A study of the multitime evolution equation with time-nonlocal conditions

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Abstract. The aim of this paper is to prove existence, uniqueness, and continuous dependence upon the data of solutions to the multitime evolution equations with nonlocal initial conditions. The proofs are based on a priori estimates established in non-classical function spaces and on the density of the range of the operator generated by the studied problem.

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1 Introduction

The classical time-dependent partial differential equations (PDEs) of mathematical physics involve evolution in one-dimensional time. Space can be multidimensional, but time stayed one dimensional until 1932, then the adjective *multitime* was introduced for the first time in physics by Dirac (1932) where he considered the multitime wave functions via m-time evolution equation and later it was used in mathematics.
Multitime evolution equation arise for example in Brownian motion [1], Transport theory (Fokker-plank-type equations), biology (age structured population dynamics)[18], wave and maxell's equations [3], [17], mechanics, physics and cosmology([24], [31]).

The important step in the theoretical study of multidimensional time problems was made by Friedman and Littman ([13], [20]) where they have proved the existence and uniqueness of the following mixed problem with two-dimensional time $u_{t_1,t_2} - Lu = F(x,t), \quad u\mid_{t_1=0}= u\mid_{t_2=0}= u\mid_{x\in\partial D}$, where L is a second order elliptic self-adjoint differential operator. The further development of the theory was elaborated in the series of papers by Brish and Yurchuk ([4], [5], [6]) and Rebbani, Zouyed and Boussetila ([22], [35], [23]) for the Mixed and Goursat problems, hyperbolic equations and recently by (Rebbani, Zouyed and Boussetila) for the multitime evolution equations with nonlocal initial conditions, all these works were studied by the energy inequality method. In [11], Dezin showed for the first time that, for the description of all solvable extensions of differential operators generated by a general differential

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equation with constant coefficients, one should use not only local but also nonlocal conditions.

The problems with nonlocal conditions in the time variable for some classes of PDEs depending on one time variable have attracted much interest in recent years, and have been studied extensively by many authors, see for instance [21], [25], Fardigola,[12], Chesalin and Yurchuk [7], [8], [9] Gordeziani and Avalishvili [15], [16], and Shakhmurov and al. [26]. However, the case of multitime equations with time-nonlocal conditions does not seem to have been widely investigated and few results are available, see, e.g., the articles by the authors Rebbani and *al* [22, 23, 35]. In the present paper, we consider time-nonlocal problems for a class of multitime evolution equations, and we will apply the same technic used in [35]. We will prove the existence and the uniqueness of the strong solution, this results are proved by the energy inequality method.

2 Assumptions and statement of the problem

Throughout this paper H will represent a complex Hilbert space, endowed with the inner product $(.,.)$ and the norm $|.|$, and $\mathcal{L}(H)$ denote the Banach algebra of bounded linear operators on H. Let T_1, $T_2 > 0$, $\Omega =]0, T_1[\times]0, T_2[= \mathcal{Q}_1 \times \mathcal{Q}_2$ be a bounded rectangle in the plane $\mathbb{R}^2$. We consider the following problem: Given the data f,φ, ψ and H, find a function $u(t_1, t_2)$ satisfying the multitime evolution equation

$$\mathcal{L}u \equiv \frac{\partial^2 u}{\partial t_1 \partial t_2} + B\left[\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2}\right] + A(t)u = f(t), \quad t \in \Omega, \tag{2.1}$$

$$\begin{aligned} \ell_{\lambda_1} u &\equiv u\,|_{t_1=0} - \lambda_1 u\,|_{t_1=T_1} = \varphi(t_2), \quad t_2 \in \mathcal{Q}_2, \\ \ell_{\lambda_2} u &\equiv u\,|_{t_2=0} - \lambda_2 u\,|_{t_2=T_2} = \psi(t_1), \quad t_1 \in \mathcal{Q}_1, \end{aligned} \tag{2.2}$$

where u and f are H-valued functions on Ω, φ (resp. ψ) is H-valued function on $\mathcal{Q}_2$ (resp. $\mathcal{Q}_1$) and satisfy the compatibility condition

$$\varphi(0) - \lambda_2\varphi(T_2) = \psi(0) - \lambda_1\psi(T_1), \tag{2.3}$$

λ_1 and λ_2 are two complex parameters, $A(t)$ is an unbounded linear operator in H, with domain of definition $\mathfrak{D}(A)$ densely defined and independent of t and $B \in \mathcal{L}(H)$. We require the following assumptions

1. The operator $A(t)$ is self-adjoint for every $t \in \overline{\Omega}$ and verifies

$$(A(t)u, u) \geq c_0\,|u|^2\,, \quad \forall u \in \mathfrak{D}(A), \tag{2.4}$$

$$A(0, t_2) = A(T_1, t_2), \quad t_2 \in \mathcal{Q}_2, \tag{2.5}$$

$$A(t_1, 0) = A(t_1, T_2), \quad t_1 \in \mathcal{Q}_1. \tag{2.6}$$

where c_0 is a positive constant not depending on u and t.

2. $\lambda_i \neq 0$ $(i = 1, 2)$ such that,

$$\alpha_i = |\lambda_i|^2 \exp(3C(T_1 + T_2)) < 1, \tag{2.7}$$

where C is a positive constant depending on B, $A(t)$ and its derivatives and λ_1, λ_2 are tow complex parameters belonging to $\mathcal{M}$, (C, $\mathcal{M}$ will be defined later).

3 Spaces and auxiliary inequalities

3.1 Abstract formulation

Let us reformulate problem $((2.1)-(2.2)=\mathcal{P})$ as the problem of solving the operator equation

$$Lu = \mathcal{F} = (f, \varphi, \psi), \tag{3.1}$$

where $L = (\mathcal{L}, \ell_{\lambda_1}, \ell_{\lambda_2})$ is generated by $(\mathcal{P})$, with domain of definition $\mathfrak{D}(L)$, the operator L is considered from the Banach space $\mathbb{E}$ into the Hilbert space $\mathbb{F}$, which will be defined later. For this operator we establish an energy inequality

$$\|u\|_{\mathbb{E}} \le k\|Lu\|_{\mathbb{F}}. \tag{3.2}$$

If the operator L is closable then we denote by $\overline{L}$ the closure of L and by $\mathfrak{D}(\overline{L})$ its domain.

Definition 3.1. A solution of the abstract equation $\overline{L}u = \mathcal{F}$ is called a strongly generalized solution of problem $(\mathcal{P})$.

Inequality (3.2) can be extended to $u \in \mathfrak{D}(\overline{L})$, that is,

$$\|u\|_{\mathbb{E}} \le k\|\overline{L}u\|_{\mathbb{F}}, \quad \forall u \in \mathfrak{D}(\overline{L}). \tag{3.3}$$

From this inequality, we obtain the uniqueness of a strong solution, if it exists, and the equality of the sets $\mathcal{R}(\overline{L})$ and $\overline{\mathcal{R}(L)}$. Thus, to prove the existence of a strong solution for any $\mathcal{F} \in \mathbb{F}$, it remains to prove that the set $\mathcal{R}(L)$ is dense in $\mathbb{F}$.

3.2 Function spaces

In this subsection, we introduce and study certain fundamental function spaces. For this purpose, let us denote by $W^r = \mathfrak{D}(A^r(0))$, $0 \le r \le 1$, the space W^r endowed with the inner product $(x, y)_r = (A^r(0)x, A^r(0)x)$ and the norm $|x|_r = |A^r(0)x|$ is a Hilbert space. We show that the operator $A(t)$ (resp. $A^{\frac{1}{2}}(t)$) is bounded from W^1 (resp. $W^{\frac{1}{2}}$) into H, i.e., $A(t)$ (resp. $A^{\frac{1}{2}}(t)) \in \mathcal{L}(W^1; H)$ (resp. $\mathcal{L}(W^{\frac{1}{2}}; H)$) (see [19]). Thus, we have the following results

Proposition 3.1. *[6] If the function $\overline{\Omega} \ni t \longmapsto A(t) \in \mathcal{L}(W^1; H)$ is continuous with respect to the topology of $\mathcal{L}(W^1; H)$, then there exist positive constants c_1 and c_2 such that*

$$c_1|u|_1 \le |A(t)u| \le c_2|u|_1, \quad \forall u \in W^1, \tag{3.4}$$

$$\sqrt{c_1}\,|u|_{\frac{1}{2}} \le |A^{\frac{1}{2}}(t)u| \le \sqrt{c_2}\,|u|_{\frac{1}{2}}, \quad \forall u \in W^{\frac{1}{2}}. \tag{3.5}$$

Lemma 3.2. *If the function $\overline{\Omega} \ni t \longmapsto A(t) \in \mathcal{L}(W^1; H)$ admits bounded derivatives with respect to t_1 and t_2 with respect to the simple topology in $\mathcal{L}(W^1; H)$, then we have the estimates*

$$\left\| \frac{\partial A(t)^{\frac{1}{2}}}{\partial t_i} A(t)^{-\frac{1}{2}} \right\|_{\mathcal{L}(H)} \le \delta \left\| \frac{\partial A(t)}{\partial t_i} A(t)^{-1} \right\|_{\mathcal{L}(H)}, \ (i = 1, 2), \tag{3.6}$$

where $\delta = \int_0^\infty \dfrac{\sqrt{s}}{(1+s)^2}ds.$ (see [19], Lemma 1.9, p. 186).

Proposition 3.3. *The operators* $\dfrac{\partial A(t)}{\partial t_i}A(t)^{-1}$, $\dfrac{\partial A(t)^{\frac{1}{2}}}{\partial t_i}A(t)^{-\frac{1}{2}}$ *are uniformly bounded, i.e.,* $\dfrac{\partial A(t)}{\partial t_i}A(t)^{-1}$, $\dfrac{\partial A(t)^{\frac{1}{2}}}{\partial t_i}A(t)^{-\frac{1}{2}} \in L_\infty(\Omega;\mathcal{L}(H)),\ (i=1,2).$

To show the estimate (3.2) we introduce the following spaces
$H^{1,1}(\Omega;W^1)$ is the space obtained by completing $\mathcal{C}^\infty(\overline{\Omega};W^1)$ with respect to the norm

$$\|u\|_{1,1}^2 = \int\limits_\Omega \left(\left|\frac{\partial^2 u}{\partial t_1 \partial t_2}\right|_1^2 + \left|\frac{\partial u}{\partial t_1}\right|_1^2 + \left|\frac{\partial u}{\partial t_2}\right|_1^2 + |u|_1^2 \right) dt.$$

Let $H^1(\mathcal{Q}_i;W^{\frac{1}{2}})$ be the obtained space by completing $\mathcal{C}^\infty(\mathcal{Q}_i;W^{\frac{1}{2}})$, $(i=1,2)$ with respect to the norms $\|\varphi\|_1^2 = \int\limits_0^{T_2}\left(\left|\varphi'\right|^2 + |\varphi|_{\frac{1}{2}}^2\right)dt_2$, $\quad \|\psi\|_1^2 = \int\limits_0^{T_1}\left(\left|\psi'\right|^2 + |\psi|_{\frac{1}{2}}^2\right)dt_1$.
By completing $\mathcal{C}^\infty(\overline{\Omega};W^1)$ with respect to the norm

$$\|u\|_{\mathbb{E}}^2 \;=\; (J(\lambda))^2 \sup_{\tau\in\Omega}\left(\|u(\tau_1,.)\|_1^2 + \|u(.,\tau_2)\|_1^2\right),$$

where $J(\lambda) = \frac{(1-\alpha_1)(1-\alpha_2)}{(1+\alpha_1)(1+\alpha_2)}$, we obtain the space $\mathbb{E}$.

Denoting by $\mathbb{F}$ the Hilbert space $L_2(\Omega;H)\times\mathcal{V}^1(\mathcal{Q}_2;W^{\frac{1}{2}})\times\mathcal{V}^1(\mathcal{Q}_1;W^{\frac{1}{2}})$, consisting of vector-valued functions $\mathcal{F}=(f,\varphi,\psi)$ for which the norm $\|\mathcal{F}\|_{\mathbb{F}}^2 = \|f\|^2 + \|\varphi\|_1^2 + \|\psi\|_1^2$, is finite. $\mathcal{V}^1(\mathcal{Q}_2;W^{\frac{1}{2}})\times\mathcal{V}^1(\mathcal{Q}_1;W^{\frac{1}{2}})$ is the closed subspace of $H^1(\mathcal{Q}_2;W^{\frac{1}{2}})\times H^1(\mathcal{Q}_1;W^{\frac{1}{2}})$ composed of elements (φ,ψ) satisfying (2.3).

To prove the existence of the strong generalized solution we need the following Hilbert structure.
Let $H^{1,1}(\Omega;H)$ be the Hilbert space obtained by completion of $\mathcal{C}^\infty(\overline{\Omega};H)$ with respect to the norm $\|u\|_{1,1}^2 = \int\limits_\Omega\left(\left|\frac{\partial^2 u}{\partial t_1\partial t_2}\right|^2 + \left|\frac{\partial u}{\partial t_1}\right|^2 + \left|\dfrac{\partial u}{\partial t_2}\right|^2 + |u|^2\right)dt.$
Let $H^1(\mathcal{Q}_2;H)$ be the Hilbert space obtained by completion of the space $\mathcal{C}^\infty(\mathcal{Q}_2;H)$ with respect to the norm $\|\varphi\|_1^2 = \|\varphi\|^2 + \|\varphi'\|^2$.
We construct $H^1(\mathcal{Q}_1;H)$ in a similar manner.
Denoting by $\mathcal{E}$ the Hilbert space $L_2(\Omega;H)\times\mathcal{V}^1(\mathcal{Q}_2;H)\times\mathcal{V}^1(\mathcal{Q}_1;H)$ composed of elements $\mathcal{F}=(f,\varphi,\psi)$ such that the norm $\|\mathcal{F}\|_{\mathcal{E}}^2 = \|f\|^2+\|\varphi\|_1^2+\|\psi\|_1^2$ is finite, where $\mathcal{V}^1(\mathcal{Q}_2;H)\times\mathcal{V}^1(\mathcal{Q}_1;H)$ is the closed subspace of $H^1(\mathcal{Q}_2;H)\times H^1(\mathcal{Q}_1;H)$ composed of elements (φ,ψ) such that $\overline{\lambda}_2\varphi(0)-\varphi(T_2) = \overline{\lambda}_1\psi(0)-\psi(T_1)$.
$H_0^{1,1}(\Omega;W^1) = \left\{u\in H^{1,1}(\Omega;W^1)\,:\,\ell_{\lambda_1}u=0,\ \ell_{\lambda_2}u=0\right\}$ is the closed subspace of $H^{1,1}(\Omega;W^1)$.
$H_0^{1,1}(\Omega;H) = \left\{u\in H^{1,1}(\Omega;H)\,:\,\ell_{\lambda_1}u=0,\ \ell_{\lambda_2}u=0\right\}$ is the closed subspace of $H^{1,1}(\Omega;H)$

$$\overline{H}_0^{1,1}(\Omega;H) = \left\{u\in H^{1,1}(\Omega;H)\,:\,\overline{\lambda}_1 u\,|_{t_1=0} - u\,|_{t_1=T_1} = 0,\ \overline{\lambda}_2 u\,|_{t_2=0} - u\,|_{t_2=T_2} = 0\right\},$$

is the closed subspace of $H^{1,1}(\Omega, H)$. We further denote

$$\mathcal{M} = \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : \ \lambda_i \neq 0 \ \textit{and} \ \alpha_i < 1, \ \ (i = 1, 2)\right\}.$$

4 Uniqueness and continuous dependence

We are now in a position to state and to prove the main theorem of this section for the operator $L = (\mathcal{L}, \ell_{\lambda_1}, \ell_{\lambda_2})$ acting from $\mathbb{E}$ into $\mathbb{F}$ with domain of definition $\mathfrak{D}(L) = H^{1,1}(\Omega; W^1) \subset \mathbb{E}$, from which we conclude the uniqueness and continuous dependence of the solution with respect to the data.

Theorem 4.1. *Let the function $\Omega \ni t \longmapsto A(t) \in \mathcal{L}(W^1; H)$ have bounded derivatives with respect to t_1 and t_2 with respect to the simple convergence topology of $\mathcal{L}(W^1; H)$ and the conditions* (2.4)*,* (2.5)*,* (2.6) *and* (2.7) *be fulfilled. Then we have*

$$\|u\|^2_{\mathbb{E}} \leq S\|Lu\|^2_{\mathbb{F}}, \quad \forall u \in H^{1,1}(\Omega; W^1), \tag{4.1}$$

where S is a positive constant independent of λ_1, λ_2 and u.

Lemma 4.2. (generalized Gronwall's lemma)
(GV1) Let $v(t_1, t_2)$ and $F(t_1, t_2)$ be two non negative integrable functions on Ω such that the function $F(t_1, t_2)$ is non-decreasing with respect to the variables t_1 and t_2. Then the inequality $v(t_1, t_2) \leq c_3 \left\{ \int_0^{t_1} v(\tau_1, t_2)\, d\tau_1 + \int_0^{t_2} v(t_1, \tau_2)\, d\tau_2 \right\} + F(t_1, t_2)$,
$(c_3 \geq 0)$, *gives* $v(t_1, t_2) \leq \exp\Big(2c_3(t_1 + t_2)\Big) F(t_1, t_2)$.
(GV2) Let $v(t_1, t_2)$ and $G(t_1, t_2)$ be two non negative integrable functions on Ω such that the function $G(t_1, t_2)$ is non-increasing with respect to the variables t_1 and t_2. Then the inequality $v(t_1, t_2) \leq c_4 \left\{ \int_{t_1}^{T_1} v(\tau_1, t_2)\, d\tau_1 + \int_{t_2}^{T_2} v(t_1, \tau_2)\, d\tau_2 \right\} + G(t_1, t_2)$,
$(c_4 \geq 0)$, *yields* $v(t_1, t_2) \leq \exp\Big(2c_4(T_1 + T_2 - t_1 - t_2)\Big) G(t_1, t_2)$.

Proof. see [35]. □

Lemma 4.3. *Let $|.|_m$ be the norm in W^m $(m = \frac{1}{2}, 1)$, g be a function of variable $t \in [0, T]$ in W^m, and let $h_i = g(0) - \lambda_i g(T)$, $(i = 1, 2)$. Then, if the condition* (2.7) *holds, we have* $\theta_3 |g(0)|^2_m - \frac{1}{2}(1 + \alpha_i)|g(T)|^2_m \leq \theta_3 \frac{(1+\alpha_i)}{(1-\alpha_i)} |h_i|^2_m$, $\quad \theta_3 = \frac{\alpha_i}{|\lambda_i|^2}$ $(i = 1, 2)$.

Proof. It sufficient to use the ε inequality with $\varepsilon = \frac{(1-\alpha_i)}{2\alpha_i}, (i = 1, 2)$. □

Lemma 4.4. [The method of continuity]
Let $\mathcal{X}_1$, $\mathcal{X}_2$ be two Banach spaces and L_0, L_1 be bounded operators from $\mathcal{X}_1$ into $\mathcal{X}_2$. For each $r \in [0, 1]$, set $L_r = (1 - r)L_0 + rL_1$ and suppose that there is a constant k such that $\|u\|_{\mathcal{X}_1} \leq k\|L_r u\|_{\mathcal{X}_2}$ for $r \in [0, 1]$. Then L_1 maps $\mathcal{X}_1$ onto $\mathcal{X}_2$ if and only if L_0 maps $\mathcal{X}_1$ onto $\mathcal{X}_2$. (see [14], Th. 5.2, p. 75).

We also need the ε-inequality: $2(a, b) \leq \varepsilon|a|^2 + \varepsilon^{-1}|b|^2$, $\varepsilon > 0$. Let us return now to the demonstration of the theorem 4.1.

Proof. The proof is based on detailed analysis of the forms $\int\limits_{\Omega_\tau} 2Re(\mathcal{L}u, \mathcal{M}u)\,dt_1dt_2$, where $\mathcal{M}u = \dfrac{\partial u}{\partial t_1} + \dfrac{\partial u}{\partial t_2}$ and $\Omega \supset \Omega_\tau = (0,\tau_1)\times(0,\tau_2),\ (0,\tau_1)\times(0,T_2),\ (0,T_1)\times(0,\tau_2),\ (\tau_1,T_1)\times(\tau_2,T_2)$ and by making use some technical elementary estimates, propositions (3.1, 3.3) and lemmas (3.6, 4.2, 4.3) we get $\|u\|^2_{\mathbb{E}} \le S\|Lu\|^2_{\mathbb{F}}, \quad \forall u \in \mathfrak{D}(L)$. □

It follows from estimation (4.1), that there is a bounded inverse operator L^{-1} on the range $\mathcal{R}(L)$ of L. However, since we have no information concerning $\mathcal{R}(L)$, except that $\mathcal{R}(L) \subset \mathbb{F}$, we must extend L so that the estimation (4.1), holds for the extension and its range is the whole space. We first show that $L : \mathbb{E} \longrightarrow \mathbb{F}$, with the domain $\mathfrak{D}(L)$, has a closure.

Proposition 4.5. *If the conditions of theorem 4.1 are satisfied, then the operator L admits a closure $\overline{L}$ with domain of definition denoted by $\mathfrak{D}(\overline{L})$.*

The solution of the equation

$$\overline{L}u = \mathcal{F}, \quad \mathcal{F} \in \mathbb{F}, \tag{4.2}$$

is called a strong generalized solution of problem $(\mathcal{P})$. Passing to the limit, we extend the inequality (4.1) to the strong generalized solution, we obtain

$$\|u\|^2_{\mathbb{E}} \le S\left\|\overline{L}u\right\|^2_{\mathbb{F}}, \quad \forall u \in \mathfrak{D}(\overline{L}), \tag{4.3}$$

from which we deduce

Corollary 4.6. *From the inequality* (4.3) *we deduce that, if the strong generalized solution exists, then this solution is unique and it depends continuously on $\mathcal{F} = (f, \varphi, \psi)$.*

Corollary 4.7. *The set of values $\mathcal{R}(\overline{L})$ of the operator $\overline{L}$ is equal to the closure $\overline{\mathcal{R}(L)}$ of $\mathcal{R}(L)$ and $(\overline{L})^{-1} = \overline{L^{-1}}$.*

This corollary allows us to claim that, to establish the existence of the strong generalized solution to problem $(\mathcal{P})$ it suffices to prove the density of the set $\mathcal{R}(L)$ in $\mathbb{F}$.

5 Existence of a solution

To show the existence, we need the following condition

Condition $(\mathcal{H})$ $\quad \Omega \ni t \longmapsto A(t) \in \mathcal{L}(\Omega; W^1)$ admits mixed derivatives

$$\frac{\partial^2 A}{\partial t_1 \partial t_2},\ \frac{\partial^2 A}{\partial t_2 \partial t_1} \text{ with } \frac{\partial A}{\partial t_1 \partial t_2}A^{-1},\ \frac{\partial A}{\partial t_2 \partial t_1}A^{-1} \in L_2(\Omega; \mathcal{L}(H)).$$

We are now, in a position to state and to prove the main result of this paper i.e., establish the density of $\mathcal{R}(L)$ in $\mathbb{F}$, who is equivalent to show that, $\mathcal{R}(L)^\perp = \{(0,0,0)\}$ for this purpose, we meet some difficulties (derivation), and to surmount these difficulties we introduces the regularization operators, so we use the regularization technique.

Definition 5.1. We put $A_\varepsilon(t) = (I + \varepsilon A(t))$, $J_\varepsilon(t) = A_\varepsilon^{-1}(t) = (I + \varepsilon A(t))^{-1}$, $R_\varepsilon(t) = A(t)(I + \varepsilon A(t))^{-1} = \frac{1}{\varepsilon}(I - J_\varepsilon(t))$, $\varepsilon > 0$, and call $R_\varepsilon(t)$ the *Yosida approximation* of $A(t)$.

Theorem 5.1. *Under the conditions of the Theorem* (4.1) *and the condition* ($\mathcal{H}$), *the set* $\mathcal{R}(L)$ *is dense in* $\mathbb{F}$.

Proof. We use the method of continuity given in the book [14]. We introduce the family of operators $L_\omega = (\mathcal{L}_\omega, \ell_{\lambda_1}, \ell_{\lambda_2})$, $\omega \in [0, 1]$, where

$$\mathcal{L}_\omega = \frac{\partial^2}{\partial t_1 \partial t_2} + \omega \mathcal{B} + A(t) = (1 - \omega)\mathcal{L}_0 + \omega \mathcal{L}_1, \quad \mathcal{B} = B\left[\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right].$$

Let's start with showing the result (i.e., $\mathcal{R}(L_0)^\perp = \{(0, 0, 0)\}$) in the case $\omega = 0$, and by means of the method of continuity, we establish the general case.

First step $\omega = \mathbf{0}$. Let $V = (v, v_1, v_2)$ be an orthogonal element to $\mathcal{R}(L_0)$. Then we have

$$\langle L_0 u, V\rangle_{\mathbb{F}} = \langle \mathcal{L}_0 u, v\rangle + \langle \ell_{\lambda_1} u, v_1\rangle + \langle \ell_{\lambda_2} u, v_2\rangle = 0, \quad \forall u \in H^{1,1}(\Omega, W^1). \tag{5.1}$$

Let's show that $V = (0, 0, 0)$, but ℓ_{λ_1}, ℓ_{λ_2} are independent and their ranges are dense, then it is sufficient to prove the following proposition

Proposition 5.2. *If for every* $v \in L_2(\Omega; H)$ *we have*

$$\langle \mathcal{L}_0 u, v\rangle = \left\langle \frac{\partial^2 u}{\partial t_1 \partial_2} + A(t)u, v\right\rangle = 0, \forall u \in H_0^{1,1}(\Omega; W^1), \text{ then } v = 0. \tag{5.2}$$

Proof. Let $w = A_\varepsilon^{-1} v$ and $h = A_\varepsilon u$, then the relation (5.2) becomes

$$\left\langle \frac{\partial^2 h}{\partial t_1 \partial t_2} - \frac{\partial}{\partial t_2}(B_{1\varepsilon}^* h) - \frac{\partial}{\partial t_1}(B_{2\varepsilon}^* h) + B_{3\varepsilon}^* h, w\right\rangle = -\langle h, Aw\rangle, \tag{5.3}$$

$h \in H_0^{1,1}(\Omega; H)$ and $B_{i\varepsilon}^* \in \mathcal{L}(H)$, $(i = 1, 2, 3)$ are given by $B_{1\varepsilon}^* = \varepsilon \frac{\partial A}{\partial t_1} A_\varepsilon^{-1} - B$, $B_{2\varepsilon}^* = \varepsilon \frac{\partial A}{\partial t_2} A_\varepsilon^{-1} - B$,$B_{3\varepsilon}^* = \varepsilon \frac{\partial^2 A}{\partial t_1 \partial t_2} A_\varepsilon^{-1} - \varepsilon B \frac{\partial A}{\partial t_1} A_\varepsilon^{-1} - \varepsilon B \frac{\partial A}{\partial t_2} A_\varepsilon^{-1}$, $(*)$ denotes the symbol of the adjoint.
Since the equation (5.3) is true for all function $h \in H_0^{1,1}(\Omega; H)$, it remains true for $h \in \mathcal{C}_0^\infty(\Omega; H)$, what gives to the sense of distributions

$$\left\langle h, \frac{\partial^2 w}{\partial t_1 \partial t_2} + B_{1\varepsilon}\frac{\partial w}{\partial t_2} + B_{2\varepsilon}\frac{\partial w}{\partial t_1}\right\rangle_{\mathcal{D}'} = -\langle h, (A + B_{3\varepsilon})w\rangle, \quad \forall h \in \mathcal{C}_0^\infty(\Omega; H). \tag{5.4}$$

We define the following operators

$$\mathfrak{D}(\widetilde{\mathcal{L}}) = \overline{H}_0^{1,1}(\Omega; H), \ \widetilde{\mathcal{L}} u = \frac{\partial^2 u}{\partial t_1 \partial t_2} + B_{2\varepsilon}\frac{\partial u}{\partial t_1} + B_{1\varepsilon}\frac{\partial u}{\partial t_2}, \tag{5.5}$$

$$\mathfrak{D}(\widetilde{\mathcal{L}}') = H_0^{1,1}(\Omega; H), \ \widetilde{\mathcal{L}}' u = \frac{\partial^2 u}{\partial t_1 \partial t_2} - \frac{\partial}{\partial t_1}(B_{2\varepsilon}^* u) - \frac{\partial}{\partial t_2}(B_{1\varepsilon}^* u). \tag{5.6}$$

According to (5.3) and (5.4), we can show that $\widetilde{\mathcal{L}}' = (\widetilde{\mathcal{L}})^*$. Let's come back to the equation (5.4), we have, for each $\varepsilon \neq 0$, w is the weak solution to the problem

$$(5.7)\qquad \begin{cases} \widetilde{\mathcal{L}}w \equiv \dfrac{\partial^2 w}{\partial t_1 \partial t_2} + B_{1\varepsilon}\dfrac{\partial w}{\partial t_2} + B_{2\varepsilon}\dfrac{\partial w}{\partial t_1} = -(B_{3\varepsilon}A^{-1}\varepsilon + AA^{-1}\varepsilon)v, \\ \widetilde{\ell}_{\lambda_1} w \equiv \overline{\lambda}_1 w\,|_{t_1=0} - w\,|_{t_1=T_1} = 0, \\ \widetilde{\ell}_{\lambda_2} w \equiv \overline{\lambda}_2 w\,|_{t_2=0} - w\,|_{t_2=T_2} = 0, \end{cases}$$

where $v \in L_2(\Omega; H)$, $B_{j\varepsilon} \in \mathcal{L}(H)$, $(j = 1, 2, 3)$.
We are going to show that, w is a solution in the strong sense of the problem (5.7) and that it verifies an a priori estimate, then we shows that $v = 0$.
To establish these results, we can show that the operator $\widetilde{L} = (\widetilde{\mathcal{L}}, \widetilde{l}_{1\mu}, \widetilde{l}_{2\mu})$ acting from $H^{1,1}(\Omega; H)$ into $\mathcal{E}$ is isomorphism

Proposition 5.3. *The operator $\widetilde{L}$ is isomorphism from $H^{1,1}(\Omega; H)$ into $\mathcal{E}$.*

Proof. We must show that $\mathcal{R}(\widetilde{L}) = \mathcal{E}$ and

$$(5.8)\qquad (i)\quad \|\widetilde{L}u\|^2_{\mathcal{E}} \leq d_1\,\|u\|^2_{1,1}, \quad \forall u \in H^{1,1}(\Omega; H),$$

$$(5.9)\qquad (ii)\quad \|u\|^2_{1,1} \leq d_2 \|\widetilde{L}u\|^2_{\mathcal{E}}, \quad \forall u \in H^{1,1}(\Omega; H),$$

where d_1 and d_2 are positive constants independent of u.

(i) It is easy to show

$$(5.10)\qquad \|\widetilde{\mathcal{L}}u\|^2 \leq 4\max(1, C^2)\,\|u\|^2_{1,1}, \quad \forall u \in H^{1,1}(\Omega; H).$$

By virtue of the continuity of the operators $\widetilde{\ell}_{\lambda_1}$, $\widetilde{\ell}_{\lambda_2}$ from $H^{1,1}(\Omega; H)$ into $H^1(\mathcal{Q}_2; H)$, $H^1(\mathcal{Q}_1; H)$ respectively and the inequality (5.10), we obtain the estimate (i).

(ii) We use the same techniques to those used to establish the estimate (4.1) in Theorem (4.1), then we establish the estimate (5.9).
From the continuity of the operator $\widetilde{L}$ and the inequality (5.10), we conclude that the operator $\widetilde{L}$ is an isomorphism from $H^{1,1}(\Omega; H)$ into the closed subspace $\mathcal{R}(\widetilde{L}) = \widetilde{L}\left(H^{1,1}(\Omega; H)\right)$.
It remains to show that $\mathcal{R}(\widetilde{L}) = \mathcal{E}$, for this purpose, we introduce the family of operators $\left\{\widetilde{L}_\eta\right\}_{\eta\in[0,\,1]}$ defined by

$$(5.11)\qquad \begin{cases} \widetilde{L}_\eta = (\widetilde{\mathcal{L}}_\eta\,, \widetilde{\ell}_{\lambda_1}, \widetilde{\ell}_{\lambda_2}), \quad \eta \in [0, 1]\,, \\ \widetilde{\mathcal{L}}_\eta u = \dfrac{\partial^2 u}{\partial t_1 \partial t_2} + \eta B_\varepsilon u\,, \text{ with } B_\varepsilon u = B_{2\varepsilon}\dfrac{\partial u}{\partial t_1} + B_{1\varepsilon}\dfrac{\partial u}{\partial t_2}, \\ \mathfrak{D}(\widetilde{L}_\eta) = H^{1,1}(\Omega, H). \end{cases}$$

We proceed by the method of continuity, we can show that $\mathcal{R}(\widetilde{L}_1) = \mathcal{R}(\widetilde{L}) = \mathcal{E}$. This proves the proposition (5.3). □

Proposition 5.4. *The operator* $\widetilde{L} = \widetilde{\mathcal{L}}$ *is closed*

Proof. The proof is similar to the proof of Proposition 4.6 in [35]. □

From the properties of the operators with closed range, it follows

$$\mathcal{N}(\widetilde{L}') = \mathcal{R}(\widetilde{L})^{\perp} = L_2(\Omega; H)^{\perp} = \{0\}\,,$$
$$\mathcal{R}(\widetilde{L}\,') = \overline{\mathcal{R}(\widetilde{L}\,')} = \mathcal{N}(\widetilde{L})^{\perp} = \{0\}^{\perp} = L_2(\Omega; H).$$

Hence $\widetilde{L}'$ is an isomorphism from $H_0^{1,1}(\Omega; H)$ into $L_2(\Omega; H)$ and it is closed in the topology of $L_2(\Omega; H)$.

Definition 5.2. We Denote by $\hat{\mathcal{L}} = \left(\widetilde{\mathcal{L}}'\right)^*$ the weak extension of the operator $\widetilde{\mathcal{L}}$ defined by

$$\left\langle \widetilde{\mathcal{L}}' u, v \right\rangle = \left\langle u, \hat{\mathcal{L}}\, v \right\rangle = \langle u, f \rangle\,, \quad \forall u \in H_0^{1,1}(\Omega, H) \text{ and } \hat{\mathcal{L}}v = f \in L_2(\Omega, H) \tag{5.12}$$

Proposition 5.5. *The weak extension* $\hat{\mathcal{L}}$ *of the operator* $\widetilde{\mathcal{L}}$ *coincides with its strong extension* $\left(\hat{\mathcal{L}}\right)' = \widetilde{\mathcal{L}}'$.

Proof. see[35]. □

From the proposition (5.5), we deduce that the weak solution to problem (5.7) coincides with its strong solution. Hence $w \in H^{1,1}(\Omega; H) \cap L_2(\Omega, W^1)$ and satisfies the problem (5.7) in the strong sense, i.e.,

$$\begin{cases} \mathfrak{D}(\mathcal{L}) = \overline{H}_0^{1,1}(\Omega; H), \\ \mathcal{L}w = \dfrac{\partial^2 w}{\partial t_1 \partial t_2} + B_{2\varepsilon}\dfrac{\partial w}{\partial t_1} + B_{1\varepsilon}\dfrac{\partial w}{\partial t_2} + Aw = -B_{3\varepsilon}w = f. \end{cases} \tag{5.13}$$

By a similar calculations to those used to establish theorem (4.1), we show

Proposition 5.6. *Under the assumptions of the theorem* (4.1) *we have the estimate*

$$\|A^{\frac{1}{2}}w\|^2 \le d_6 \|B_{3\varepsilon}w\|^2, \quad \forall w \in \overline{H}_0^{1,1}(\Omega; H), \tag{5.14}$$

from (5.14) and (2.4) it follows $\|w\|^2 \le \dfrac{1}{c_0}\|A^{\frac{1}{2}}w\|^2 \le \dfrac{d_6}{c_0}\|B_{3\varepsilon}w\|^2$, replacing w by $A_\varepsilon^{-1}v$ in the last inequality, we obtain

$$\|A_\varepsilon^{-1}v\|^2 \le \frac{d_6}{c_0}\|B_{3\varepsilon}A_\varepsilon^{-1}v\|^2. \tag{5.15}$$

We have

$$\begin{aligned} (B_{3\varepsilon}^*)^* A_\varepsilon^{-1} v \quad \le \quad & \left\{ \left\| (I - A_\varepsilon^{-1}) \left(\frac{\partial^2 A}{\partial t_2 \partial t_1} A^{-1} \right)^* (A_\varepsilon^{-1} v - v) \right\| + \left\| (I - A_\varepsilon^{-1}) \left(\frac{\partial^2 A}{\partial t_1 \partial t_2} A^{-1} \right)^* v \right\| \right. \\ + \quad & 2\, \|B\|_{\mathcal{L}(H)}^{\frac{1}{2}} \left\| (I - A_\varepsilon^{-1}) \left(\frac{\partial^2 A}{\partial t_2 \partial t_1} A^{-1} \right)^* (A_\varepsilon^{-1} v - v) \right\| \\ + \quad & \left. \left\| (I - A_\varepsilon^{-1})^* (B \frac{\partial A}{\partial t_1} A^{-1})^* v \right\| + \left\| (I - A_\varepsilon^{-1})^* (B \frac{\partial A}{\partial t_2} A^{-1})^* v \right\| \right\} \to 0, \quad \varepsilon \to 0. \end{aligned}$$

While taking account of the last inequality, and while passing to the limit in (5.15), when $\varepsilon \longrightarrow 0$ and applying the properties of A_ε^{-1}, we obtain $v = 0$. This completes the proof of proposition (5.2). □

Let us go back now to (5.1), by virtue of proposition (5.2), we obtain $\langle \ell_{\lambda_1} u, v_1 \rangle_0 + \langle \ell_{\lambda_2} u, v_2 \rangle_0 = 0$. Since ℓ_{λ_1}, ℓ_{λ_2} are independent and the ranges of the operators ℓ_{λ_1}, ℓ_{λ_2} are dense in the corresponding spaces, we obtain $v_1 = v_2 = 0$. Hence $V = (0, 0, 0)$, therefor $\overline{\mathcal{R}(L_\omega)} = \mathbb{F}$ for $\omega = 0$.
Second step $\omega \neq 0$. We need the following lemma

Lemma 5.7. *The operator $(L_1 - L_0)$ is bounded, and we have*

$$\|(L_1 - L_0)u\|_{\mathcal{E}} \leq k\|u\|_{\mathbb{E}}, \tag{5.16}$$

where the constant k does not depend on u .

Proof. The equation $\overline{L}_\omega u = \mathcal{F}$ can be written as

$$u + (\omega - \omega_0)(\overline{L}_{\omega_0})^{-1}\overline{(L_1 - L_0)}u = (\overline{L}_{\omega_0})^{-1}\mathcal{F}. \tag{5.17}$$

From (4.3) and (5.16) we have $\|(\overline{L}_{\omega_0})^{-1}\mathcal{F}\|_{\mathbb{E}} \leq \sqrt{S}\|\mathcal{F}\|_{\mathcal{E}}$, and $\|(\overline{L}_{\omega_0})^{-1}\overline{(L_1 - L_0)}u\|_{\mathbb{E}} \leq m\|u\|_{\mathbb{E}}$, where $m = k\sqrt{S}$. Let $|\omega - \omega_0| \leq \rho < \frac{1}{m}$, putting $\Lambda = (\omega - \omega_0)(\overline{L}_{\omega_0})\,\overline{(L_1 - L_0)}$ and $N = (\overline{L}_{\omega_0})^{-1}\mathcal{F}$, (5.17) can be written as $u + \Lambda u = N$.
Observing that $||\Lambda|| = \sup\limits_{u \in \mathfrak{D}(\overline{L}_\lambda)} \dfrac{||\Lambda u||_1}{||u||_1} < 1$. The Neumann series $u = \sum_{n=0}^{\infty}(-\Lambda)^n N$ is then a solution to equation (5.17). We have thus proved that if $\mathcal{R}(\overline{L}_{\omega_0}) = \mathbb{F}$ and $|\omega - \omega_0| \leq \rho < \dfrac{1}{m}$, then $\mathcal{R}(\overline{L}_\omega) = \mathbb{F}$. Proceeding step by step in this way we establish that $\mathcal{R}(\overline{L_\omega}) = \mathbb{F}$ for every $\omega \in [0, 1]$. For the case $\omega = 1$, we have $\mathcal{R}(\overline{L}) = \mathbb{F}$. The proof of theorem (5.1) is achieved. □

Theorem 5.8. *For every element $\mathcal{F} = (f, \varphi, \psi) \in \mathbb{F}$ there exists a unique strong generalized solution $u = (\overline{L})^{-1}\mathcal{F} = (\overline{L^{-1}})\mathcal{F}$ to problem $(\mathcal{P})$ satisfying the estimate*

$$\|u\|_{\mathbb{E}}^2 \leq S\|Lu\|_{\mathbb{F}}^2, \quad \forall u \in H^{1,1}(\Omega; W^1),$$

where S is a positive constant independent of λ_1, λ_2 and u.

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