Solving Bonnet problems to construct families of surfaces

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Abstract. In this work, the authors study Bonnet Problems using Cartan moving frames and associated structure equations. The Cartan structural forms are written in terms of the first and second fundamental forms, and the Lax system is consequently reinterpreted; orthonormal moving frames are obtained solutions to this Bonnet-Lax system, via numerical integration. Certain classifications of families of surfaces are provided in terms of the first and second fundamental forms, given certain prescribed invariants. Numerical applications (an improved Runge Kutta method) applied to this theoretical framework produced a fast way to visualize families of surfaces under investigation. We provide a few examples and visual models.

M.S.C. 2010: 42A20, 42A32.
Key words: Bonnet problem; Bonnet theorem; CMC surface, associate family; family of surfaces.

1 Introduction

Pierre Bonnet (1819-1892) had numerous and significant contributions to the field of Differential Geometry, including three famous results: the Gauss-Bonnet theorem (computing the path integral of the geodesic curvature along a closed curve); an upper bound result for the diameter of a complete Riemannian manifold when the Ricci curvature is bounded from below (what is known today as the Bonnet-Myers theorem proved in 1941); and ultimately what the geometric community called Bonnet's Theorem, a fundamental result proved by Bonnet around 1860, whose statement is as short as all the major philosophical statements, namely:

Balkan Journal of Geometry and Its Applications, Vol.16, No.2, 2011, pp. 70-80.
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Theorem 1.1. *(Bonnet's Theorem) The first and second fundamental forms determine an immersed surface up to rigid motions*[1]*.*

Solving a Bonnet problem usually means obtaining an explicit formula for the immersion of a surface, starting from given first and second fundamental forms. The problem is considered open, as impossible to achieve in general, even for the case of a regular immersion in the 3-d Euclidean space. However, Cartan's theory on structure equations, together with solving a Lax system numerically by an improved Runge-Kutta method, provides us with a consistent method of achieving this goal to its best extent. This represents the first part of our research note.

The second part of this work focuses on families of surfaces which naturally arise in the study of Bonnet problems, by considering certain invariants (such as mean curvature H, or Gauss curvature K, or both). One of the families that we obtain as a bi-product of this procedure is the family of associate surface which preserve both mean curvature and Gauss curvature (i.e., preserve the principal curvatures). However, several other families occur in the process, of which, to our knowledge, some were never classified or named before.

Several classics (e.g., works of Cartan and Chern) brought contributions to the topic of studying Bonnet problems from specific view points.

In the past few decades, several authors brought new insights and significant progress to the general study of Bonnet problems (such as, George Kamberov in [7] and Alexander Bobenko in [3] and other related papers).

Our research meshes well with the preexisting work in this field, without significant overlaps and without any contradictions.

2 Bonnet's Theorem

Consider an immersed surface in Euclidean 3-space parametrized via the map

$$f : D \subset \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(u,v) = (f^1(u,v), f^2(u,v), f^3(u,v)),$$

where D represents an open simply connected domain in plane. The first and second fundamental forms corresponding to this immersion can be expressed as:

$$I = \langle df, df \rangle = Edu^2 + 2Fdudv + Gdv^2 \tag{2.1}$$

$$II = -\langle df, dN \rangle = ldu^2 + 2mdudv + ndv^2, \tag{2.2}$$

where $E = \langle f_u, f_u \rangle$, $F = \langle f_u, f_v \rangle$ and $G = \langle f_v, f_v \rangle$, $l = \langle f_{uu}, N \rangle$, $m = \langle f_{uv}, N \rangle$ and $n = \langle f_{vv}, N \rangle$ represent the coefficients of the first and second fundamental forms, respectively. The shape operator can be expressed in terms of the first and second fundamental forms in matrix form as follows:

$$S = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

The principal curvatures k_1, k_2 represent the eigenvalues of the shape operator. One can express the Gaussian and mean curvatures in terms of the coefficients of the first

[1] Euclidean motions, roto-translations

and second fundamental forms,

$$H = \frac{k_1 + k_2}{2} = \frac{1}{2}\frac{En - 2Fm + Gl}{EG - F^2} \tag{2.3}$$

$$K = k_1 k_2 = \frac{ln - m^2}{EG - F^2}. \tag{2.4}$$

Theorem 2.1. *(Bonnet Theorem) The first and second fundamental forms determine an immersed surface up to rigid motions*[2].

The most natural context in which Bonnet's problem can be studied is that of Cartan's: Moving Frames, Equivalence Method and Structure Equations. We are recalling the following important result, which can be found in [4].

We apply the method of moving frames to the special case of surfaces in $\mathbb{R}^3$. Let $x : M^2 \to \mathbb{R}^3$ be an immersion of a two-dimensional differentiable manifold in $\mathbb{R}^3$. For each point $p \in M$, an inner product $\langle , \rangle_p$ is defined in T_pM by the rule:

$$\left\langle v^1, v^2 \right\rangle_p = \left\langle dx_p(v^1), dx_p(v^2) \right\rangle_{x(p)}, \tag{2.5}$$

where the inner product in the right hand side is the canonical inner product of $\mathbb{R}^3$. It is straightforward to check that $\langle , \rangle_p$ is differentiable and defines a Riemannian metric in M^2, called the *metric induced by the immersion x*.

We will study the local geometry of M around a point $p \in M$. Let $U \subset M$ be a neighborhood of p such that the restriction $x|_U$ is an embedding. Let $V \subset \mathbb{R}^3$ be a neighborhood of p in $\mathbb{R}^3$ such that $V \cap x(M) = x(U)$ and such that we can choose in V an adapted moving frame e^1, e^2, e^3. Therefore, when restricted to $x(U)$, e^1 and e^2 span the tangent bundle to $x(U)$.

Each vector field e^i is a differentiable map into $\mathbb{R}^3$. The differential at $p \in D$, $(de^i)_p : \mathbb{R}^3 \to \mathbb{R}^3$, is a linear map. Thus, for each p and each $v \in \mathbb{R}^n$ we can write

$$(de^i)_p(v) = \sum_j (\omega^i_j)_p(v) e^j.$$

The expressions $(\omega^i_j)_p(v)$, above defined, depend linearly on v. Thus $(\omega^i_j)_p$ is a linear form in $\mathbb{R}^3$ and, since e^i is a differentiable vector field, ω^i_j is a differential 1-form. So, we write the above as

$$de^i = \sum_j \omega^i_j e^j. \tag{2.6}$$

The forms ω^i_j so defined are called *connection forms* corresponding to the moving frame e^i.

The map $x : U \to V$ induces forms $x^*(\omega_i)$, $x^*(\omega^i_j)$ on U. Since x^* commutes with d and $\wedge$, such forms satisfy Cartan's equations. For all $q \in U$ and all $v \in T_qM$, it follows that

$$x^*(\omega_3)(v) = \omega_3(dx(v)) = 0.$$

[2] Euclidean motions, roto-translations

Following [4], by a well-motivated abuse of notation, we identify $x^*(\omega_i) = \omega_i$, and $x^*(\omega^i_j) = \omega^i_j$. $x|_U$ represents an embedding and these restricted forms satisfy Cartan's structure equations, where $\omega_3 = 0$.

In V we have, associated to the frame e^i, the coframe forms ω_i and the connection forms $\omega^i_j = -\omega^j_i$, $i, j = 1, 2, 3$ which satisfy the structure equations:

$$d\omega_1 = \omega_2 \wedge \omega^2_1, \quad d\omega_2 = \omega_1 \wedge \omega^1_2, \quad d\omega_3 = \omega_1 \wedge \omega^1_3 + \omega_2 \wedge \omega^2_3,$$
$$d\omega^1_2 = \omega^1_3 \wedge \omega^3_2, \quad d\omega^1_3 = \omega^1_2 \wedge \omega^2_3, \quad d\omega^2_3 = \omega^2_1 \wedge \omega^1_3.$$

The fourth listed equation among these structure equations is called *the Gauss equation.* The fifth and the sixth (last two) equations are called *the Codazzi-Mainardi-Peterson equations.*

Now, let $f : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of an immersed surface such that $(u, v) \in D$ are orthogonal, $E = \langle f_u, f_u \rangle$, $F = 0$ and $G = \langle f_v, f_v \rangle$. Then, we can choose an orthonormal frame $e^1 = \frac{f_u}{||f_u||}$, $e^2 = \frac{f_v}{||f_v||}$ in D and given the moving frame e^i we will define differential 1-forms ω_i which is the associated coframe to e^i by the conditions $\omega_i(e^j) = \delta_{ij}$. The following expressions are an immediate consequence of the coframe conditions:

$$\omega_1 = \sqrt{E}du, \qquad \omega_2 = \sqrt{G}dv.$$

Next,

$$d\omega_3 = \omega_1 \wedge \omega^1_3 + \omega_2 \wedge \omega^2_3 = 0,$$

hence, by Cartan's lemma,

$$\omega^1_3 = h_{11}\omega_1 + h_{12}\omega_2, \tag{2.7}$$
$$\omega^2_3 = h_{21}\omega_1 + h_{22}\omega_2, \tag{2.8}$$

where $h_{ij} = h_{ji}$ are differentiable functions in U. Remark that the matrix h can be reinterpreted as a shape operator, and it corresponds to the Weingarten operator $-dN(v) = S(v)$. It is very important to remark (see [8]) that the shape operator S is related to both the first and second fundamental forms through the formula $II(v, w) = \langle S(v), w \rangle_g$, where the inner product is considered with respect to the first fundamental form $I \equiv g$, and so $S = I^{-1}II$.

It is known that *this matrix in general is not symmetric, even though the shape operator is symmetric with respect to the inner products on the tangent spaces* ([8, p. 177]). On the other hand, Cartan's Lemma yields a matrix h which is symmetric, and represents a symmetrization of the matrix S. This aspect, among others, shows how natural, and much more convenient, it is to work with Cartan forms instead of the classical approach to differential geometry.

In matrix form, the (similar) matrices S and h can be represented as (in the case when the parametrization is orthogonal, $F = 0$) as follows:

$$S = \begin{pmatrix} \frac{l}{E} & \frac{m}{E} \\ \frac{m}{G} & \frac{n}{G} \end{pmatrix}, \qquad h = \begin{pmatrix} \frac{l}{E} & \frac{m}{\sqrt{EG}} \\ \frac{m}{\sqrt{EG}} & \frac{n}{G} \end{pmatrix}.$$

Remark that matrices S and h have the same eigenvalues, namely the principal curvatures k_1 and k_2. Consequently, H and K are invariants with respect to this symmetrization.

This allows us to rewrite equations (2.7), (2.8) for any parametrized surface where f_u and f_v are orthogonal:

$$\omega_3^1 = \frac{l}{\sqrt{E}}du + \frac{m}{\sqrt{E}}dv, \tag{2.9}$$

$$\omega_3^2 = \frac{m}{\sqrt{G}}du + \frac{n}{\sqrt{G}}dv \tag{2.10}$$

One more connection form needs to be determined, namely $\omega_2^1 = -\omega_1^2$, which can be obtained from the Theorem of Levi-Civita.

Lemma 2.2. *In isothermal coordinates, the Levi-Civita connection has the following expression:*

$$\omega_2^1 = -\frac{(\sqrt{E})_v}{\sqrt{G}}du + \frac{(\sqrt{G})_u}{\sqrt{E}}dv. \tag{2.11}$$

Proof of the Lemma. The Theorem of Levi-Civita states that on any 2-dimensional manifold M, for any open set U in M where a moving frame e^1, e^2 is defined, together with its associated frame ω^1, ω^2, there exists a unique one-form $\omega_1^2 = -\omega_2^1$ which satisfies the following conditions:

$$\omega_1^2 \wedge \omega_2 = d\omega_1, \qquad \omega_2^1 \wedge \omega_1 = d\omega_2.$$

On the other hand, by definition, for any differential 1-form $\omega = \sum_i a_i dx_i$, we have:

$$d\omega = \sum_i da_i \wedge dx_i.$$

The previously-stated expressions of $d\omega_1$ and $d\omega_2$, together with the expressions of the coframe $\omega_1 = \sqrt{E}du$, and respectively, $\omega_2 = \sqrt{G}dv$, lead us to the following expression of the Levi-Civita connection form ω_2^1:

$$\omega_2^1 = -\frac{(\sqrt{E})_v}{\sqrt{G}}du + \frac{(\sqrt{G})_u}{\sqrt{E}}dv.$$

This concludes the proof. □

Hence, by the equation (2.6) we can construct a system of equations (classically called Lax System of equations):

$$d\mathcal{F} = \mathcal{F}\Omega, \tag{2.12}$$

where $\mathcal{F} =< e^1, e^2, e^3 >$ represents the orthonormal frame matrix whose vectors e^i are ordered as column vectors, and Ω represents the transposed Cartan matrix. The matrix Ω is written in standard matrix notation as: $\Omega = (\Omega_{ij}) = (\omega_i^j)$, $i, j = 1, 2, 3$, where i represents the row index and j represents the column index.

The Maurer-Cartan form Ω is valued in the Lie algebra so$(3, \mathrm{R})$ and so $\omega_j^i = -\omega_i^j$. This form, corresponding to a smooth immersion of prescribed first fundamental form and second fundamental form, can be explicitly written $\Omega = \mathcal{F}^{-1} \cdot d\mathcal{F}$ writes as:

$$\Omega = \begin{pmatrix} 0 & \frac{(\sqrt{E})_v}{\sqrt{G}} & -\frac{l}{\sqrt{E}} \\ -\frac{(\sqrt{E})_v}{\sqrt{G}} & 0 & -\frac{m}{\sqrt{G}} \\ \frac{l}{\sqrt{E}} & \frac{m}{\sqrt{G}} & 0 \end{pmatrix} du + \begin{pmatrix} 0 & -\frac{(\sqrt{G})_u}{\sqrt{E}} & -\frac{m}{\sqrt{E}} \\ \frac{(\sqrt{G})_u}{\sqrt{E}} & 0 & -\frac{n}{\sqrt{G}} \\ \frac{m}{\sqrt{E}} & \frac{n}{\sqrt{G}} & 0 \end{pmatrix} dv.$$

and

$$\mathcal{F}_u = \mathcal{F}\begin{pmatrix} 0 & \frac{(\sqrt{E})_v}{\sqrt{G}} & -\frac{l}{\sqrt{E}} \\ -\frac{(\sqrt{E})_v}{\sqrt{G}} & 0 & -\frac{m}{\sqrt{G}} \\ \frac{l}{\sqrt{E}} & \frac{m}{\sqrt{G}} & 0 \end{pmatrix} \tag{2.13}$$

$$\mathcal{F}_v = \mathcal{F}\begin{pmatrix} 0 & -\frac{(\sqrt{G})_u}{\sqrt{E}} & -\frac{m}{\sqrt{E}} \\ \frac{(\sqrt{G})_u}{\sqrt{E}} & 0 & -\frac{n}{\sqrt{G}} \\ \frac{m}{\sqrt{E}} & \frac{n}{\sqrt{G}} & 0 \end{pmatrix} \tag{2.14}$$

where E, F, G and l, m, n represent the coefficient functions of the first and second fundamental forms, respectively.

Further, we can write the following explicit formulas:

$$\mathcal{F}_u = \left\langle \frac{f_{uu}\sqrt{E} - f_u(\sqrt{E})_u}{E}, \frac{f_{vu}\sqrt{G} - f_v(\sqrt{G})_u}{G}, (\frac{f_u}{\sqrt{E}} \times \frac{f_v}{\sqrt{G}})_u \right\rangle,$$

$$\mathcal{F}_v = \left\langle \frac{f_{uv}\sqrt{E} - f_u(\sqrt{E})_v}{E}, \frac{f_{vv}\sqrt{G} - f_v(\sqrt{G})_v}{G}, (\frac{f_u}{\sqrt{E}} \times \frac{f_v}{\sqrt{G}})_v \right\rangle.$$

On the Lax system of differential equations, one needs to ask: Are there any solutions, if so how many?

The answer is given by Picard in case of a single ordinary differential equation (ODE) which is classically known as Picard's Theorem.

Let's consider our system of PDE:

$$\begin{aligned} \mathcal{F}_u &= \mathcal{F}A(u,v) \\ \mathcal{F}_v &= \mathcal{F}B(u,v) \end{aligned} \tag{2.15}$$

where $A(u,v), B(u,v) \in so(3)$, $Adu + Bdv = \Omega$.

We can attempt to solve the Lax system in a neighborhood of $(0,0)$ by solving a succession of ODE's. In each step, Picard's Theorem implies the existence and uniqueness of a solution to the corresponding ODE system, depending only on the initial value. At every step, the compatibility condition for the Lax system is verified, namely $F_{uv} = F_{vu}$; this condition reduces to:

$$BA + A_v = AB + B_u \tag{2.16}$$

which can be rewritten as $A_v - B_u - [A,B] = 0$.

In the following theorem, we will show that the compatibility condition for the Lax system is equivalent to the Gauss-Codazzi Mainardi equations.

Theorem 2.3. *Let $Edu^2 + Gdv^2$ represent a positive definite bilinear form, called metric tensor (I) , and $ldu^2 + 2mdudv + ndv^2$ represent a symmetric and positive definite bilinear form denoted as (II). Assume that all coefficients of these forms are at least class C^2. Assume that together, these coefficients satisfy the Gauss-Codazzi Mainardi equations in a simply connected open subset of $\mathbb{R}^2$. Then, there exists a unique solution of the system*

$$\mathcal{F}_u = \mathcal{F}A(u,v), \qquad \mathcal{F}_v = \mathcal{F}B(u,v), \tag{2.17}$$

where $A(u,v), B(u,v) \in so(3)$

$$A = \begin{pmatrix} 0 & \frac{(\sqrt{E})_v}{\sqrt{G}} & -\frac{l}{\sqrt{E}} \\ -\frac{(\sqrt{E})_v}{\sqrt{G}} & 0 & -\frac{m}{\sqrt{G}} \\ \frac{l}{\sqrt{E}} & \frac{m}{\sqrt{G}} & 0 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 & -\frac{(\sqrt{G})_u}{\sqrt{E}} & -\frac{m}{\sqrt{E}} \\ \frac{(\sqrt{G})_u}{\sqrt{E}} & 0 & -\frac{n}{\sqrt{G}} \\ \frac{m}{\sqrt{E}} & \frac{n}{\sqrt{G}} & 0 \end{pmatrix},$$

depending only on the choice of the initial value. This solution represents the orthonormal moving frame of a surface immersion in $\mathbb{R}^3$ that admits (I) and (II) as first and second fundamental forms, respectively. This surface is unique up to a rigid motion in space.

Conversely, if this Lax system admits a solution $\mathcal{F}$, then it corresponds to a surface immersion whose Gauss-Codazzi-Mainardi equations represent the compatibility conditions of the Lax system.

Proof. We know that the existence and uniqueness of the system depends on the compatibility condition. Let's recall the compatibility condition

$$AB - BA = A_v - B_u \tag{2.18}$$

Here after some computations, we will have the right and left hand side of the equation (2.18) as

$$AB - BA = \begin{pmatrix} 0 & -\frac{ln-m^2}{\sqrt{EG}} & -\frac{(\sqrt{E})_v n}{G} - \frac{(\sqrt{G})_u m}{\sqrt{EG}} \\ \frac{ln-m^2}{\sqrt{EG}} & 0 & \frac{(\sqrt{E})_v m}{\sqrt{EG}} + \frac{(\sqrt{G})_u n}{E} \\ \frac{(\sqrt{E})_v n}{G} + \frac{(\sqrt{G})_u m}{\sqrt{EG}} & -\frac{(\sqrt{E})_v m}{\sqrt{EG}} - \frac{(\sqrt{G})_u n}{E} & 0 \end{pmatrix}$$

$$A_v - B_u = \begin{pmatrix} 0 & (\frac{(\sqrt{E})_v}{\sqrt{G}})_v + (\frac{(\sqrt{G})_u}{\sqrt{E}})_u & (-\frac{l}{\sqrt{E}})_v + (\frac{m}{\sqrt{E}})_u \\ -(\frac{(\sqrt{E})_v}{\sqrt{G}})_v - (\frac{(\sqrt{G})_u}{\sqrt{E}})_u & 0 & -(\frac{m}{\sqrt{G}})_v + (\frac{n}{\sqrt{G}})_u \\ (\frac{l}{\sqrt{E}})_v - (\frac{m}{\sqrt{E}})_u & (\frac{m}{\sqrt{G}})_v - (\frac{n}{\sqrt{G}})_u & 0 \end{pmatrix}$$

Since both matrices are skew-symmetric, then we will have the following equations in order to satisfy the compatibility condition:

$$ln - m^2 = \frac{-E_{vv} - G_{uu}}{2} + \frac{G_u E_u}{4E} + \frac{E_v^2}{4E} + \frac{G_u^2}{4G} + \frac{E_v G_v}{4G} \tag{2.19}$$

$$l_v - m_u = \frac{lE_v}{2E} + m\left(\frac{G_u}{2G} - \frac{E_u}{2E}\right) + \frac{nE_v}{2G} \tag{2.20}$$

$$m_v - n_u = -\frac{lG_u}{2E} + m\left(\frac{G_v}{2G} - \frac{E_v}{2G}\right) - \frac{nG_u}{2G} \tag{2.21}$$

which represent one of the many equivalent forms of the Gauss-Codazzi-Mainardi equations. In this group of three equations, the first equation is called Gauss equation, while the second and third are known as Codazzi-Mainardi. Through the classical literature, the Gauss-Codazzi-Mainardi equations are usually expressed in terms of Riemann-Christoffel symbols, thus making their expression extremely sophisticated. However, one can easily verify, after some heavy computation by hand or using mathematical software, that the expressions above are equivalent to the Gauss-Codazzi-Mainardi equations given in terms of Riemann-Christoffel symbols. □

Remark A. It follows from the above equation and Bonnet theorem that if we prescribe the smooth functions E, F, G, l, m, n which satisfy the required compatibility equations, as coefficients of the first and second fundamental forms of an immersion, then a solution to the above Lax system is found. It represents a moving orthonormal frame $\mathcal{F}$ corresponding to a smooth surface that is unique, up to roto-translations.

Remark B.
An important particular case is that of curvature line coordinates, which exist on all differentiable 2-manifolds, away from singularities. This means that the first and the second fundamental forms, (I) and (II), are diagonalizable simultaneously. In this case, the Gauss-Codazzi-Mainardi equations reduce to the following equations:

$$l_v = \frac{E_v}{2}\left(\frac{l}{E} + \frac{n}{G}\right), \qquad n_u = \frac{G_u}{2}\left(\frac{l}{E} + \frac{n}{G}\right).$$

It is important to note that the expression $\left(\frac{l}{E} + \frac{n}{G}\right) = H$ actually represents the mean curvature function.

Remark C.
An important particular subcase of the curvature line coordinate parameterization is that of (surfaces which) admit isothermic coordinates. Isothermic coordinates mean isothermal (conformal) coordinates which also represents curvature lines ($E = G$, $F = 0$ and $m = 0$). Known surfaces that admit isothermic coordinates include all constant mean curvature (CMC) surfaces, Bonnet surfaces, quadrics, and a few other special families.

3 Families of isometric surfaces in curvature line coordinates

In the previous section, we formulated the Bonnet Problem in terms of coefficients of the first and second fundamental forms. An important geometric application consists in studying families of surfaces corresponding to prescribed invariants. We will characterize surfaces via the coefficients of their fundamental forms. We remark that these coefficients are not independent: namely, they need to satisfy compatibility conditions coming from the Gauss-Codazzi-Mainardi equations.

Consider an isometric family of smooth surfaces in conformal coordinates, immersed into the Euclidean 3-space (all members of the family having the same first fundamental form given by $I := E(du^2 + dv^2)$). The formula that expresses the Gauss curvature K is provided by Gauss' 'Theorema Egregium' (the Remarkable Theorem), that is actually the same with the *Gauss Equation.*

Of course, the expression $K = \frac{ln - m^2}{det I}$ is the most common expression of the Gauss curvature, and it was adopted due to its simplicity, but this formula makes us often forget that $ln - m^2$ (and hence K) can be exclusively written in terms of E, F and G, which is a rephrasing of *Gauss' Remarkable Theorem* which states that the Gaussian curvature is preserved by isometries.

The following result is easy to prove in conformal coordinates, but it is valid in any surface coordinates:

Lemma 3.1. *Consider an isometric family of surfaces as defined above, in Riemannian metric $E(u,v)(du^2+dv^2)$, with the same mean curvature H function (that is, the surface transformation preserves the metric and the mean curvature). Then, the modulus of Hopf differential $|Q|$ will be invariant.*

Proof. In writing down the expressions for H, K (2.3),(2.4), which are direct consequences of the Gauss-Codazzi-Mainardi equations, we will get

$$l = 2HE - n \tag{3.1}$$

$$m^2 = -n^2 + 2HEn - KE^2 \tag{3.2}$$

Considering the modulus of the Hopf coefficient, we have by definition $|Q| = (l-n)^2 + 4m^2$. Further, substituting (3.1) and (3.2) into $|Q|$, we obtain $|Q| = (l-n)^2+4m^2 = 4E^2(H^2-K)$. Since E, H, K are invariant, so is $|Q|$. □

Remark that k_1 and k_2, the principal curvatures, are also invariant as solutions of the equation

$$k^2 - 2Hk + K = 0 \tag{3.3}$$

Note the discriminant is $\Delta = H^2 - 4K \geq 0$, which appears in the expression of the Hopf coefficient $|Q|$.

Note that we obtained a well-known family of associate surfaces with respect to the following rotational transformation on the Hopf differential: $Q \to e^{it}Q$. Due to the parameter e^{it} on S^1, this is also called a **1-parameter family of associate surfaces**.

Remark. The isometric family of surfaces which preserves the mean curvature function coincides with the *family of associate surfaces*, which by definition preserves the Hopf differential.

This family is classically well-studied [5], but not from a constructional view point. Our constructional approach involves solving a Bonnet-type problem for the following invariants: given metric, and mean curvature.

Construction algorithm for an associate family of isometric surfaces in curvature line coordinates on an open, simply connected domain

a) Choose a smooth surface of Riemannian metric $E(du^2+dv^2)$ and differentiable mean curvature function $H(u,v)$. Compute K determined by the metric, from the Gauss equation. H and K determine k_1 and k_2, the principal curvatures, at every point.

b) Choose an appropriate smooth function $n(u,v)$ within the range[3]

$$[min(k_1E, k_2E), max(k_1E, k_2E)] \subset \mathbb{R}.$$

c) Based on equation $l = 2HE - n$, compute the function $l(u,v)$.

d) Compute $m(u,v)$ from equation 3.2, considering both possible solutions. A straightforward computation shows that the obtained functions E, l, m, n verify the Gauss-Codazzi-Mainardi equations.

e) Plug the functions $E = G, l, m, n$ into the Lax matrices from 2.15, and solve the Lax system numerically, with appropriate initial conditions.

[3]This condition is imposed by equation (3.2).

f) Use the solution obtained above (moving frame) in order to obtain the explicit immersion formula, either using Sym's formula for associated surfaces, or with direct numerical integration using Picard's theorem.

Remark 1. The same procedure can be used in order to construct isometric families whose mean curvature represents an explicit function of the Gauss curvature $H = H(K)$ or viceversa. One particular example is that of Weingarten surfaces, that is, smooth surfaces in the Euclidean space whose curvatures verify a linearity condition of the type $aH + bK + c = 0$, where a, b, c represent given real numbers. One would indeed start again from the Riemannian metric, which determines the Gauss curvature K at every point, which in turn determines the mean curvature $H = H(K)$ at every point; the rest of the construction remains the same as in the previous algorithm.

Remark 2. Observe that for the case of isothermic coordinates (e.g., for the case of CMC surfaces), one can obtain the functions l, n, m at the same time, by solving the Gauss-Codazzi-Mainardi equations simultaneously, as they are easy to solve compared to the general ones:

$$\begin{aligned} &ln = \frac{-E_{vv} - E_{uu}}{2} + \frac{(E_u)^2 + (E_v)^2}{2E} \\ &l_v = E_v \cdot H, \qquad n_u = E_u \cdot H. \end{aligned}$$

In particular, if H is constant, one immediately obtains $l = HE + \alpha(u)$ and $n = HE + \beta(v)$, and the rest of the conditions subsequently follow.

4 Visual examples

We are concluding this report with a few numerical/visual examples of associate surfaces that were obtained using previously-stated Construction Algorithm for Families of Associate Surfaces.

Fig.1. Associated family of cylinder

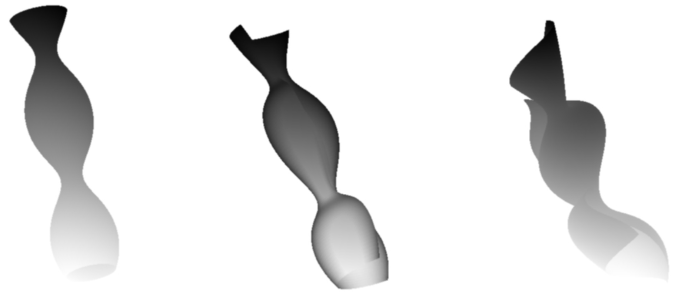

Fig.2. Associated family of unduloid

Acknowledgement. The research of this paper was supported by the NSF grant DMS-0908177.

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