Laplacians on the tangent bundle of Finsler manifold

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Abstract. Let M be a smooth manifold with a Finsler metric F and $\tilde{g}$ be the naturally induced Riemann metric on the slit tangent bundle $\tilde{M}$. The Weitzenböck formulas of the horizontal Laplacian $\triangle_h$ and the vertical Laplacian $\triangle_v$ are obtained in terms of the Cartan connection of (M, F). The relationship between the Hodge-Laplace operator $\triangle$ of $\tilde{g}$ and the operators $\triangle_h, \triangle_v, \triangle_{mix}$ are investigated. As application, the relationship between the eigenvalues of $\triangle$ and $\triangle_h, \triangle_v$ are established, and a Bochner-type vanishing theorem of horizontal differential form on $\tilde{M}$ is obtained.

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1 Introduction

Let M be a real manifold of dimension m, we denote by TM its tangent bundle, and $\pi : TM \to M$ the canonical projection, the cotangent bundle of M is denoted by T^*M. We assume that M is endowed with a Finsler metric F in the sense of [1], see also [3] and [7]. It is known that there is not a canonical way to define Laplacian on (M, F), we refer to [2] for more details. Usually Laplacians on Finsler manifolds are constructed either on the base manifold M or the slit tangent bundle $\tilde{M} = TM - \{o\}$, where o denotes the zero section of TM. In [8], Munteanu obtained the Weitzenböck formulas of horizontal and vertical Laplacians for 1-forms on a Riemann vector bundle $p : E \to M$ with compact fiber spaces $\mathcal{F} = p^{-1}(x)$ for $x \in M$. Very recently, this idea was developed in [10] to investigate vanishing theorem of holomorphic forms on Kähler Finsler manifolds. Note that since most curvature components of Finsler metrics depend not only on base point $x \in M$ but also on direction $y \in T_xM$ [9], and a Finsler metric F on M naturally induces a Riemann metric $\tilde{g}$ on $\tilde{M}$, it is natural to define a Laplacian on the tangent bundle TM of a Finsler manifold (M, F). The purpose of this paper is to introduce the horizontal and vertical Laplacians of a Finsler metric on the slit tangent bundle $\tilde{M}$ and relate it to the Hodge-Laplace operator $\triangle$ of the Riemann metric $\tilde{g}$ on $\tilde{M}$. It is known that $\triangle$ is not a type preserving operator when acting on differential forms of type (p, q) on $\tilde{M}$, i.e., those forms which are

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horizontal type of p and vertical type of q on $\tilde{M}$. Nevertheless, there are three components of $\triangle$, i.e., the horizonal Laplacian $\triangle_h$, the vertical Laplacian $\triangle_v$ and the mixed part $\triangle_{mix}$, which are type preserving [8]. Our results shows that both the local expressions of $\triangle_h$ and $\triangle_v$ depend on the choice of the Finsler connection associated to (M, F). Moreover, the horizontal Laplacian $\triangle_h$ depends only on the horizontal curvature components of F, the vertical Laplacian $\triangle_v$ depends only on the vertical curvature components of F, the mixed Laplacian $\triangle_{mix}$ depends only on the Riemann curvature of F and is independent of the choice of Finsler connections associated to (M, F). We obtain the Weitzenböck formula of $\triangle_h$ and $\triangle_v$ in terms of the Cartan connection of (M, F), and obtain a Bochner-type vanishing theorem of horizontal differential forms which are compactly supported in $\tilde{M}$.

2 Preliminaries

In this section, we shall recall some basic facts of Finsler manifold. We refer to [1] for more details. Let F be a Finsler metric on a real manifold M of dimension m. Denote by $\mathcal{V}$ the vertical vector bundle of M, then there is a Riemann metric g on $\mathcal{V}$ which is induced by F, and a unique good vertical connection $\nabla : \mathcal{X}(\mathcal{V}) \to \mathcal{X}(T^*\tilde{M} \otimes \mathcal{V})$ called the Cartan connection associated to (M, F). Note that associated to ∇, there are horizontal bundle $\mathcal{H}$ and the horizontal map $\Theta : \mathcal{V} \to \mathcal{H}$. Using Θ one can transfers the natural Riemann metric g on $\mathcal{V}$ to $\mathcal{H}$, and get a Riemann metric $\tilde{g}$ on the whole bundle $T\tilde{M}$, and consequently a linear connection, still denoted by ∇, on $\mathcal{X}(T\tilde{M})$ and $\mathcal{X}(T^*\tilde{M})$. Note that one can also consider the Berwald connection associated to (M, F), and it was shown in [5] that the Berwald connection and the Cartan connection play a different role when one considers the properties of adapted coordinates system related to them.

Locally let $(x^i) = (x^1, \cdots, x^m)$ be the local coordinates on M and $(x^i, y^i) = (x^1, \cdots, x^m, y^1, \cdots, y^m)$ be the naturally induced coordinates on TM. Denote by $\{\delta_i, \dot{\partial}_i\}$ the local frame for $T\tilde{M}$ and $\{dx^i, \delta y^i\}$ the dual frame for $T^*\tilde{M}$, here

$$\delta_i = \frac{\partial}{\partial x^i} - \Gamma^j_{;i}\frac{\partial}{\partial y^j}, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i}, \quad \delta y^i = dy^i + \Gamma^i_{;j}dx^j,$$

and $\Gamma^j_{;i}$ is the nonlinear connection coefficients associated to the Cartan connection ∇. It is known that the Cartan connection ∇ satisfy

$$\text{(2.1)} \quad \nabla_{\delta_k}\dot{\partial}_j = \Gamma^i_{j;k}\dot{\partial}_i, \ \nabla_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i, \ \nabla_{\delta_k}\delta_j = \Gamma^i_{j;k}\delta_i, \ \nabla_{\dot{\partial}_k}\delta_j = C^i_{jk}\delta_i,$$

$$\text{(2.2)} \quad \nabla_{\delta_k}dx^i = -\Gamma^i_{j;k}dx^j, \quad \nabla_{\delta_k}\delta y^i = -\Gamma^i_{j;k}\delta y^j,$$

$$\text{(2.3)} \quad \nabla_{\dot{\partial}_k}dx^i = -C^i_{jk}dx^j, \quad \nabla_{\dot{\partial}_k}\delta y^i = -C^i_{jk}\delta y^j,$$

where in the above equations, $\Gamma^i_{j;k}$ and C^i_{jk} are the horizontal and vertical connection coefficients of the Cartan connection. That is,

$$\text{(2.4)} \qquad \Gamma^i_{j;k} = \frac{1}{2}g^{hi}[\delta_j(g_{hk}) + \delta_k(g_{jh}) - \delta_h(g_{jk})], \ \ C^i_{jk} = \frac{1}{2}g^{hi}\dot{\partial}_j(g_{hk}),$$

with $g_{hk} = \frac{1}{2}\dot{\partial}_h\dot{\partial}_k(F^2)$. It is clear that

$$\text{(2.5)} \qquad \Gamma^i_{j;k} = \Gamma^i_{k;j}, \quad C^i_{jk} = C^i_{kj}.$$

Note that $\{dx^i, \delta y^i\}$ is a local frame for $T^*\tilde{M}$, it follows that the differential forms on $\tilde{M}$ can be represented in terms of $\{dx^i, \delta y^i\}$. In [8], a differential form φ of type (p,q) on $\tilde{M}$ is expressed in terms of $\{dx^i, \delta y^i\}$ as follows:

$$\varphi = \frac{1}{p!q!}\varphi_{I_pJ_q} dx^{I_p} \wedge \delta y^{J_q} \tag{2.6}$$

where $I_p = i_1 \cdots i_p, J_q = j_1 \cdots j_q$, $dx^{I_p} = dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \delta y^{J_q} = \delta y^{j_1} \wedge \cdots \wedge \delta y^{j_q}$, and $\varphi_{I_pJ_q}$ are anti-symmetric in their indexes in I_p and J_q, respectively. In the following, we denote by $\wedge_c^{p,q}(\tilde{M})$ the space of differential forms on $\tilde{M}$ which are of type (p,q) and the coefficients $\varphi_{I_pJ_q}$ are compactly supported in $\tilde{M}$. Note that there is a naturally point-wise inner product $\langle\cdot,\cdot\rangle$ defined in $\wedge_c^{p,q}(\tilde{M})$ for every $(x,y) \in \tilde{M}$. More precisely, for $\varphi, \psi \in \wedge_c^{p,q}(\tilde{M})$,

$$\langle\varphi,\psi\rangle = \frac{1}{p!q!}\varphi_{i_1\cdots i_pj_1\cdots j_q}\psi_{r_1\cdots r_ps_1\cdots s_q} g^{i_1r_1}\cdots g^{i_pr_p} g^{j_1s_1}\cdots g^{j_qs_q},$$

where the above equation is evaluated at $(x,y) \in \tilde{M}$. The point-wise inner product $\langle\cdot,\cdot\rangle$ gives rise to a global inner product $(\cdot,\cdot)$ defined in $\wedge_c^{p,q}(\tilde{M})$. That is,

$$(\varphi,\psi) = \int_{\tilde{M}} \langle\varphi,\psi\rangle dV \tag{2.7}$$

whenever the integral of (2.7) exists. Here $dV = \det(g_{ij}) dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n$ is the natural volume form of the Riemann metric $\tilde{g}$ on $\tilde{M}$. Note that

$$df = \delta_i(f)dx^i + \dot{\partial}_i(f)\delta y^i \tag{2.8}$$

for $f \in C^\infty(\tilde{M})$ and

$$d(\delta y^i) = -\frac{1}{2}R^i_{jk} dx^j \wedge dx^k - G^i_{jk} dx^j \wedge \delta y^k, \tag{2.9}$$

where $R^i_{jk} = \delta_k(\varGamma^i_{;j}) - \delta_j(\varGamma^i_{;k})$ and $G^i_{jk} = \dot{\partial}_j(\varGamma^i_{;k})$. Thus for $\varphi \in \wedge_c^{p,q}(\tilde{M})$ we have $d\varphi = d_h\varphi + d_v\varphi + d_{mix}\varphi$, where $d_h\varphi \in \wedge_c^{p+1,q}(\tilde{M}), d_v\varphi \in \wedge_c^{p,q+1}(\tilde{M})$ and $d_{mix}\varphi \in \wedge_c^{p+2,q-1}(\tilde{M})$. Let d_h^*, d_v^* and d_{mix}^* be the formal adjoints of d_h, d_v and d_{mix}, respectively with respect to the global inner product (2.7), i.e.,

$$(d_h\varphi,\psi) = (\varphi, d_h^*\psi), (d_v\varphi,\psi) = (\varphi, d_v^*\psi), (d_{mix}\varphi,\psi) = (\varphi, d_{mix}^*\psi). \tag{2.10}$$

Let $\mathbf{i}$ be the substitution operator and $\mathbf{e}$ be the wedge product operator. The operators $\mathbf{i}$ and $\mathbf{e}$ satisfy

$$\mathbf{i}(\delta_j)dx^k = \delta_j^k,\ \ \mathbf{i}(\delta_j)\delta y^k = 0,\ \ \mathbf{i}(\dot{\partial}_j)dx^k = 0,\ \ \mathbf{i}(\dot{\partial}_j)\delta y^k = \delta_j^k, \tag{2.11}$$

$$\mathbf{e}(dx^i)\varphi = dx^i \wedge \varphi,\ \ \mathbf{e}(\delta y^i)\varphi = \delta y^i \wedge \varphi \tag{2.12}$$

for $\varphi \in \wedge_c^{p,q}(\tilde{M})$, and are anti-commutative, i.e.,

$$\mathbf{i}(X)\mathbf{i}(Y)\varphi = -\mathbf{i}(Y)\mathbf{i}(X)\varphi, \quad \mathbf{e}(\phi)\mathbf{e}(\omega)\varphi = -\mathbf{e}(\omega)\mathbf{e}(\phi)\varphi \tag{2.13}$$

for $X, Y \in \mathcal{X}(T\tilde{M})$ and $\phi, \omega \in \mathcal{X}(T^*\tilde{M})$. In the calculation of d_h, d_v, d_{mix} and their formal adjoint d_h^*, d_v^*, d_{mix}^* in terms of the Cartan connection ∇, equalities (2.2)-(2.3) and (2.11)-(2.13) will be used repeatedly without explicit statement.

3 Levi-Civita connection of $\tilde{g}$

Lemma 3.1. *Let D be the Levi-Civita connection of the Riemann metric $\tilde{g}$ on $\tilde{M}$. Then in terms of the local frame $\{\delta_i, \dot{\partial}_i\}$ and its dual frame $\{dx^i, \delta y^i\}$, we have*

$$D_{\delta_k}\delta_j = \Gamma^i_{j;k}\delta_i - \Big(C^i_{jk} + \frac{1}{2}R^i_{jk}\Big)\dot{\partial}_i, \tag{3.1}$$

$$D_{\delta_k}\dot{\partial}_j = \Big(C^i_{jk} + \frac{1}{2}g^{li}R^s_{lk}g_{sj}\Big)\delta_i + \Gamma^i_{j;k}\dot{\partial}_i, \tag{3.2}$$

$$D_{\dot{\partial}_j}\delta_k = \Big(C^i_{jk} + \frac{1}{2}g^{li}R^s_{lk}g_{sj}\Big)\delta_i - L^i_{jk}\dot{\partial}_i, \tag{3.3}$$

$$D_{\dot{\partial}_k}\dot{\partial}_j = L^i_{jk}\delta_i + C^i_{jk}\dot{\partial}_i, \tag{3.4}$$

$$D_{\delta_k}dx^i = -\Gamma^i_{j;k}dx^j - \Big(C^i_{jk} + \frac{1}{2}g^{li}R^s_{lk}g_{sj}\Big)\delta y^j, \tag{3.5}$$

$$D_{\delta_k}\delta y^i = \Big(C^i_{jk} + \frac{1}{2}R^i_{jk}\Big)dx^j - \Gamma^i_{j;k}\delta y^j, \tag{3.6}$$

$$D_{\dot{\partial}_k}dx^i = -\Big(C^i_{jk} + \frac{1}{2}g^{li}R^s_{lj}g_{sk}\Big)dx^j - L^i_{jk}\delta y^j, \tag{3.7}$$

$$D_{\dot{\partial}_k}\delta y^i = L^i_{jk}dx^j - C^i_{jk}\delta y^j. \tag{3.8}$$

Proof. We only need to prove (3.1)-(3.4) since one can obtain (3.5)-(3.8) by the following formula

$$D_X\omega(Y) = X(\omega(Y)) - \omega(D_XY), \quad \forall X, Y \in \mathcal{X}(T\tilde{M}), \omega \in \mathcal{X}(T^*\tilde{M}).$$

As well known, the Levi-Civita connection D of $\tilde{g}$ is characterized by the following Koszul formula

$$2\tilde{g}(D_XY, Z) = X\tilde{g}(Y, Z) + Y\tilde{g}(Z, X) - Z\tilde{g}(X, Y) + \tilde{g}([X, Y], Z) - \tilde{g}([Y, Z], X) + \tilde{g}([Z, X], Y).$$

Using the facts that

$$\tilde{g}(\delta_i, \delta_j) = g_{ij}, \quad \tilde{g}(\dot{\partial}_i, \dot{\partial}_j) = g_{ij}, \quad \tilde{g}(\delta_i, \dot{\partial}_j) = 0,$$

we have

$$D_{\delta_k}\delta_j = \Gamma^i_{j;k}\delta_i - \Big(C^i_{jk} + \frac{1}{2}R^i_{jk}\Big)\dot{\partial}_i, \tag{3.9}$$

$$D_{\delta_k}\dot{\partial}_j = \Big(C^i_{jk} + \frac{1}{2}g^{li}R^s_{lk}g_{sj}\Big)\delta_i + \frac{1}{2}g^{li}\Big[\delta_k(g_{jl}) + g_{tl}G^t_{kj} - g_{tj}G^t_{kl}\Big]\dot{\partial}_i, \tag{3.10}$$

$$D_{\dot{\partial}_k}\dot{\partial}_j = -\frac{1}{2}g^{li}\Big[\delta_l(g_{kj}) - g_{sk}G^s_{jl} - g_{sj}G^s_{kl}\Big]\delta_i + C^i_{jk}\dot{\partial}_i. \tag{3.11}$$

Since $g_{tl}G^t_{kj} - g_{tj}G^t_{kl} = \delta_j(g_{lk}) - \delta_l(g_{jk})$, it follows that

$$\frac{1}{2}g^{li}\Big[\delta_k(g_{jl}) + g_{tl}G^t_{kj} - g_{tj}G^t_{kl}\Big] = \frac{1}{2}g^{li}\Big[\delta_k(g_{jl}) + \delta_j(g_{lk}) - \delta_l(g_{jk})\Big] = \Gamma^i_{j;k}. \tag{3.12}$$

Next, if we denote by $g_{jl;k} := \delta_k(g_{jl}) - g_{tl}G^t_{jk} - g_{jt}G^t_{lk}$ and $L^i_{jk} := G^i_{jk} - \Gamma^i_{j;k}$. Then it is easy to check that

$$g_{jl;k} = g_{jk;l}, \quad L^i_{jk} = L^i_{kj}, \tag{3.13}$$

consequently we have

$$L^i_{jk}=-\frac{1}{2}g^{li}g_{jl;k}=-\frac{1}{2}g^{li}\Big[\delta_l(g_{kj})-g_{sk}G^s_{jl}-g_{sj}G^s_{kl}\Big]. \tag{3.14}$$

This completes (3.2) and (3.4). By the torsion-freeness of D, we get (3.3). □

4 Laplacians in terms of Cartan connection

Let d be the exterior differential operator on $\tilde{M}$ and d^* be the formal adjoint of d with respect to the global inner product $(\cdot,\cdot)$ defined by (2.7). In this section we shall first give a representation of d and d^* in terms of the horizontal and vertical covariant derivatives of the Cartan connection ∇. Then we shall derive the horizontal Laplacian $\triangle_h$ and the vertical Laplacian $\triangle_v$ in terms of ∇, respectively.

Lemma 4.1. *Let (M,F) be a real Finsler manifold with the Cartan connection ∇. Then*

$$\begin{aligned} d=&\mathbf{e}(dx^s)\nabla_{\delta_s}-L^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\dot{\partial}_t)\\ &+\mathbf{e}(\delta y^s)\nabla_{\dot{\partial}_s}-C^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_t)-\frac{1}{2}R^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(dx^s)\mathbf{i}(\dot{\partial}_t). \end{aligned} \tag{4.1}$$

Proof. It is easy to check that (4.1) holds for every $\varphi\in\wedge^{p,q}_c(\tilde{M})$. □

Lemma 4.2. *Let $L^k_{ji}=G^k_{ji}-\mathit{\Gamma}^k_{j;i}$. Then $g^{ki}L^h_{ji}=g^{hi}L^k_{ji}$.*

Proof. In deed, it follows from (3.13) and (3.14) that

$$g^{hi}L^k_{ji}=-\frac{1}{2}g^{hi}g^{lk}g_{jl;i}=-\frac{1}{2}g^{hl}g^{ik}g_{ji;l}=-\frac{1}{2}g^{ki}g^{lh}g_{jl;i}=g^{ki}L^h_{ji}.$$

□

Lemma 4.3. *Let (M,F) be a real Finsler manifold with the Cartan connection ∇. Let ∇ also denote the induced linear connection on $T\tilde{M}$ and $T^*\tilde{M}$, respectively. Then for every $\varphi\in\wedge^{p,q}_c(\tilde{M})$,*

$$\nabla_{\delta_s}\mathbf{i}(\delta_k)\varphi = \mathbf{i}(\delta_k)\nabla_{\delta_s}\varphi+\mathit{\Gamma}^l_{k;s}\mathbf{i}(\delta_l)\varphi, \tag{4.2}$$

$$\nabla_{\delta_s}\mathbf{i}(\dot{\partial}_h)\varphi = \mathbf{i}(\dot{\partial}_h)\nabla_{\delta_s}\varphi+\mathit{\Gamma}^l_{h;s}\mathbf{i}(\dot{\partial}_l)\varphi, \tag{4.3}$$

$$\nabla_{\dot{\partial}_s}\mathbf{i}(\delta_k)\varphi = \mathbf{i}(\delta_k)\nabla_{\dot{\partial}_s}\varphi+C^l_{ks}\mathbf{i}(\delta_l)\varphi, \tag{4.4}$$

$$\nabla_{\dot{\partial}_s}\mathbf{i}(\dot{\partial}_h)\varphi = \mathbf{i}(\dot{\partial}_h)\nabla_{\dot{\partial}_s}\varphi+C^l_{hs}\mathbf{i}(\dot{\partial}_l)\varphi, \tag{4.5}$$

$$\nabla_{\delta_s}\mathbf{e}(dx^k)\varphi = \mathbf{e}(dx^k)\nabla_{\delta_s}\varphi-\mathit{\Gamma}^k_{l;s}\mathbf{e}(dx^l)\varphi, \tag{4.6}$$

$$\nabla_{\delta_s}\mathbf{e}(\delta y^k)\varphi = \mathbf{e}(\delta y^k)\nabla_{\delta_s}\varphi-\mathit{\Gamma}^k_{l;s}\mathbf{e}(\delta y^l)\varphi, \tag{4.7}$$

$$\nabla_{\dot{\partial}_s}\mathbf{e}(dx^k)\varphi = \mathbf{e}(dx^k)\nabla_{\dot{\partial}_s}\varphi-C^k_{ls}\mathbf{e}(dx^l)\varphi, \tag{4.8}$$

$$\nabla_{\dot{\partial}_s}\mathbf{e}(\delta y^k)\varphi = \mathbf{e}(\delta y^k)\nabla_{\dot{\partial}_s}\varphi-C^k_{ls}\mathbf{e}(\delta y^l)\varphi. \tag{4.9}$$

Proof. This is a direct calculation by using (2.2)-(2.3) and (2.11)-(2.13). □

Lemma 4.4. *Let (M, F) be a real Finsler manifold with the Cartan connection ∇, and d^* be the formal adjoint of the exterior differential operator d on $\tilde{M}$ with respect to the inner product $(\cdot, \cdot)$ defined by (2.7). Then*

$$\begin{aligned} d^* = & - g^{ki}\mathbf{i}(\delta_k)\nabla_{\delta_i} + g^{ki}L^h_{ji}\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_h) + g^{ki}L^h_{ik}\mathbf{i}(\delta_h) \\ & - g^{ki}\mathbf{i}(\dot{\partial}_k)\nabla_{\dot{\partial}_i} - g^{ki}\Big(C^h_{ki} + \frac{1}{2}R^h_{ki}\Big)\mathbf{i}(\dot{\partial}_h) + g^{ki}C^h_{ji}\mathbf{e}(dx^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_h) \\ & - \frac{1}{2}g^{ki}g^{lh}R^s_{li}g_{sj}\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\delta_h). \end{aligned} \tag{4.10}$$

Proof. In terms of the local frame $\{dx^i, \delta y^i\}$ for $T^*\tilde{M}$, the dual operator d^* of d can be expressed in terms of the Levi-Civita connection D of $\tilde{g}$ in an invariant form:

$$d^*\varphi = -g^{ki}\mathbf{i}(\delta_k)D_{\delta_i}\varphi - g^{ki}\mathbf{i}(\dot{\partial}_k)D_{\dot{\partial}_i}\varphi, \quad \varphi \in \wedge^{p,q}_c(\tilde{M}). \tag{4.11}$$

Using this and (3.5)-(3.8) and Lemma 4.2, one gets (4.10). □

Theorem 4.5. *Let (M, F) be a real Finsler manifold with the Cartan connection ∇. Let φ be a differential form of type (p, q) which is compactly supported in $\tilde{M}$. Then*

$$\begin{aligned} \triangle_h\varphi \;=\; & -g^{ki}\nabla_{\delta_i}\nabla_{\delta_k}\varphi - g^{ki}\mathbf{e}(dx^s)\mathbf{i}(\delta_k)(\nabla_{\delta_s}\nabla_{\delta_i} - \nabla_{\delta_i}\nabla_{\delta_s})\varphi + g^{ki}G^h_{ik}\nabla_{\delta_h}\varphi \\ & +g^{ki}\Big(L^h_{ji|s} + L^h_{js|i} + L^t_{ji}L^h_{st} - L^t_{js}L^h_{it}\Big)\mathbf{e}(dx^s)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_h)\varphi \\ & +g^{ki}L^h_{ik|s}\mathbf{e}(dx^s)\mathbf{i}(\delta_h)\varphi + g^{ki}\Big(L^t_{ks|i} - L^h_{is}L^t_{hk} - L^h_{ik}L^t_{hs}\Big)\mathbf{e}(\delta y^s)\mathbf{i}(\dot{\partial}_t)\varphi \\ & -g^{ki}L^h_{ji}L^t_{ks}\mathbf{e}(\delta y^j)\mathbf{e}(\delta y^s)\mathbf{i}(\dot{\partial}_h)\mathbf{i}(\dot{\partial}_t)\varphi. \end{aligned}$$

Proof. By (4.1) and (4.10), we get

$$\begin{aligned} d_h \;&=\; \mathbf{e}(dx^s)\nabla_{\delta_s} - L^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\dot{\partial}_t), \\ d^*_h \;&=\; -g^{ki}\mathbf{i}(\delta_k)\nabla_{\delta_i} + g^{ki}L^h_{ji}\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_h) + g^{ki}L^h_{ik}\mathbf{i}(\delta_h). \end{aligned}$$

Thus

$$\begin{aligned} d_h \circ d^*_h \;=\; & -\mathbf{e}(dx^s)\nabla_{\delta_s}g^{ki}\mathbf{i}(\delta_k)\nabla_{\delta_i} + \mathbf{e}(dx^s)\nabla_{\delta_s}g^{ki}L^h_{ji}\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_h) \\ & +\mathbf{e}(dx^s)\nabla_{\delta_s}g^{ki}L^h_{ik}\mathbf{i}(\delta_h) + g^{ki}L^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\dot{\partial}_t)\mathbf{i}(\delta_k)\nabla_{\delta_i} \\ & -g^{ki}L^h_{ji}L^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\dot{\partial}_t)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_h) \\ & -g^{ki}L^h_{ik}L^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\dot{\partial}_t)\mathbf{i}(\delta_h). \end{aligned}$$

Since the Cartan connection ∇ is horizontal metrical, it follows that

$$g^{ki}{}_{|s} = \delta_s(g^{ki}) + g^{ti}\varGamma^k_{t;s} + g^{kt}\varGamma^i_{t;s} = 0, \tag{4.12}$$

where $_|$ denotes the horizontal covariant derivative of ∇. Using Lemma 4.3 and (4.12), we obtain

$$\begin{aligned} d_h \circ d_h^* &= g^{kt}\Gamma^i_{t;s}\mathbf{e}(dx^s)\mathbf{i}(\delta_k)\nabla_{\delta_i} - g^{ki}\mathbf{e}(dx^s)\mathbf{i}(\delta_k)\nabla_{\delta_s}\nabla_{\delta_i} \\ &+g^{ki}P^h_{ji|s}\mathbf{e}(dx^s)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_h) + g^{ki}L^h_{ji}\mathbf{e}(dx^s)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_h)\nabla_{\delta_s} \\ &+g^{ki}L^h_{ik|s}\mathbf{e}(dx^s)\mathbf{i}(\delta_h) + g^{ki}L^h_{ik}\mathbf{e}(dx^s)\mathbf{i}(\delta_h)\nabla_{\delta_s} \\ &-g^{ki}L^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_t)\nabla_{\delta_i} - g^{ki}L^h_{ji}L^j_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_h) \\ &-g^{ki}L^h_{ji}L^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_t)\mathbf{i}(\dot{\partial}_h) \\ &+g^{ki}L^h_{ik}L^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_h)\mathbf{i}(\dot{\partial}_t). \end{aligned}$$

On the other hand, it is easy to check that

$$\begin{aligned} d_h^* \circ d_h &= g^{ki}\Gamma^s_{k;i}\nabla_{\delta_s} - g^{ki}\Gamma^s_{l;i}\mathbf{e}(dx^l)\mathbf{i}(\delta_k)\nabla_{\delta_s} - g^{ki}\nabla_{\delta_i}\nabla_{\delta_k} + g^{ki}\mathbf{e}(dx^s)\mathbf{i}(\delta_k)\nabla_{\delta_i}\nabla_{\delta_s} \\ &+g^{ki}L^t_{ks|i}\mathbf{e}(\delta y^s)\mathbf{i}(\dot{\partial}_t) + g^{ki}L^t_{rs|i}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_t) \\ &+g^{ki}L^t_{ks}\mathbf{e}(\delta y^s)\mathbf{i}(\dot{\partial}_t)\nabla_{\delta_i} + g^{ki}L^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_t)\nabla_{\delta_i} \\ &-g^{ki}L^h_{ji}\mathbf{e}(\delta y^j)\mathbf{i}(\dot{\partial}_h)\nabla_{\delta_k} - g^{ki}L^h_{ji}\mathbf{e}(dx^s)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_h)\nabla_{\delta_s} \\ &+g^{ki}L^h_{ji}L^t_{kh}\mathbf{e}(\delta y^j)\mathbf{i}(\dot{\partial}_t) - g^{ki}L^h_{ji}L^t_{ks}\mathbf{e}(\delta y^j)\mathbf{e}(\delta y^s)\mathbf{i}(\dot{\partial}_h)\mathbf{i}(\dot{\partial}_t) \\ &+g^{ki}L^h_{ji}L^t_{rh}\mathbf{e}(dx^r)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_t) \\ &+g^{ki}L^h_{ji}L^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^j)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_h)\mathbf{i}(\dot{\partial}_t) \\ &+g^{ki}L^h_{ik}\nabla_{\delta_h} - g^{ki}L^h_{ik}\mathbf{e}(dx^s)\mathbf{i}(\delta_h)\nabla_{\delta_s} \\ &-g^{ki}L^h_{ik}L^t_{hs}\mathbf{e}(\delta y^s)\mathbf{i}(\dot{\partial}_t) - g^{ki}L^h_{ik}L^t_{rs}\mathbf{e}(dx^r)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_h)\mathbf{i}(\dot{\partial}_t), \end{aligned}$$

where $L^t_{rs|i}$ denote the horizontal covariant derivative of L^t_{rs} with respect to ∇. Now sum $d_h \circ d_h^*$ and $d_h^* \circ d_h$ together and rearrange the resulted terms and indexes, we obtain the expression of $\triangle_h$ in Theorem 4.5. □

Remark 4.6. *The horizontal Laplacian for functions $f \in C^\infty(\tilde{M})$ was first derived in [4], and the horizontal and vertical Laplacians for 1-forms on the total space E of Riemann vector bundle $p : E \to M$ was first derived in [8] under the assumption that the fiber space $\mathcal{F} = p^{-1}(x), x \in M$, is compact.*

Note that if F is a Landsberg metric then $L^i_{jk} = 0$. Thus we have

Corollary 4.7. *Let (M, F) be a Landsberg manifold with the Cartan connection ∇, and φ be a differential form of type (p, q) which is compactly supported in $\tilde{M}$. Then*

$$\triangle_h\varphi = -g^{ki}\nabla_{\delta_i}\nabla_{\delta_k}\varphi - g^{ki}\mathbf{e}(dx^s)\mathbf{i}(\delta_k)(\nabla_{\delta_s}\nabla_{\delta_i} - \nabla_{\delta_i}\nabla_{\delta_s})\varphi + g^{ki}G^h_{ik}\nabla_{\delta_h}\varphi.$$

Corollary 4.8. *Let (M, F) be a real Finsler manifold with the Cartan connection ∇, and φ be a horizontal differential form of type p which is compactly supported in $\tilde{M}$. Then*

$$\begin{aligned} \triangle_h\varphi &= -g^{ki}\nabla_{\delta_i}\nabla_{\delta_k}\varphi - g^{ki}\mathbf{e}(dx^s)\mathbf{i}(\delta_k)(\nabla_{\delta_s}\nabla_{\delta_i} - \nabla_{\delta_i}\nabla_{\delta_s})\varphi \\ &+g^{ki}G^h_{ik}\nabla_{\delta_h}\varphi + g^{ki}L^h_{ik|s}\mathbf{e}(dx^s)\mathbf{i}(\delta_h)\varphi. \end{aligned}$$

Theorem 4.9. *Let (M,F) be a real Finsler manifold with the Cartan connection ∇, and φ be a differential form of type (p,q) which is compactly supported in $\tilde{M}$. Then*

$$\begin{aligned}\triangle_v\varphi &= -g^{ki}\nabla_{\dot\partial_i}\nabla_{\dot\partial_k}\varphi - g^{ki}\mathbf{e}(\delta y^s)\mathbf{i}(\dot\partial_k)\Big(\nabla_{\dot\partial_s}\nabla_{\dot\partial_i}-\nabla_{\dot\partial_i}\nabla_{\dot\partial_s}\Big)\varphi - \frac{1}{2}g^{ki}R^h_{ki}\nabla_{\dot\partial_h}\varphi\\ &\quad -g^{ki}\Big(C^h_{ki}+\frac{1}{2}R^h_{ki}\Big)_{\|s}\mathbf{e}(\delta y^s)\mathbf{i}(\dot\partial_h)\varphi\\ &\quad -g^{hi}\Big[2C^k_{ji\|s}-(C^t_{ji}C^k_{ts}-C^t_{js}C^k_{ti})\Big]\mathbf{e}(dx^j)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot\partial_h)\varphi\\ &\quad -g^{ki}\Big[C^a_{sk\|i}+\frac{1}{2}R^h_{ki}C^a_{sh}+(C^h_{ki}C^a_{sh}-C^h_{si}C^a_{kh})\Big]\mathbf{e}(dx^s)\mathbf{i}(\delta_a)\varphi\\ &\quad -g^{ki}C^t_{ji}C^h_{st}\mathbf{e}(dx^j)\mathbf{e}(dx^s)\mathbf{i}(\delta_k)\mathbf{i}(\delta_h)\varphi.\end{aligned}$$

Proof. By Lemma 4.1 and Lemma 4.4, we have

$$d_v = \mathbf{e}(\delta y^s)\nabla_{\dot\partial_s} - C^a_{st}\mathbf{e}(dx^s)\mathbf{e}(\delta y^t)\mathbf{i}(\delta_a) \tag{4.13}$$

and

$$d_v^* = -g^{ki}\mathbf{i}(\dot\partial_k)\nabla_{\dot\partial_i} - g^{ki}\Big(C^h_{ki}+\frac{1}{2}R^h_{ki}\Big)\mathbf{i}(\dot\partial_h) + g^{ki}C^h_{ji}\mathbf{e}(dx^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot\partial_h). \tag{4.14}$$

Since ∇ is v-metrical, it follows that the vertical covariant derivatives of g^{ki} vanish identically, i.e.,

$$g^{ki}{}_{\|s} = \dot\partial_s(g^{ki}) + g^{ti}C^k_{ts} + g^{kt}C^i_{ts} \equiv 0.$$

Using this fact and the properties of the operators $\mathbf{i}$ and $\mathbf{e}$, we obtain

$$\begin{aligned}d_v\circ d_v^* &= g^{ti}C^k_{st}\mathbf{e}(\delta y^s)\mathbf{i}(\dot\partial_k)\nabla_{\dot\partial_i} - g^{ki}\mathbf{e}(\delta y^s)\mathbf{i}(\dot\partial_k)\nabla_{\dot\partial_s}\nabla_{\dot\partial_i}\\ &\quad -g^{ki}\Big(C^h_{ki}+\frac{1}{2}R^h_{ki}\Big)_{\|s}\mathbf{e}(\delta y^s)\mathbf{i}(\dot\partial_h) - g^{ki}\Big(C^h_{ki}+\frac{1}{2}R^h_{ki}\Big)\mathbf{e}(\delta y^s)\mathbf{i}(\dot\partial_h)\nabla_{\dot\partial_s}\\ &\quad +\Big[-g^{ki}C^h_{ji\|s}+g^{ti}C^k_{js}\Big(C^h_{ti}+\frac{1}{2}R^h_{ti}\Big)-g^{ki}C^t_{js}C^h_{ti}\Big]\mathbf{e}(dx^j)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot\partial_h)\\ &\quad -g^{ki}C^h_{ji}\mathbf{e}(dx^j)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot\partial_h)\nabla_{\dot\partial_s} + g^{hi}C^k_{js}\mathbf{e}(dx^j)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot\partial_h)\nabla_{\dot\partial_i}\\ &\quad -g^{ki}C^a_{st}C^h_{ji}\mathbf{e}(dx^s)\mathbf{e}(dx^j)\mathbf{e}(\delta y^t)\mathbf{i}(\delta_a)\mathbf{i}(\delta_k)\mathbf{i}(\dot\partial_h).\end{aligned}$$

By similar calculations, we have

$$\begin{aligned}d_v^*\circ d_v &= -\frac{1}{2}g^{ki}R^h_{ki}\nabla_{\dot\partial_h} - g^{ki}\nabla_{\dot\partial_i}\nabla_{\dot\partial_k} + g^{ki}\mathbf{e}(\delta y^s)\mathbf{i}(\dot\partial_k)\nabla_{\dot\partial_i}\nabla_{\dot\partial_s}\\ &\quad -g^{kt}C^i_{st}\mathbf{e}(\delta y^s)\mathbf{i}(\dot\partial_k)\nabla_{\dot\partial_i} + g^{ki}\Big(C^h_{ki}+\frac{1}{2}R^h_{ki}\Big)\mathbf{e}(\delta y^s)\mathbf{i}(\dot\partial_h)\nabla_{\dot\partial_s}\\ &\quad +\Big(g^{at}C^i_{st}-g^{ki}C^a_{sk}\Big)\mathbf{e}(dx^s)\mathbf{i}(\delta_a)\nabla_{\dot\partial_i}\\ &\quad +\Big[-g^{ki}C^a_{sk\|i}+g^{ki}C^h_{si}C^a_{kh}-g^{ki}\Big(C^h_{ki}+\frac{1}{2}R^h_{ki}\Big)C^a_{sh}\Big]\mathbf{e}(dx^s)\mathbf{i}(\delta_a)\\ &\quad +\Big[-g^{hi}C^k_{js\|i}-g^{ti}\Big(C^h_{ti}+\frac{1}{2}R^h_{ti}\Big)C^k_{js}+g^{ti}C^h_{ji}C^k_{ts}\Big]\mathbf{e}(dx^j)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot\partial_h)\\ &\quad -g^{ki}C^t_{ji}C^h_{st}\mathbf{e}(dx^j)\mathbf{e}(dx^s)\mathbf{i}(\delta_k)\mathbf{i}(\delta_h) - g^{hi}C^k_{js}\mathbf{e}(dx^j)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot\partial_h)\nabla_{\dot\partial_i}\\ &\quad +g^{ki}C^h_{ji}\mathbf{e}(dx^j)\mathbf{e}(\delta y^s)\mathbf{i}(\delta_k)\mathbf{i}(\dot\partial_h)\nabla_{\dot\partial_s} + g^{ki}C^h_{ji}C^a_{st}\mathbf{e}(dx^j)\mathbf{e}(dx^s)\mathbf{e}(\delta y^t)\mathbf{i}(\delta_k)\mathbf{i}(\delta_a)\mathbf{i}(\dot\partial_h).\end{aligned}$$

Note that $g^{ki}C^h_{ji} = g^{hi}C^k_{ji}$ and $g^{ki}{}_{\|s} = 0$, thus

$$\begin{aligned} & g^{ki}C^h_{ji\|s} + g^{ki}C^t_{js}C^h_{ti} + g^{hi}C^k_{js\|i} - g^{ti}C^h_{ji}C^k_{ts} \\ = \;& g^{hi}\Big(C^k_{ji\|s} + C^k_{js\|i}\Big) - g^{hi}\Big(C^t_{ji}C^k_{ts} - C^t_{js}C^k_{ti}\Big). \end{aligned}$$

Now it is easy to check that $C^k_{ji\|s} = C^k_{js\|i}$. Thus sum $d_v \circ d_v^*$ and $d_v^* \circ d_v$ together and rearrange the resulted terms, we obtain $\triangle_v$ in Theorem 4.9. □

Corollary 4.10. *Let ∇ be the Cartan connection associated to a real Finsler manifold (M, F), and φ be a horizontal differential form of type p which is compactly supported in $\tilde{M}$. Then*

$$\begin{aligned} \triangle_v\varphi \;=\; & -g^{ki}\nabla_{\dot\partial_i}\nabla_{\dot\partial_k}\varphi - \frac{1}{2}g^{ki}R^h_{ki}\nabla_{\dot\partial_h}\varphi \\ & -g^{ki}\Big(C^a_{sk\|i} - C^h_{si}C^a_{kh} + C^h_{ki}C^a_{sh} + \frac{1}{2}R^h_{ki}C^a_{sh}\Big)\mathbf{e}(dx^s)\mathbf{i}(\delta_a)\varphi \\ & -g^{ki}C^t_{ji}C^h_{st}\mathbf{e}(dx^j)\mathbf{e}(dx^s)\mathbf{i}(\delta_k)\mathbf{i}(\delta_h)\varphi. \end{aligned}$$

5 The mixed-type operator

In this section, we shall derive the local expression of $\triangle_{mix} = d_{mix} \circ d^*_{mix} + d^*_{mix} \circ d_{mix}$.

Theorem 5.1. *Let φ be a differential form of type (p, q) which is compactly supported in $\tilde{M}$. Then*

$$\begin{aligned} \triangle_{mix}\varphi = & \frac{1}{4}g^{ki}g^{lh}R^j_{ac}R^s_{li}g_{sj}\mathbf{e}(dx^a)\mathbf{e}(dx^c)\mathbf{i}(\delta_k)\mathbf{i}(\delta_h)\varphi \\ & + \frac{1}{2}g^{ki}g^{lh}R^t_{hk}R^s_{li}g_{sj}\mathbf{e}(\delta y^j)\mathbf{i}(\dot\partial_t)\varphi \\ & + g^{ki}g^{lh}R^t_{hc}R^s_{li}g_{sj}\mathbf{e}(dx^c)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot\partial_t)\varphi. \end{aligned} \tag{5.1}$$

Proof. On the one hand,

$$\begin{aligned} d_{mix} \circ d^*_{mix} \;=\; & \frac{1}{4}g^{ki}g^{lh}R^j_{ac}R^s_{li}g_{sj}\mathbf{e}(dx^a)\mathbf{e}(dx^c)\mathbf{i}(\delta_k)\mathbf{i}(\delta_h) \\ & -\frac{1}{4}g^{ki}g^{lh}R^t_{ac}R^s_{li}g_{sj}\mathbf{e}(dx^a)\mathbf{e}(dx^c)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\delta_h)\mathbf{i}(\dot\partial_t). \end{aligned}$$

On the other hand,

$$d^*_{mix} \circ d_{mix} \;=\; \frac{1}{4}g^{ki}g^{lh}R^t_{ac}R^s_{li}g_{sj}\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\delta_h)\mathbf{e}(dx^a)\mathbf{e}(dx^c)\mathbf{i}(\dot\partial_t).$$

Using the relationship $\mathbf{i}(\delta_h)\mathbf{e}(dx^a) = \delta^a_h - \mathbf{e}(dx^a)\mathbf{i}(\delta_h)$ repeatedly, we get

$$\begin{aligned} \mathbf{i}(\delta_k)\mathbf{i}(\delta_h)\mathbf{e}(dx^a)\mathbf{e}(dx^c) \;=\; & \delta^a_h\delta^c_k - \delta^a_h\mathbf{e}(dx^c)\mathbf{i}(\delta_k) - \delta^a_k\delta^c_h + \delta^a_k\mathbf{e}(dx^c)\mathbf{i}(\delta_h) + \delta^c_h\mathbf{e}(dx^a)\mathbf{i}(\delta_k) \\ & -\delta^c_k\mathbf{e}(dx^a)\mathbf{i}(\delta_h) + \mathbf{e}(dx^a)\mathbf{e}(dx^c)\mathbf{i}(\delta_k)\mathbf{i}(\delta_h). \end{aligned}$$

Thus

$$\begin{aligned}
d^*_{mix}\circ d_{mix} &= \frac{1}{4}g^{ki}g^{lh}(R^t_{hk}-R^t_{kh})R^s_{li}g_{sj}\mathbf{e}(\delta y^j)\mathbf{i}(\dot{\partial}_t)\\
&+\frac{1}{4}g^{ki}g^{lh}R^t_{hc}R^s_{li}g_{sj}\mathbf{e}(dx^c)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_t)\\
&-\frac{1}{4}g^{ki}g^{lh}R^t_{kc}R^s_{li}g_{sj}\mathbf{e}(dx^c)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_h)\mathbf{i}(\dot{\partial}_t)\\
&-\frac{1}{4}g^{ki}g^{lh}R^t_{ah}R^s_{li}g_{sj}\mathbf{e}(dx^a)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\dot{\partial}_t)\\
&+\frac{1}{4}g^{ki}g^{lh}R^t_{ak}R^s_{li}g_{sj}\mathbf{e}(dx^a)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_h)\mathbf{i}(\dot{\partial}_t)\\
&+\frac{1}{4}g^{ki}g^{lh}R^t_{ac}R^s_{li}g_{sj}\mathbf{e}(dx^a)\mathbf{e}(dx^c)\mathbf{e}(\delta y^j)\mathbf{i}(\delta_k)\mathbf{i}(\delta_h)\mathbf{i}(\dot{\partial}_t).
\end{aligned}$$

Sum $d_{mix}\circ d^*_{mix}$ and $d^*_{mix}\circ d_{mix}$ together, and use the fact $R^i_{jk}=-R^i_{kj}$, we get (5.1). □

Remark 5.2. *It follows from Theorem 4.5 and Theorem 4.9 that the local expressions of $\triangle_h$ and $\triangle_v$ depend on the choice of Finsler connection associated to (M,F), while the expression of $\triangle_{mix}$ is independent of the choice of Finsler connections associated to (M,F).*

Corollary 5.3. *Let φ be a horizontal form of type p which is compactly supported in $\tilde{M}$. Then*

$$\triangle_{mix}\varphi = \frac{1}{4}g^{ki}g^{lh}R^j_{ac}R^s_{li}g_{sj}\mathbf{e}(dx^a)\mathbf{e}(dx^c)\mathbf{i}(\delta_k)\mathbf{i}(\delta_h)\varphi.$$

Especially, for every horizontal 1-form $\varphi=\varphi_i dx^i$ which is compactly supported in $\tilde{M}$, we have $\triangle_{mix}\varphi\equiv 0$.

6 Some properties of the type preserving operators

Let $\triangle$ be the Hodge-Laplace operator of $\tilde{g}$ on $\tilde{M}$. It is clear that $\triangle,\triangle_h,\triangle_v,\triangle_{mix}$ satisfy

$$(6.1)\qquad (\triangle\varphi,\psi)=(\triangle_h\varphi,\psi)+(\triangle_v\varphi,\psi)+(\triangle_{mix}\varphi,\psi),$$

which implies that $\triangle\varphi=0$ if and only if

$$(6.2)\qquad \triangle_h\varphi=0,\triangle_v\varphi=0,\triangle_{mix}\varphi=0,\quad \forall\varphi\in\wedge^{p,q}_c(\tilde{M}).$$

This is essentially arisen from the type-preserving property of $\triangle_h,\triangle_v$ and $\triangle_{mix}$. Especially if $f\in C^\infty_c(\tilde{M})$, then

$$(6.3)\qquad 0\le(df,df)=(\triangle f,f)=(\triangle_h f,f)+(\triangle_v f,f),$$

which implies that $\triangle f=0$ if and only if $\triangle_h f=0$ and $\triangle_v f=0$.

Proposition 6.1. *Let λ, λ_h and λ_v be constants such that*

$$\triangle f = \lambda f, \quad \triangle_h f = \lambda_h f, \quad \triangle_v f = \lambda_v f,$$

Then $\lambda \geq 0, \lambda_h \geq 0, \lambda_v \geq 0$ and

$$\lambda = \lambda_h + \lambda_v. \tag{6.4}$$

Proof. It is clear. □

Theorem 6.2. *Let f be a scalar field on $\tilde{M}$ which is not constant and satisfies $\triangle_h f = \lambda_h f, \triangle_v f = \lambda_v f$ with λ_h, λ_v being a constant such that $\lambda_h^2 + \lambda_v^2 \neq 0$. Then the constant $\lambda =: \lambda_h + \lambda_v$ must be positive.*

Proof. It is easy to check that

$$\frac{1}{2}\triangle_h f^2 = f\triangle_h f - g^{ij}(\nabla_{\delta_i} f)(\nabla_{\delta_j} f), \quad \frac{1}{2}\triangle_v f^2 = f\triangle_v f - g^{ij}(\nabla_{\dot{\partial}_i} f)(\nabla_{\dot{\partial}_j} f).$$

Since f is compactly supported in $\tilde{M}$, thus

$$0 = \int_{\tilde{M}} \frac{1}{2}\triangle f^2 dV = \int_{\tilde{M}} \Big[f\triangle f - g^{ij}(\nabla_{\delta_i} f)(\nabla_{\delta_j} f) - g^{ij}(\nabla_{\dot{\partial}_i} f)(\nabla_{\dot{\partial}_j} f)\Big]dV. \tag{6.5}$$

Substituting the identity $\triangle f = \triangle_h f + \triangle_v f$ in (6.5), we get

$$\int_{\tilde{M}} \Big[\lambda f^2 - g^{ij}(\nabla_{\delta_i} f)(\nabla_{\delta_j} f) - g^{ij}(\nabla_{\dot{\partial}_i} f)(\nabla_{\dot{\partial}_j} f)\Big]dV = 0.$$

If on the contrary $\lambda < 0$, then the above equality implies that $\nabla_{\delta_i} f = \nabla_{\dot{\partial}_i} f \equiv 0$, that is, f is a constant. This is a contradiction to the assumption. □

Theorem 6.3. *There exists no horizontal 1-form $\varphi = \varphi_i dx^i$ with compact support in $\tilde{M}$ which satisfies relations*

$$g^{ki}\Big(\varphi_{l|k|i} - G^h_{ki}\varphi_{l|h}\Big) + g^{ki}\Big(\varphi_{l\|k\|i} + \frac{1}{2}R^h_{ki}\varphi_{l\|h}\Big) = T_{tl}\varphi^t \tag{6.6}$$

and

$$T_{tl}\varphi^t\varphi^l \geq 0 \tag{6.7}$$

unless we have

$$\varphi_{l|k} = 0, \ \ \varphi_{l\|k} = 0 \tag{6.8}$$

and then automatically $T_{tl}\varphi^t\varphi^l = 0$.

Proof. Denote $\|\varphi\|^2 := g^{lt}\varphi_l\varphi_t$ and $\varphi^l := g^{lt}\varphi_t$. Then it is easy to check that

$$\begin{aligned}\triangle_h\|\varphi\|^2 &= -2g^{ki}\Big(\varphi_{l|k|i} - G^h_{ki}\varphi_{l|h}\Big)\varphi^l - 2g^{ki}g^{lt}\varphi_{l|i}\varphi_{t|k},\\ \triangle_v\|\varphi\|^2 &= -2g^{ki}\Big(\varphi_{l\|k\|i} + \frac{1}{2}R^h_{ki}\varphi_{l\|h}\Big)\varphi^l - 2g^{ki}g^{lt}\varphi_{l\|k}\varphi_{t\|i}.\end{aligned}$$

Thus if condition (6.6) and (6.7) are satisfied, we have

$$-\frac{1}{2}\triangle\|\varphi\|^2 = T_{tl}\varphi^t\varphi^l + g^{ki}g^{lt}(\varphi_{l|i}\varphi_{t|k} + \varphi_{l\|k}\varphi_{t\|i}) \geq 0,$$

from which we get (6.8). □

Remark 6.4. *Note that a function $f \in C^\infty(\tilde{M})$ which is both horizontal and vertical parallel with respect to the Cartan connection ∇ is necessary a constant. It follows from Theorem 6.3 that the only exceptions are vector fields which are both horizontal and vertical parallel, and there are no such vector fields other than zero if the quadratic form $T_{tl}\varphi^t\varphi^l$ is positive definite.*

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