.1). Moreover the following relationships are defined:

$$\begin{aligned} d\omega_{xy} + \sum_z \varepsilon_z \omega_{xz} \wedge \omega_{zy} &= \Omega_{xy}, \\ \Omega_{xy} &= \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{x}yk\bar{l}} \omega_k \wedge \overline{\omega}_l, \end{aligned} \tag{2.10}$$

where Ω_{xy} is called the *normal curvature form* of M.

For the Riemannian curvature tensors R and R' of M and M' respectively, it follows from the third equation of (2.1) and (2.9) that the Gauss equation

$$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \sum_x \varepsilon_x h^x_{jk} \overline{h}^x_{il} \tag{2.11}$$

holds and by means of (2.2), (2.9) and (2.10) we can see that

$$R_{\overline{x}yk\bar{l}} = R'_{\overline{x}yk\bar{l}} + \sum_j \varepsilon_j h^x_{kj} \overline{h}^y_{jl}.$$

It is easy to compute that the components of the Ricci tensor S and the scalar curvature r of M satisfy the identities, respectively

$$S_{\bar{j}i} = \sum_k \varepsilon_k R'_{\overline{k}ki\bar{j}} - (h_{\bar{j}i})^2, \tag{2.12}$$

$$r = 2\sum_{j,k} \varepsilon_j \varepsilon_k R'_{\overline{j}jk\overline{k}} - 2h_2, \tag{2.13}$$

where $(h_{\bar{j}i})^2 = \sum_{r,x} \varepsilon_r \varepsilon_x h^x_{ir} \overline{h}^x_{rj}$ and $h_2 = \sum_i \varepsilon_i (h_{\bar{i}i})^2$.

Hereafter, let the ambient space be an indefinite complex space form $M' = M^{n+p}_{s+t}(c')$. Then from (2.8) and (2.11)-(2.13), we say that

$$R_{\bar{i}jk\bar{l}} = \frac{c'}{2}\varepsilon_j \varepsilon_k (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x \varepsilon_x h^x_{jk} \overline{h}^x_{il}, \tag{2.14}$$

$$S_{\bar{j}i} = \frac{(n+1)c'}{2}\varepsilon_i \delta_{ij} - (h_{\bar{j}i})^2, \tag{2.15}$$

$$r = n(n+1)c' - 2h_2. \tag{2.16}$$

3 Several results on an indefinite Kähler manifold

Let M be a complex $n(>2)$-dimensional indefinite Kähler manifold. The Krupka-type curvature tensor $\hat{C}$ with components $\hat{C}_{\bar{i}jk\bar{l}}$ of M is given by

$$\begin{aligned} \hat{C}_{\bar{i}jk\bar{l}} = {} & R_{\bar{i}jk\bar{l}} - \frac{1}{n}(\varepsilon_j \delta_{ij} S_{k\bar{l}} + \varepsilon_k S_{\bar{i}j}\delta_{kl}) - \frac{2(n-2)}{n(2n-1)}\varepsilon_j \delta_{jl} S_{\bar{i}k} \\ & + \frac{(n+2)r}{2n^2(n+1)}\varepsilon_j \varepsilon_k \delta_{ij}\delta_{kl} - \frac{(n+4)r}{2n^2(n+1)(2n-1)}\varepsilon_j \varepsilon_k \delta_{ik}\delta_{jl}, \end{aligned} \tag{3.1}$$

which may be found in [5]. Let $\hat{S}$ denote the Ricci contraction of $\hat{C}$, that is,

$$\hat{S}_{k\bar{l}} = \sum_i \varepsilon_i \hat{C}_{\bar{i}ik\bar{l}}. \tag{3.2}$$

From (3.1) and (3.2), it is clear that

$$\hat{S}_{k\bar{l}} = -\frac{2(n-2)}{n(2n-1)}(S_{k\bar{l}} - \frac{r}{2n}\varepsilon_k \delta_{kl}). \tag{3.3}$$

Summing up the equation (3.3) for k and l and taking account of $r = 2\sum_k \varepsilon_k S_{k\bar{k}}$, we obtain

$$\sum_k \varepsilon_k \hat{S}_{k\bar{k}} = 0. \tag{3.4}$$

If the Ricci contraction $\hat{S}$ vanishes everywhere i.e., $\hat{S}_{k\bar{l}} = 0$ and $n > 2$, then we obtain $S_{k\bar{l}} = \frac{r}{2n}\varepsilon_k \delta_{kl}$ because of (3.3). Since this equation represents the first Chern class, it follows that r is constant. Thus M is Einstein by means of (2.4). Conversely, if M is Einstein, then we see that $\hat{S}_{k\bar{l}} = 0$ with the aid of (2.4) and (3.3).

Thus we get the following lemma.

Lemma 3.1. *Let M be an indefinite Kähler manifold of complex dimension $n(n > 2)$. Then the Ricci contraction $\hat{S}$ of the Krupka-type curvature tensor $\hat{C}$ of M vanishes everywhere if and only if M is Einstein.*

Remark 3.1. The real version of lemma 3.1 was proved by S. Funabashi, Y.-M. Kim, the first and third authors ([4]).

The Bochner curvature tensor B with components $B_{\bar{i}jk\bar{l}}$ of the indefinite Kähler manifold is given by

$$\begin{aligned} B_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} & \\ & - \frac{1}{n+2}(\varepsilon_j \delta_{ij} S_{k\bar{l}} + \varepsilon_k S_{\bar{i}j}\delta_{kl} + \varepsilon_k \delta_{ik} S_{j\bar{l}} + \varepsilon_j S_{\bar{i}k}\delta_{jl}) \\ & + \frac{r}{2(n+1)(n+2)}\varepsilon_j \varepsilon_k (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}), \end{aligned} \tag{3.5}$$

which was introduced by S. Bochner ([3]). Thus, from (3.1) and (3.5), we know that

$$\begin{aligned} \hat{C}_{\bar{i}jk\bar{l}} = \ & B_{\bar{i}jk\bar{l}} - \frac{2(n-2)}{n(2n-1)}\varepsilon_j \delta_{jl} S_{\bar{i}k} \\ & + \frac{1}{n+2}\{\varepsilon_k \delta_{ik} S_{j\bar{l}} + \varepsilon_j S_{\bar{i}k}\delta_{jl} - \frac{(n^2-n+4)r}{n^2(2n-1)}\varepsilon_j \varepsilon_k \delta_{ik}\delta_{jl}\} \\ & - \frac{2}{n(n+2)}(\varepsilon_j \delta_{ij} S_{k\bar{l}} + \varepsilon_k S_{\bar{i}j}\delta_{kl} - \frac{r}{n}\varepsilon_j \varepsilon_k \delta_{ij}\delta_{kl}). \end{aligned} \tag{3.6}$$

If $n > 2$, then by means of (3.3) and the last equation (3.6), we obtain

$$\begin{aligned} \hat{C}_{\bar{i}jk\bar{l}} = B_{\bar{i}jk\bar{l}} & + \varepsilon_j \delta_{jl} \hat{S}_{\bar{i}k} \\ & - \frac{n(2n-1)}{2(n+2)(n-2)}(\varepsilon_k \delta_{ik} \hat{S}_{j\bar{l}} + \varepsilon_j \hat{S}_{\bar{i}k}\delta_{jl}) \\ & + \frac{2n-1}{(n+2)(n-2)}(\varepsilon_j \delta_{ij} \hat{S}_{k\bar{l}} + \varepsilon_k \hat{S}_{\bar{i}j}\delta_{kl}). \end{aligned} \tag{3.7}$$

Assume that $\hat{C} = B$ and $n > 2$. Then the equation (3.7) reduces to

$$\begin{aligned} & n(\varepsilon_k \delta_{ik} \hat{S}_{j\bar{l}} + \varepsilon_j \hat{S}_{\bar{i}k}\delta_{jl}) - 2(\varepsilon_j \delta_{ij} \hat{S}_{k\bar{l}} + \varepsilon_k \hat{S}_{\bar{i}j}\delta_{kl}) \\ & - \frac{2(n+2)(n-2)}{2n-1}\varepsilon_j \delta_{jl} \hat{S}_{\bar{i}k} = 0. \end{aligned}$$

Summing up the above equation for i and k and making use of (3.4), we get $\hat{S}_{j\bar{l}} = 0$. Conversely, if $\hat{S}_{j\bar{l}} = 0$ and $n > 2$, then the equation (3.7) implies $\hat{C} = B$.

Furthermore owing to Lemma 3.1, we can have

Proposition 3.2. *Let M be an indefinite Kähler manifold of complex dimension $n(n > 2)$. Then the Krupka-type curvature tensor is equal to the Bochner curvature tensor on M if and only if M is Einstein.*

The conformal curvature tensor C with components $C_{\bar{i}jk\bar{l}}$ of M is given by

$$\begin{aligned} C_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} &- \frac{1}{n}(\varepsilon_j \delta_{ij} S_{\bar{l}k} + \varepsilon_k S_{\bar{i}j}\delta_{kl}) \\ &+ \frac{(n+2)r}{2n^2(n+1)}\varepsilon_j\varepsilon_k\delta_{ij}\delta_{kl} - \frac{r}{2n(n+1)}\varepsilon_j\varepsilon_k\delta_{ik}\delta_{jl}, \end{aligned} \tag{3.8}$$

which was introduced in [6].

The last equation (3.8) combined with (3.1) yields

$$\hat{C}_{\bar{i}jk\bar{l}} = C_{\bar{i}jk\bar{l}} - \frac{2(n-2)}{n(2n-1)}(S_{\bar{i}k} - \frac{r}{2n}\varepsilon_k\delta_{ik})\varepsilon_j\delta_{jl}. \tag{3.9}$$

Assume that $\hat{C} = C$ and $n > 2$. Then the equation (3.9) gives $S_{\bar{i}k} = \frac{r}{2n}\varepsilon_k\delta_{ik}$. Conversely, if $S_{\bar{i}k} = \frac{r}{2n}\varepsilon_k\delta_{ik}$, then we say that $\hat{C} = C$ by means of (3.9).

Thus we obtain

Proposition 3.3. *Let M be an indefinite Kähler manifold of complex dimension $n(n > 2)$. Then the Krupka-type curvature tensor is equal to the conformal curvature tensor on M if and only if M is Einstein.*

Remark 3.2. Let M be an indefinite Kähler manifold of complex dimension 2. Then the Krupka-type curvature tensor is equal to the conformal curvature tensor on M.

Remark 3.3. Making use of the Proposition 3.2 in [10], we can also prove the above Proposition 3.3.

Remark 3.4. Let M be an indefinite Kähler manifold of complex dimension $n(n > 2)$. The Weyl curvature tensor W with components $W_{\bar{i}jk\bar{l}}$ is defined by

$$W_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{1}{n+1}(\varepsilon_j\delta_{ij}S_{k\bar{l}} + \varepsilon_k\delta_{ik}S_{j\bar{l}}).$$

It is easy to know that the Weyl curvature tensor is equal to the Bochner curvature tensor if and only if M is Einstein.

Remark 3.5. Let M be an indefinite Kähler manifold of complex dimension $n(n > 2)$ and let Z be a concircular curvature tensor is defined in [12]. Then the concircular curvature tensor is equal to the Weyl curvature tensor if and only if M is Einstein.

Remark 3.6. Let M be an indefinite Kähler manifold of complex dimension $n(n > 2)$. Then any two tensors among B, C, $\hat{C}$, W and Z are equal to each other if and only if M is Einstein. In fact, with the help of Proposition 3.2, 3.3, Remark 3.4 and 3.5, we can see that our assertion is true.

The components $\hat{S}_{\bar{i}jk}$ and $\hat{S}_{\bar{i}j\bar{k}}$ of the covariant derivative of the Ricci contraction $\hat{S}$ of the Krupka-type curvature tensor $\hat{C}$ are defined by

$$\sum_k \varepsilon_k(\hat{S}_{\bar{i}jk}\omega_k + \hat{S}_{\bar{i}j\bar{k}}\overline{\omega}_k) = d\hat{S}_{\bar{i}j} - \sum_k \varepsilon_k(\hat{S}_{\bar{k}j}\overline{\omega}_{ki} + \hat{S}_{\bar{i}k}\omega_{kj}). \tag{3.10}$$

Since we see that $dr = \sum_j \varepsilon_j(r_j\omega_j + r_{\bar{j}}\overline{\omega}_j)$, taking account of (2.1), (2.6), (3.3) and (3.10), we get

$$\begin{aligned}&\sum_k \varepsilon_k(\hat{S}_{\bar{i}jk}\omega_k + \hat{S}_{\bar{i}j\bar{k}}\overline{\omega}_k)\\ &= -\frac{2(n-2)}{n(2n-1)}\sum_k \varepsilon_k\{S_{\bar{i}jk}\omega_k + S_{\bar{i}j\bar{k}}\overline{\omega}_k - \frac{1}{2n}\varepsilon_j\delta_{ij}(r_k\omega_k + r_{\bar{k}}\overline{\omega}_k)\}\end{aligned}$$

so that

$$\begin{aligned}\hat{S}_{\bar{i}jk} &= -\frac{n(n-2)}{n(2n-1)}(S_{\bar{i}jk} - \frac{1}{2n}\varepsilon_j\delta_{ij}r_k),\\ \hat{S}_{\bar{i}j\bar{k}} &= -\frac{n(n-2)}{n(2n-1)}(S_{\bar{i}j\bar{k}} - \frac{1}{2n}\varepsilon_j\delta_{ij}r_{\bar{k}}).\end{aligned} \tag{3.11}$$

Assume that the Ricci contraction $\hat{S}$ is parallel, i.e., $\hat{S}_{\bar{i}jk} = 0$ and $\hat{S}_{\bar{i}j\bar{k}} = 0$. If $n > 2$, then from (3.11) it turns out to be

$$S_{\bar{i}jk} = \frac{1}{2n}\varepsilon_j\delta_{ij}r_k, \qquad S_{\bar{i}j\bar{k}} = \frac{1}{2n}\varepsilon_j\delta_{ij}r_{\bar{k}}. \tag{3.12}$$

Then the last equation (3.12) coupled with (2.7) reduces to $r_k = 0$ and $r_{\bar{k}} = 0$. Substituting these equations into (3.12), we obtain $S_{\bar{i}jk} = 0$ and $S_{\bar{i}j\bar{k}} = 0$, that is, the Ricci tensor is parallel.

Conversely, if the Ricci tensor is parallel, then $r_k = 0$ and $r_{\bar{k}} = 0$, and consequently $\hat{S}_{\bar{i}jk} = 0$ and $\hat{S}_{\bar{i}j\bar{k}} = 0$ with the help of (3.11).

Thus we established the following

Proposition 3.4. *Let M be an indefinite Kähler manifold of complex dimension $n(n > 2)$. Then the Ricci contraction of the Krupka-type curvature tensor is parallel if and only if the Ricci tensor is parallel.*

Remark 3.7. The real version of proposition 3.4 was proved by S. Funabashi, Y.-M. Kim, the first and third authors ([4]).

Owing to Proposition 3.4 and Theorem due to the second author ([9]), the following result is immediate

Corollary 3.5. *Let M be an indefinite Kaehlerian manifold of complex dimension $n(n > 2)$. Then the following assertions are equivalent to each other:*

(1) *the Ricci contraction of the Krupka-type curvature tensor of M is parallel,*
(2) *M has harmonic curvature,*
(3) *the Ricci tensor of M is cyclic-parallel.*

Let M be an indefinite Kähler manifold of complex dimension $n(n > 2)$. The components $\hat{C}_{\bar{i}jk\bar{l}m}$ and $\hat{C}_{\bar{i}jk\bar{l}\overline{m}}$ of the covariant derivative of the Krupka-type curvature tensor $\hat{C}$ are defined by

$$\begin{aligned}\sum_m \varepsilon_m(\hat{C}_{\bar{i}jk\bar{l}m}\omega_m + \hat{C}_{\bar{i}jk\bar{l}\overline{m}}\overline{\omega}_m) = d\hat{C}_{\bar{i}jk\bar{l}} - \sum_m \varepsilon_m(\hat{C}_{\bar{i}jk\bar{l}}\overline{\omega}_{mi} \\ + \hat{C}_{\bar{i}mk\bar{l}}\omega_{mj} + \hat{C}_{\bar{i}jm\bar{l}}\omega_{mk} + \hat{C}_{\bar{i}jk\overline{m}}\overline{\omega}_{ml}).\end{aligned} \tag{3.13}$$

Since $dr = \sum_j \varepsilon_j(r_j\omega_j + r_{\bar{j}}\overline{\omega}_j)$, it follows from (2.1), (2.5), (2.6), (3.1) and (3.13) that

$$\begin{aligned}\hat{C}_{\bar{i}jk\bar{l}m} = R_{\bar{i}jk\bar{l}m} &- \frac{1}{n}(\varepsilon_j\delta_{ij}S_{k\bar{l}m} + \varepsilon_k S_{\bar{i}jm}\delta_{kl}) \\ &- \frac{2(n-2)}{n(2n-1)}\varepsilon_j\delta_{jl}S_{\bar{i}km} + \frac{(n+2)r_m}{2n^2(n+1)}\varepsilon_j\varepsilon_k\delta_{ij}\delta_{kl} \\ &- \frac{(n+4)r_m}{2n^2(n+1)(2n-1)}\varepsilon_j\varepsilon_k\delta_{ik}\delta_{jl},\end{aligned} \tag{3.14}$$

$$\begin{aligned}\hat{C}_{\bar{i}jk\bar{l}\overline{m}} = R_{\bar{i}jk\bar{l}\overline{m}} &- \frac{1}{n}(\varepsilon_j\delta_{ij}S_{\bar{l}k\overline{m}} + \varepsilon_k S_{\bar{i}j\overline{m}}\delta_{kl}) \\ &- \frac{2(n-2)}{n(2n-1)}\varepsilon_j\delta_{jl}S_{\bar{i}k\overline{m}} + \frac{(n+2)r_{\overline{m}}}{2n^2(n+1)}\varepsilon_j\varepsilon_k\delta_{ij}\delta_{kl} \\ &- \frac{(n+4)r_{\overline{m}}}{2n^2(n+1)(2n-1)}\varepsilon_j\varepsilon_k\delta_{ik}\delta_{jl}.\end{aligned} \tag{3.15}$$

If the Krupka-type curvature tensor of M is parallel, then we kkow that $\hat{S}_{j\bar{l}m} = 0$ and $\hat{S}_{j\bar{l}\overline{m}} = 0$, that is, the Ricci contraction $\hat{S}$ is parallel. Thus, making use of Proposition 3.4, we say that the Ricci tensor is parallel, provided $n > 2$, which together with (3.11) yields $r_m = 0 = r_{\overline{m}}$. Hence using (3.14) and (3.15), we obtain $R_{\bar{i}jk\bar{l}m} = 0$ and $R_{\bar{i}jk\bar{l}\overline{m}} = 0$, that is, M is locally symmetric. Conversely, if M is locally symmetric, then we get $S_{j\bar{l}m} = 0, S_{j\bar{l}\overline{m}} = 0, r_m = 0$ and $r_{\overline{m}} = 0$. Thus, from (3.14) and (3.15), we can see that the Krupka-type curvature tensor of M is parallel.

Hence we have proved

Theorem 3.6. *Let M be an indefinite Kähler manifold of complex dimension $n(n > 2)$. Then M is locally symmetric if and only if the Krupka-type curvature tensor of M is parallel.*

Moreover from Theorem 3.6 and Theorem in [10], we conclude

Theorem 3.7. *Let M be an indefinite Kähler manifold of complex dimension $n(n > 2)$. Then the conformal curvature tensor of M is parallel if and only if the Krupka-type curvature tensor of M is parallel.*

4 Indefinite complex submanifolds

This section is concerned with indefinite complex submanifold of an indefinite complex space form.

Let $M' = M^{n+p}_{s+t}(c')$ be an indefinite complex space form of index $2(s+t)$ $(0 \leq s \leq n,\ \ 0 \leq t \leq p)$. Then we can easily see that the Krupka-type curvature tensor on M' vanishes.

In this discussion we introduce a theorem (Theorem 3.6 in [10]).

Theorem A. *Let M' be an $(n+1)$-dimensional indefinite Käehler manifold of index $2(s+t), t = 0$ or 1, and with vanishing conformal curvature tensor, and let M be an indefinite complex hypersurface of index $2s$ of M' $(n > 2)$. Then the following assertions are equivalent to each other:*

(1) *M has the vanishing conformal curvature tensor,*

(2) *M is totally geodesic.*

Consequently, owing to the above Theorem A and Proposition 3.3, we are ready to prove the following

Theorem 4.1. *Let M' be an $(n+1)$-dimensional indefinite complex space form of index $2(s+t), t = 0$ or 1, and let M be an indefinite complex hypersurface of index $2s$ of M' $(n > 2)$. Then the following assertions are equivalent to each other :*

(1) *M has the vanishing Krupka-type curvature tensor,*

(2) *M is totally geodesic.*

Proof. Since the Krupka-type curvature tensor on M' vanishes, we know that M' is Einstein by Lemma 3.1 and so the Krupka-type curvature tensor is equal to the conformal curvature tensor on M' by Proposition 3.3. Hence the conformal curvature tensor on M' vanishes.

Assume that M has the vanishing Krupka-type curvature tensor. Then M is Einstein due to Lemma 3.1, which impliesthat the Krupka-type curvature tensor is equal to the conformal curvature tensor on M because of Proposition 3.3. Hence the conformal curvature tensor on M vanishes. From Theorem A, we have M is totally geodesic.

Conversely, assume that M is totally geodesic, then the conformal curvature tensor on M vanishes by means of Theorem A, and so we have M is Einstein using the lemma in [10]. Thus Proposition 3.3 implies that the Krupka-type curvature tensor is equal to the conformal curvature tensor field on M. Therefore M has the vanishing Krupka-type curvature tensor. □

Acknowledgements. This work was supported by the 2013 Inje University research grant.

References

[1] R. Aiyama, J.-H. Kwon and H. Nakagawa, *Indefinite complex submanifolds of an indefinite complex space form*, J. Ramanujan Math. Soc. **2** (1987), 43-67.

[2] M. Barros and A. Romero, *Indefinite Kähler anifolds*, Math. Ann. **281** (1982), 55-62.

[3] S. Bochner, *Curvature and Betti number II*, Ann. of Math. **50** (1949), 77-93.

[4] S. Funabashi, H. S. Kim, Y.-M. Kim and J. S. Pak, *Traceless component of the conformal curvature tensor in Kähler manifold*, Czecho. Math. J. **56(131)** (2006), 857-874.

[5] S. Funabashi, H. S. Kim, Y.-M. Kim and J. S. Pak, *F-traceless component of the conformal curvature tensor on Kähler manifold*, Bull. Korean Math. Soc. **44-4** (2007), 795-806.

[6] H. Kitahara, K. Matsuo and J. S. Pak, *A conformal curvature tensor field on Hermitian manifolds*, J. Korean Math. Soc. **27** (1990), 7-17.

[7] H. Kitahara, K. Matsuo and J. S. Pak, *Appendium: A conformal curvature tensor field on Hermitian manifolds*, Bull. Korean Math. Soc. **27**, 1 (1990), 27-30.

[8] D. Krupka, *The trace decomposition problems*, Beiträge zur Algebra und Geometrie Contributions to Algebra and Geometry **36** (1995), 303-315.

[9] J.-H. Kwon, *A Theorem on Indefinite Kaehlerian Manifolds*, J. Science Education, Taegu Univ. **1** (1998), 45-49.

[10] J.-H. Kwon, W.-H. Sohn and K.-H. Cho, *Indefinite Kaehlerian manifolds with parallel conformal curvature tensor field*, Comm. Korean Math. Soc. **8** (1993), 499-509.

[11] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[12] S. Tachibana, *On the Bochner curvature tensor*, Natural Science Report, Ochanomizu Univ. **18-1** (1967), 15-19.

[13] J. A. Wolf, *Spaces of constant curvature*, Mc. Graw-Hill, New York, 1967.

Author's address:

Hyang Sook Kim
Department of Mathematics, Institute of Basic Science, Inje University, Kimhae 621-749, Korea.
E-mail: mathkim@inje.ac.kr

Jung-Hwan Kwon
Daegu University, Gyeongsan 712-714, Korea.
E-mail: jhkwon@daegu.ac.kr

Jin Suk Pak
Kyungpook National University, Daegu 702-701, Korea.
E-mail: jspak@knu.ac.kr